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The Fréchet variation and a generalization for multiple Fourier series of the Jordan test. (**)

1. – Introduction.

Recent discoveries by the authors of certain new properties of the FRÉCHET variation of a function f defined over a p-dimensional interval $I^{(p)}$ make it possible to give a more natural generalization of the JORDAN test for the convergence of a simple FOURIER series of a function f. The function f is supposed defined in a Cartesian p-space $R^{(p)}$ and to have the period 2π in each coordinate. Moreover f is supposed to satisfy conditions $\widehat{F}$ over the closed p-interval $I^{(p)}[\mathbf{0}, 2\pi \boldsymbol{i}]$ with vertices $\mathbf{0} = (0,..., 0)$ and $\boldsymbol{i} = (1,..., 1)$. *Conditions* $\widehat{F}$ require that the FRÉCHET variation $P^{(p)}[g, I^{(p)}]$ of g over $I^{(p)}$ be finite, and that the FRÉCHET variation be finite over a set of sections of $I^{(p)}$ which includes an r-section parallel to each coordinate r-plane $(r = 1,..., p)$.

A first consequence of these hypotheses is as follows. See MORSE and TRANSUE (5) [hereafter referred to as MT(5)]. Let $s = (s^{(1)},..., s^{(p)})$ be the coordinates of a point in $R^{(p)}$. Let a be a particular point in $R^{(p)}$. The $(p-1)$-planes $[s^{(i)} = a^{(i)}]$ $i = 1,..., p$ separate $R^{(p)}$ into 2^p open sectors S_a (generalizing octants in 3-space). The above function f tends to a definite « sector limit » as $s \to a$ in any one sector S_a. These 2^p limits may all be different at a given point a.

The theorem generalizing the JORDAN theorem follows.

THEOREM 1(a). *If f has the period 2π in each coordinate in $R^{(p)}$ and satisfies conditions $\widehat{F}$ over $I^{(p)}[\mathbf{0}, 2\pi \boldsymbol{i}]$ then the* FOURIER *series for f converges in the sense of* PRINGSHEIM *to the mean $\bar{f}(s)$ of the sector limits of f at any given point s.* (b) *If in addition f is continuous the* FOURIER *series converges uniformly to f.*

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The condition that $P^{(p)}[f, I^{(p)}]$ be finite is much less restrictive than the condition that the VITALI variation $V^{(p)}[f, I^{(p)}]$ be finite. See ADAMS and CLARKSON, and MT(7). Thus our conditions are less restrictive than the classical conditions of HARDY-KRAUSE. See GERGEN where the classical tests are compared and extended. The properties of the FRÉCHET variation established in MT(5), (7) and (9) permit a lightening of the 2-dimensional tests of DINI-YOUNG, YOUNG-POLLARD, DE LA VALLEÉ POUSSIN-YOUNG, LEBESGUE-GERGEN, TONELLI and GERGEN. In MT(10) we prove that our modification of the GERGEN test is *definitely less restrictive* than each of the above tests. This is somewhat surprising since the classical theory, using the VITALI variation, includes a proof by GERGEN that his conditions are *not more* restrictive than those of the three other LEBESGUE type tests admitted, but there seems to be no proof that the GERGEN test is *less* restrictive.

The classical bound $\frac{2V}{\pi n}$ for the absolute values $|a_n|$, $|b_n|$ of the FOURIER coefficients when $p = 1$, has here its appropriate generalization when $p > 1$ and when the FRÉCHET variation replaces the JORDAN variation. For the case $p = 1$ see ZYGMUND, p. 19.

The advances made in the theory of the FRÉCHET variation in MT(5) which appear essential in the application of this theory to multiple FOURIER series center around the following topics (See § 2).

(1). *The property that* $P^{(r)}[f, Q^{(r)}] \to 0$, *as an arbitrary r-segment* $Q^{(r)}$ *tends to a point* $a \in R^{(p)}$, $Q^{(r)}$ *remaining in a sector* S_a *with vertex at* a.

(2). *The existence of the sector limits of* f.

(3). *The existence of a transform* g *of* f *which equals* f *on the maximal open subinterval* $I_0^{(p)}$ *of* $I^{(p)}$ *and which has « canonical » boundary values.*

(4). *The existence of a left decomposition* $g = \sum_{\sigma} g_\sigma$ *of the transform* g *of* f.

(5). *The existence of a « variation modulus » common to the above functions* g_σ, *when* g *is continuous.*

(6). *An inequality replacing the second law of the mean when multiple integrals of the* DIRICHLET *type are concerned.*

We shall enlarge on these points in § 2.

2. - The Fréchet variation.

The space $R^{(p)}$ is referred to coordinates $s = [s^{(1)}, \ldots, s^{(p)}]$. Let K_r represent an interval for $s^{(r)}$ chosen from one of the intervals

$$(2.0) \qquad (a^{(r)}, b^{(r)}), \quad (a^{(r)}, b^{(r)}], \quad [a^{(r)}, b^{(r)}), \quad [a^{(r)}, b^{(r)}]$$

with $a^{(r)} < b^{(r)}$, including (excluding) an end point adjacent to a square (round) parenthesis. We shall permit K_r to reduce to the point $a^{(r)}$. By an *n-segment* in $R^{(p)}$ is meant the set of points s in a Cartesian product

$$Q^{(n)} = K_1 \times K_2 \times \ldots \times K_p \tag{2.1}$$

where n is the number of the K_i which are intervals and not merely points $a^{(i)}$. We suppose that $0 < n \leqq p$.

When $n = p$ we term $Q^{(p)}$ a *p-interval* in $R^{(p)}$. When $n = p$ and each interval K_r is of the form $(a^{(r)}, b^{(r)})$ or $[a^{(r)}, b^{(r)}]$, $Q^{(p)}$ is open or closed respectively and we write

$$Q^{(p)} = I^{(p)}(a, b)\,, \qquad Q^{(p)} = I^{(p)}[a, b]\,.$$

Let $Z_r Q^{(n)}$ denote the orthogonal projection of $Q^{(n)}$, as represented by (2.1), onto the $(p-1)$-plane $[s^{(r)} = a^{(r)}]$. By the *left boundary* of $Q^{(n)}$ is meant the

$$\text{Union } Z_r Q^{(n)} \tag{2.2}$$

where r ranges over the integers for which K_r in (2.1) is an interval and not merely a point.

By an *m-face*, $m > 0$, of the n-segment (2.1) is meant an m-segment $F^{(m)}$ $(0 < m \leqq n)$ of the form

$$F(m) = K_1' \times \ldots \times K_p' \tag{2.3}$$

in which $K_r' = a^{(r)}$ if $K_r = a^{(r)}$, while K_r' is either an end point of K_r or one of the intervals (2.0) in case K_r is one of the intervals (2.0). In particular $Q^{(n)}$ is included as one of its own faces. It will also be convenient to term a vertex of $Q^{(n)}$ a *0-face* of $Q^{(n)}$.

Suppose that g maps $Q^{(n)}$ into $R^{(1)}$. We refer the reader to MT(5) for the definition of the Fréchet variation $P^{(n)}[g, Q^{(n)}]$ of g over $Q^{(n)}$. We say that g satisfies *conditions* $\widehat{F}$ over $Q^{(n)}$ if $P^{(n)}[g, Q^{(n)}]$ is finite, and if to an arbitrary r-face of $Q^{(n)}$, $0 < r < n$ there corresponds a parallel r-section $H^{(r)}$ of $Q^{(n)}$ such that $P^{(r)}[g, H^{(r)}]$ is finite.

We shall summarize those results of MT(5) which are needed.

Theorem 2.1. *If g satisfies $\widehat{F}$ over a general interval $I^{(p)} \subset R^{(p)}$ the following is true.* (1). *The values $|g(s)|$ are bounded over $I^{(p)}$ and g is L-measureable.* (2). *The points of discontinuity of g lie on at most a countable set of $(p-1)$-sections of $I^{(p)}$ parallel to the coordinate planes.* (3). *If a is any point in $I^{(p)}$ and S_a a sector (open) which intersects $I^{(p)}$, then as $s \to a$ in S_a, $g(s)$ tends to a unique limit.* (4). *The variation $P^{(r)}[g, Q^{(r)}]$ is uniformly bounded* *over r-segments* $Q^{(r)} \subset I^{(p)}$, $r = 1, \ldots, p$. *See* MT(5), *Theorems* 3.1, 5.1, 8.4, 8.7 *and Cor.* 3.1.

Most of the results in MT(5) are stated for an interval determined by the points

$$\mathbf{0} = (0, \ldots, 0), \qquad \boldsymbol{i} = (1, \ldots 1) \tag{2.4}$$

in $R^{(p)}$. It is clear that there exists a 1-1 linear transformation mapping $I^{(p)}[a, b]$ onto $I^{(p)}[\mathbf{0}, \boldsymbol{i}]$ with $\mathbf{0}$ and $\boldsymbol{i}$ the images of a and b respectively. By virtue of such a mapping the results of MT(5) can be carried over to the more general intervals determined by the vertices a and b.

Before defining canonical *left boundary values* the limit g^r must be defined. Let

$$g(s^{(1)}, \ldots, s^{(r-1)}, s^{(r)} + u, s^{(r+1)}, \ldots, s^{(p)})$$

be defined for a fixed point s, with u on a 1-interval $(0, e)$. We then set

$$g(s^{(1)}, \ldots, s^{(r-1)}, s^{(r)}+, s^{(r+1)}, \ldots, s^{(p)}) = g^r(s) \qquad (r = 1, \ldots, p) \tag{2.5}$$

whenever the limit implied in (2.5) exists, regardless as to whether $g(s)$ is defined or not.

DEF. *Let g satisfy $\widehat{F}$ over $I^{(p)}[a, b]$. Then g will be said to have canonical left boundary values if, for $r = 1, \ldots, p$,*

$$g^r(s) = g(s) \qquad [for \ s \in Z_r I^{(p)}]$$

and null left boundary values if $g^r(s) = g(s) = 0$ for these same values of s.

THEOREM 2.2. *If k satisfies $\widehat{F}$ over the open interval $I_0^{(p)} = I^{(p)}(a, b)$ there exists an extension g of k over* (*) $\overline{I_0^{(p)}}$ *such that $g(s) = k(s)$ over $I_0^{(p)}$, while g has canonical left boundary values and*

$$P^{(r)}[g, Q^{(r)}] = P^{(r)}[g, Q^{(r)} \cap I_0^{(p)}] \qquad (r = 1, \ldots, p) \tag{2.6}$$

for any r-segment $Q^{(r)} \subset \overline{I_0^{(p)}}$ intersecting $I_0^{(p)}$. As a consequence g satisfies $\widehat{F}$ over $\overline{I_0^{(p)}}$. MT(5), Theorem 6.4.

Let σ be any closed r-face $(p \geqq r \geqq 0)$ of $I^{(p)}[a, b]$ incident with a. Let s be any point of $I^{(p)}$ and s_σ the orthogonal projection of s into σ, with $s_\sigma = a$ if $\sigma = a$.

DEF. *A function g which satisfies $\widehat{F}$ over $I^{(p)}[a, b]$ and has canonical left boundary values will be said to be a left σ-function g_σ over $I^{(p)}$ if $g(s) = g(s_\sigma)$*

(*) $\overline{I_0^{(p)}}$ is the closure of $I_0^{(p)}$.

for $s \in I^{(p)}$ *and if* $g(s_\sigma) = 0$ *for* s *on the left boundary of* σ. *If* $\sigma = a$ $g_\sigma(s) = g_\sigma(a)$ *for* $s \in I^{(p)}$.

THEOREM 2.3. *A function* g *which satisfies* $\widehat{F}$ *over* $I^{(p)}[a, b]$ *and has canonical left boundary values admits a unique decomposition*

$$g = \sum_\sigma g_\sigma , \tag{2.7}$$

in which g_σ *is a left* σ*-function corresponding to an arbitrary closed face* σ *of* $I^{(p)}[a, b]$ *incident with* a, *including* a *and* $I^{(p)}[a, b]$ *as faces.* MT(5), Theorems 7.1 and 7.2.

We term the sum $\sum_\sigma g_\sigma$ of the theorem the *left decomposition* of g.

DEF. Let g satisfy $\widehat{F}$ over a general interval $I^{(p)}$. A function ρ with values $\rho(\eta) > 0$ defined for $\eta > 0$ is called a *variation modulus* of g over $I^{(p)}$ if $\rho(\eta) \to 0$ as $\eta \to 0+$, and if for any r-segment $Q^{(r)} \subset I^{(p)}$

$$P^{(r)}[g, Q^{(r)}] < \rho(\eta) \qquad (r = 1,\ldots, p)$$

whenever the maximum length of the 1-faces of $Q^{(r)}$ *is less than* η.

THEOREM 2.4. *A function* g *which satisfies* $\widehat{F}$ *and is continuous over* $I^{(p)}[a, b]$ *admits a variation modulus* ρ. *Moreover* $2^p\rho$ *is a variation modulus common to the functions* g_σ *in a left decomposition of* g. MT(5), Theorem 7.3.

We shall now give an inequality applicable to the multiple DIRICHLET integral and which acts as a replacement (when $p > 1$) for the second law of the mean when $p = 1$.

Let φ_r, $r = 1,\ldots, p$, be in L with values $\varphi_r(t)$ over the 1-interval $(a^{(r)}, b^{(r)})$.

Set

$$M_r = \max_c \left| \int_c^{b^{(r)}} \varphi_r(u)\, du \right| \qquad (r = 1,\ldots, p) \tag{2.8}$$

for $a^{(r)} \leqq c \leqq b^{(r)}$. Let g satisfy $\widehat{F}$ over $I_0^{(p)} = I^{(p)}(a, b)$. The L-integral

$$J_0 = \int_{a^{(1)}}^{b^{(1)}} \ldots \int_{a^{(p)}}^{b^{(p)}} \varphi_1(s^{(1)}) \ldots \varphi_p(s^{(p)}) g(s^{(1)},\ldots, s^{(p)})\, ds^{(1)} \ldots ds^{(p)} \tag{2.9}$$

is well defined by virtue of Theorem 2.1 (1). Theorem 9.2 of MT(5) gives the following.

THEOREM 2.5. *If* g *satisfies* $\widehat{F}$ *over* $I^{(p)}[a, b]$ *and has null left boundary values then*

$$| J_0 | \leqq P^{(p)}[g, I^{(p)}(a, b)] M_1 \ldots M_p . \tag{2.10}$$

The conclusion of this theorem holds equally well if g satisfies $\widehat{F}$ over $I^{(p)}(a, b) = I_0^{(p)}$ and if $g^r(s) = 0$ for each point $s \in Z_r I_0^{(p)}$, $r = 1,\dots, p$. But a g which satisfies this modified set of conditions has an extension over $\overline{I_0^{(p)}}$ which satisfies the conditions of the theorem so that no essential advantage comes from the modified form of statement.

THEOREM 2.6. *If g satisfies $\widehat{F}$ over $I^{(p)}(\mathbf{0}, \mathbf{i})$ then as $\eta \to 0$ with $0 < \eta < 1$*

$$P^{(p)}[g, I^{(p)}(\mathbf{0}, \eta\mathbf{i})] \to 0 \qquad \text{[Cf. MT(5), Cor. 6.2].}$$

3. - The Dirichlet integral.

We shall need certain known formulas. For brevity set

$$\frac{\sin\left(r + \frac{1}{2}\right)t}{\sin \frac{t}{2}} = \varphi(r, t) \qquad (r = 0, 1, 2,\dots; t \neq 0)$$

and recall that

$$\left| \int_c^d \varphi(r, t)\, dt \right| < \frac{8}{2r + 1} \operatorname{cosec} \frac{c}{2} \qquad (0 < c < d \leqq \pi), \tag{3.1}$$

$$\int_0^\pi \varphi(r, t)\, dt = \pi . \tag{3.2}$$

Cf. HOBSON, Vol. II, p. 510. KÜSTERMAN gives the formula

$$\left| \int_c^d \varphi(r, t)\, dt \right| \leqq \pi^2 \qquad (0 \leqq c < d \leqq \pi) \tag{3.3}$$

established as follows. On setting $t = 2u$ and $2r + 1 = m$ in the integral (3.3), when $r = 1, 2,\dots,$

$$2 \left| \int_{\frac{c}{2}}^{\frac{d}{2}} \frac{\sin mu}{\sin u}\, du \right| \leqq 2 \int_0^{\frac{\pi}{m}} \frac{\sin mu}{\sin u}\, du \leqq 2 \int_0^{\frac{\pi}{m}} \frac{mu}{\sin u}\, du \leqq \int_0^{\frac{\pi}{m}} m\pi\, du = \pi^2. \tag{3.4}$$

Let $[a^{(i)}, b^{(i)}]$ be a subinterval of $[0, \pi]$, $i = 1,\dots, p$. Suppose that g is in L over $I^{(p)}[\mathbf{0}, \pi\mathbf{i}]$. Let $\mathbf{n}$ represent the set of integers $[n^{(1)},\dots, n^{(p)}]$ with $n^{(i)} \geqq 0$. Set

$$(2\pi)^p D^{(p)}[a, b, \mathbf{n}, g] = \int_{a^{(1)}}^{b^{(1)}} \dots \int_{a^{(p)}}^{b^{(p)}} g(t^{(1)},\dots, t^{(p)}) \varphi(n^{(1)}, t^{(1)}) \dots \varphi(n^{(p)}, t^{(p)})\, dt^{(1)} \dots dt^{(p)} . \tag{3.5}$$

We state a major lemma.

LEMMA 3.1. *Suppose that g satisfies $\widehat{F}$ over $I^{(p)}[\mathbf{0}, \pi\boldsymbol{i}]$ and has null left boundary values. If η is any constant such that $0 < \eta < \pi$ then*

$$(3.6) \qquad |D^{(p)}[\mathbf{0}, \pi\boldsymbol{i}, \boldsymbol{n}, g]| \leqq C_1(p)P^{(p)}[g, I^{(p)}(\mathbf{0}, \eta\boldsymbol{i})] + C_2(p)\frac{\operatorname{cosec}(\eta/2)}{\min_i [2n^{(i)}+1]} P^{(p)}[g, I^{(p)}(\mathbf{0}, \pi\boldsymbol{i})]$$

where $C_1(p)$ and $C_2(p)$ are constants depending only on p.

The utility of the lemma depends upon the following. The first term on the right of (3.6) tends to zero as $\eta \to 0$ in accordance with Theorem 2.6, while for fixed $\eta > 0$, the second term on the right of (3.6) tends to zero as $\min_i n^{(i)} \uparrow \infty$.

Let $[a^{(i)}, b^{(i)}]$ be chosen as $[0, \eta]$ or as $[\eta, \pi]$ for each $i = 1, \dots, p$. One thus has 2^p choices of the pair of points a, b, in $R^{(p)}$. To establish (3.6) observe that

$$(3.7) \qquad D^{(p)}[\mathbf{0}, \pi\boldsymbol{i}, \boldsymbol{n}, g] = \sum_{a, b} D^{(p)}[a, b, \boldsymbol{n}, g]$$

summing over the above 2^p choices of the pair of points a, b. We write (3.7) in the form

$$(3.8) \qquad D^{(p)}[\mathbf{0}, \pi\boldsymbol{i}, \boldsymbol{n}, g] = D^{(p)}[\mathbf{0}, \eta\boldsymbol{i}, \boldsymbol{n}, g] + \sum_{a, b}{}' D^{(p)}[a, b, \boldsymbol{n}, g]$$

where the prime indicates that the integral $D^{(p)}[\mathbf{0}, \eta\boldsymbol{i}, \boldsymbol{n}, g]$ has been omitted from the sum, $\sum'$. On making use of (2.10) and (3.3) we see that

$$(3.9) \qquad (2\pi)^p D^{(p)}[\mathbf{0}, \eta\boldsymbol{i}, \boldsymbol{n}, g] \leqq \pi^{2p} P^{(p)}[g, I^{(p)}(\mathbf{0}, \eta\boldsymbol{i})] .$$

Consider a typical integral $D^{(p)}[a, b, \boldsymbol{n}, g]$ in the sum $\sum'$ in (3.8). We shall use (2.10) to evaluate this integral. For at least one value of r on the range $1, \dots, p$ $[a^{(r)}, b^{(r)}] = [\eta, \pi]$ for a given choice of a, b. We apply (2.10) as follows. Let k_r be the characteristic function of the interval $[a^{(r)}, b^{(r)}]$. Identify the function φ_r of (2.9) with (*) $k_r\varphi(n^{(r)}, \cdot)$. With M_r defined as in (2.8) and for each r on the range $1, \dots p$

$$(3.10)' \qquad M_r \leqq \frac{8 \operatorname{cosec}(\eta/2)}{2n^{(r)}+1} \qquad \left[\text{when } [a^{(r)}, b^{(r)}] = [\eta, \pi]\right],$$

$$(3.10)'' \qquad M_r \leqq \pi^2 \qquad \left[\text{when } [a^{(r)}, b^{(r)}] = [0, \eta]\right]$$

by virtue of (3.1) and (3.3) respectively. When the points a and b are such

(*) The function over $(0, \pi]$ with values $\varphi(n^{(r)}, t)$ for fixed $n^{(r)}$ is denoted by $\varphi(n^{(r)}, \cdot)$.

that there is at least one integer $r = 1, 2, \dots, p$ for which $[a^{(r)}, b^{(r)}] = [\eta, \pi]$ our generalized law of the mean (2.10) gives the result: for this r

$$(2\pi)^p D^{(p)}[a, b, \boldsymbol{n}, g] \leqq \frac{8 \operatorname{cosec} (\eta/2)}{2n^{(r)} + 1} (\pi^2)^{p-1} P^{(p)}[g, I^{(p)}(\boldsymbol{0}, \pi \boldsymbol{i})] . \tag{3.11}$$

Relation (3.6) follows from (3.8), (3.9) and (3.11).

LEMMA 3.2. *Under the hypothesis of Lemma* 3.1

$$D^{(p)}[\boldsymbol{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g] \to 0 \qquad \text{as } \min \; [n^{(1)}, \dots, n^{(p)}] \uparrow \infty .$$

Let d be a prescribed positive constant. Let $\eta > 0$ in (3.6) be chosen so small that the first term on the right of (3.6) is less than $d/2$. This is possible by virtue of Theorem 2.6. With η so chosen we require that $\min [n^{(1)}, \dots, n^{(p)}]$ exceed an integer N so large that the second term on the right of (3.6) is less than $d/2$. For $\min [n^{(1)}, \dots, n^{(p)}] > N$

$$D^{(p)}[\boldsymbol{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g] < d$$

and Lemma 3.2 follows.

4. - Proof of Theorem 1(a).

The function f is supposed to satisfy $\widehat{F}$ over $I^{(p)}[\boldsymbol{0}, 2\pi \boldsymbol{i}]$ and to have the period 2π in each of its arguments. With $\boldsymbol{n} = [n^{(1)}, \dots, n^{(p)}]$ as previously, let $S^{(p)}[\boldsymbol{n}, s)$ denote the sum at the point s of the terms of the FOURIER series of which involve $\sin rs^{(i)}$ and $\cos ms^{(i)}$ for $r, m = 0, 1, \dots, n^{(i)}$ $(i=1, \dots, p)$. To represent $S^{(p)}(\boldsymbol{n}, s)$ by the DIRICHLET integral let $\boldsymbol{c}$ be a set $[c^{(1)}, \dots, c^{(p)}]$ of constants $c^{(i)} = \pm 1$, $(i = 1, \dots, p)$. For arbitrary points s and $t \in R^{(p)}$ set

$$\sum_{\boldsymbol{c}} f[s^{(1)} + c^{(1)}t^{(1)}, \dots, s^{(p)} + c^{(p)}t^{(p)}] = F^{(s)}(t) \tag{4.0}$$

summing over all admissible $\boldsymbol{c}$. The classical formula for $S^{(p)}(\boldsymbol{n}, s)$ becomes

$$S^{(p)}(\boldsymbol{n}, s) = D^{(p)}[\boldsymbol{0}, \pi \boldsymbol{i}, \boldsymbol{n}, F^{(s)}] . \tag{4.1}$$

From the fact (*) that f satisfies $\widehat{F}$ over $I^{(p)}[\boldsymbol{0}, 2\pi \boldsymbol{i}]$ it follows that $F^{(s)}$ satisfies $\widehat{F}$ over $I^{(p)}[\boldsymbol{0}, \pi \boldsymbol{i}]$. It follows from Theorem 2.1 (3) that $F^{(s)}(t)$ tends to a definite limit $F^{(s)}(\boldsymbol{0}+)$, as $t \to 0$ with $t^{(i)} > 0$ $(i = 1, \dots, p)$. From the definition of $F^{(s)}$ we see that $F^{(s)}(\boldsymbol{0}+) = 2^p \bar{f}(s)$ where $\bar{f}(s)$ is the mean of

(*) One uses the periodicity of f as well.

the 2^p sector limits of f at s. To show that $S^{(p)}(\boldsymbol{n}, s)$ converges to $\bar{f}(s)$ as $\min_i n^{(i)} \uparrow \infty$ it is accordingly sufficient to prove the following lemma.

LEMMA 4.1. *If a function k satisfies $\widehat{F}$ over $I_0^{(p)} = I^{(p)}(\mathbf{0}, \pi \boldsymbol{i})$ then as $\min_i n^{(i)} \uparrow \infty$*

$$2^p D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, k] \to k(\mathbf{0}+) \tag{4.2}$$

where $k(\mathbf{0}+)$ is the limit of $k(t)$ as $t \to \mathbf{0}$ with $t^{(i)} > 0$ $(i = 1, \ldots, p)$.

According to Theorem 2.2 there is an extension g of k over $\overline{I_0^{(p)}} = Q$ such that $g(s) = k(s)$ for $s \in I_0^{(p)}$ while g has canonical left boundary values and satisfies $\widehat{F}$ over Q. Since

$$k(\mathbf{0}+) = g(\mathbf{0}+) = g(\mathbf{0}),$$

$$D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, k] = D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g]$$

it is sufficient to establish the relation

$$2^p D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g] \to g(\mathbf{0}) \tag{4.3}$$

as $\min_i n^{(i)} \uparrow \infty$.

According to Theorem 2.3 g admits a left decomposition over Q

$$g = \sum_\sigma g_\sigma$$

where g_σ is a left σ-function over Q and σ ranges over all closed faces of Q incident with $\mathbf{0}$. Let $r(\sigma)$ be the dimension of σ. In case $r(\sigma) = 0$, $g_\sigma(s) \equiv g(\mathbf{0})$ so that in this case

$$2^p D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g_\sigma] \equiv g(\mathbf{0}) \qquad [r(\sigma) = 0] \tag{4.4}$$

as one sees with the aid of (3.2). Moreover

$$D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g_\sigma] \to 0 \qquad [\text{when } r(\sigma) = p] \tag{4.5}$$

by virtue of Lemma 3.2, since $\sigma = Q$ in this case, and g_σ has null left boundary values.

In the case in which $0 < r(\sigma) < p$ we shall refer to the DIRICHLET integral $D^{(r)}$ in the coordinate r-space $R_\sigma^{(r)}$ in which σ lies, understanding that the coordinates in $R_\sigma^{(r)}$ are those in $s = [s^{(1)}, \ldots, s^{(p)}]$ which vary over σ. To avoid ambiguity let $\mathbf{0}_\sigma$ and $\boldsymbol{i}_\sigma$ represent the origin and point with unit coordinates in $R_\sigma^{(r)}$ and let $\boldsymbol{n}_\sigma$ be the orthogonal projection into $R_\sigma^{(r)}$ of $n = [n^{(1)}, \ldots, n^{(p)}]$ regarded as a point in $R^{(p)}$. Recall that the function g_σ is independent over Q of those coordinates $s^{(i)}$ which are constant over σ. Let $g_\sigma \mid \sigma$ be the function defined by g_σ over σ. It follows directly from the definition of the DIRICHLET integral that

$$(4.6) \qquad (2\pi)^p D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g_\sigma] = (2\pi)^r D^{(r)}[\mathbf{0}_\sigma, \pi \boldsymbol{i}_\sigma, \boldsymbol{n}_\sigma, g_\sigma \mid \sigma] \prod_i \int_0^\pi \varphi(n^{(i)}, u)\, du$$

where the product Π is taken over those integers i for which $s^{(i)}$ is constant over σ. Moreover

$$\prod_i \int_0^\pi \varphi(n^{(i)}, u)\, du = \pi^{p-r}$$

in accordance with (3.2).

The specific values of these constants is immaterial. For

$$D^{(r)}(\mathbf{0}_\sigma, \pi \boldsymbol{i}_\sigma, \boldsymbol{n}_\sigma, g_\sigma \mid \sigma] \to 0 \qquad [r > 0]$$

as $\min_i n^{(i)} \uparrow \infty$, since $g_\sigma \mid \sigma$ has null left boundary values relative to σ and Lemma 3.2 applies with $p = r$. Hence

$$(4.7) \qquad D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g_\sigma] \to 0 \qquad (0 < r(\sigma) \leqq p)$$

as $\min_i n^{(i)} \uparrow \infty$. Thus

$$2^p D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, g] \to g(\mathbf{0})$$

in accordance with (4.4), (4.5) and (4.7). The Lemma follows.

5. – Proof of Theorem 1(b).

Recall the definition of the function $F^{(s)}$. For $c^{(i)} = \pm 1$, for fixed $a \in R^{(p)}$ and $t \in I^{(p)}[\mathbf{0}, \pi \boldsymbol{i}]$

$$(5.1) \qquad F^{(a)}(t) = \sum_c f[a^{(1)} + c^{(1)}t^{(1)}, \ldots, a^{(p)} + c^{(p)}t^{(p)}] .$$

We shall prove the following.

(A). *For fixed a and c as above let φ_c^a be the function with values*

$$\varphi_c^a(t) = f[a^{(1)} + c^{(1)}t^{(1)}, \ldots, a^{(p)} + c^{(p)}t^{(p)}]$$

with $t \in I^{(p)}[\mathbf{0}, \pi \boldsymbol{i}]$. A variation modulus of f over $I^{(p)}[\mathbf{0}, 4\pi \boldsymbol{i}]$ is a variation modulus of φ_c^a.

Let G be the group of translations of $R^{(p)}$ under which a point s is replaced by a point

$$(s^{(1)} + 2n^{(1)}\pi, \ldots, s^{(p)} + 2n^{(p)}\pi) \tag{5.2}$$

where $n^{(i)}$ is a rational integer. Given $a^{(i)}$ there exists a constant $b^{(i)}$ such that

$$\pi < b^{(i)} \leqq 3\pi$$

and $b^{(i)} \equiv a^{(i)} \mod 2\pi$. There accordingly exists a transformation of G under which $\varphi_c^a = \varphi_c^b$. As t ranges over $I^{(p)}[\mathbf{0}, \pi \boldsymbol{i}]$ the point

$$[b^{(1)} + c^{(1)}t^{(1)}, \ldots, b^{(p)} + c^{(p)}t^{(p)}],$$

ranges over a p-interval

$$J_{b,c}^{(p)} \subset I^{(p)}[\mathbf{0}, 4\pi \boldsymbol{i}].$$

It is clear that a variation modulus of f over $J_{b,c}^{(p)}$ is a variation modulus of φ_c^b and hence of φ_c^a. This implies (A).

We continue with the following.

(B). *Let ρ be a variation modulus of f over $I^{(p)}[\mathbf{0}, 4\pi \boldsymbol{i}]$. The functions $F_\sigma^{(a)}$ in a left decomposition of $F^{(a)}$ over $I^{(p)}[\mathbf{0}, \pi \boldsymbol{i}]$ have $4^p\rho$ as a common variation modulus.*

It follows from (A) that $2^p\rho$ is a variation modulus of $F^{(a)}$, since there are 2^p functions φ_c^a summed in (5.1) to obtain $F^{(a)}$. According to Theorem 2.4 $2^p(2^p\rho)$ is then a variation modulus common to the functions $F_\sigma^{(a)}$ of a left decomposition of $F^{(a)}$. This establishes (B).

To prove the uniform convergence of $S^{(p)}(\boldsymbol{n}, s)$ to the mean $\bar{f}(s)$ we make use of (B) and (3.6).

Recall that when σ is the origin $\mathbf{0}$

$$D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, F_\sigma^{(s)}] = \bar{f}(s) \qquad \text{[Cf. (4.4)].}$$

Hence from (4.1)

$$| S^{(p)}(\boldsymbol{n}, s) - \bar{f}(s) | \leqq \Big| \sum_\sigma{}' D^{(p)}[\mathbf{0}, \pi \boldsymbol{i}, \boldsymbol{n}, F_\sigma^{(s)}] \Big| \tag{5.3}$$

where $\sum_\sigma{}'$ is a sum over all closed left faces σ of $I^{(p)}[\mathbf{0}, \pi \boldsymbol{i}]$, omitting the face $\mathbf{0}$. Let N_p be the number of such faces. It follows from (3.6) and (B) that for $0 < \eta < \pi$,

$$| S^{(p)}(\boldsymbol{n}, s) - \bar{f}(s) | \leqq C_1(p)\lambda(\eta) + \frac{C_2(p) \operatorname{cosec} (\eta/2)}{\min_i [2n^{(i)} + 1]} \lambda(\pi) \tag{5.4}$$

where $\lambda(\eta) = N_p[4^p\rho(\eta)]$. Relation (5.4) clearly implies the uniform convergence of $S^{(p)}(\mathbf{n}, s)$ to $\bar{f}(s)$.

This completes the proof of Theorem 1(b).

6. - Bounds for the Fourier coefficients of k.

Suppose that k satisfies $\widehat{F}$ over $I_0^{(p)} = I^{(p)}(\mathbf{0}, 2\pi\mathbf{i})$. Let the integers $1,\dots, p$ taken in an arbitrary order be written in the form

$$m_1,\dots, m_i;\ n_1,\dots, n_j;\ r_1,\dots, r_\nu$$

where i, j, ν are non-negative integers with $i + j + \nu = p$. Let

$$M(m_1),\dots, M(m_i);\quad N(n_1),\dots, N(n_j)$$

be positive integers. The coefficient of the product (*)

$$\varphi_{MN}(s) = \prod_{\alpha, \beta} \cos M(m_\alpha)s^{(m_\alpha)} \sin N(n_\beta)s^{(n_\beta)} \tag{6.1}$$

$\alpha = 1,\dots, i$; $\beta = 1,\dots, j$, in the FOURIER series of k will be denoted by

$$A_k[M(m_1),\dots, M(m_i);\ N(n_1),\dots, N(n_j)] \tag{6.2}$$

and will be given by the p-fold integral

$$\frac{1}{2^\nu\pi^p}\int_0^{2\pi}\dots\int_0^{2\pi} k(s)\varphi_{MN}(s)\, ds^{(1)} \dots ds^{(p)}\,. \tag{6.3}$$

THEOREM 6.1. *In case* $\nu = 0$ *the coefficient* (6.2) *satisfies the relation*

$$A_k \leq \left(\frac{2}{\pi}\right)^p \frac{P^{(p)}[k, I_0^{(p)}]}{\prod_{\alpha,\beta} M(m_\alpha)N(n_\beta)} \qquad \begin{Bmatrix} \alpha = 1,\dots, i \\ \beta = 1,\dots, j \end{Bmatrix}. \tag{6.4}$$

To establish (6.4) let g be the extension of k over $\overline{I_0^{(p)}}$, as given in Theorem 2.2, and let

$$g = \sum_\sigma g_\sigma$$

be a left decomposition of g over $\overline{I_0^{(p)}}$. The integral (6.3) is unchanged in

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(*) We take the product (6.1) as 1 when $i = j = 0$. When $i = 0$ and $j = p$ α ranges over a null set, while $\cos M(m_\alpha)s^{(m_\alpha)}$ in (6.1), and $M(m_\alpha)$ in (6.4), is to be replaced by 1. Similarly when $j = 0$ and $i = p$.

value if k is replaced by g. Let $r(\sigma)$ be the dimension of σ. If k is replaced by g_σ in (6.3) the resulting integral will vanish unless $r(\sigma) = p$; in fact, in every case in which $r(\sigma) < p$, $g_\sigma(s)$ is independent of at least one of the coordinates $s^{(a)}$, and corresponding to this coordinate, (6.3) has a null factor of the general form of one of the two integrals

$$\int_0^{2\pi} \cos\,[M(m_a)s^{(a)}]\,ds^{(a)}\,, \qquad \int_0^{2\pi} \sin\,[N(n_a)s^{(a)}]\,ds^{(a)}\,.$$

When $r(\sigma) = p$, an application of (2.10) to (6.3) gives (6.4) on recalling that

$$\max_c \left| \int_c^{2\pi} \cos m\alpha\,d\alpha \right| \leqq \frac{2}{m}$$

$$\max_c \left| \int_c^{2\pi} \sin n\alpha\,d\alpha \right| \leqq \frac{2}{n} \qquad (0 \leqq c \leqq 2\pi)$$

for m and n arbitrary positive integers, and that

$$P^{(p)}[g_\sigma, \overline{I_0^{(p)}}] = P^{(p)}[g, \overline{I_0^{(p)}}] = P^{(p)}[k, I_0^{(p)}] \qquad \text{[Cf. (2.6)]}$$

when $r(\sigma) = p$.

The case $p > \nu > 0$. In this case we introduce the ν-fold integral

$$(6.5) \qquad \frac{1}{(2\pi)^\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} k(s^{(1)},\ldots, s^{(p)})\,ds^{(r_1)} \ldots ds^{(r_\nu)}$$

noting that this integral defines a function $J^{(p-\nu)}$ over an interval $I_0^{(p-\nu)}$ of points with coordinates

$$[s^{(m_1)},\ldots, s^{(m_i)};\ s^{(n_1)},\ldots, s^{(n_j)}] \qquad (i + j + \nu = p)$$

such that

$$0 < s^{(\alpha)} < 2\pi \qquad (\alpha = m_1,\ldots, m_i)\,,$$

$$0 < s^{(\beta)} < 2\pi \qquad (\beta = n_1,\ldots, n_j)\,.$$

The function $J_0^{(p-\nu)}$ satisfies $\widehat{F}$ over $I_0^{(p-\nu)}$. This can be seen as follows. Let $Q^{(\nu)}$ be any ν-segment in $I_0^{(p-\nu)}$. Let B be a bound for the Fréchet variation $P^{(m)}[k, H^{(m)}]$ for all m-segments $H^{(m)} \subset I_0^{(p)}$ and for all m on the range $1,\ldots, p$. If one forms the general finite sum $\sum$ whose sup. by definition is $P^{(\nu)}[J^{(p-\nu)}, Q^{(\nu)}]$, [Cf. Equation (2.10) of MT(5)] operating on (6.5) under the integral sign, one finds that

$$\left|\sum\right| \leqq \frac{1}{(2\pi)^{\nu}} \int_0^{2\pi} \cdots \int_0^{2\pi} B \, ds^{(r_1)} \ldots ds^{(r_\nu)} = B .$$

On taking the appropriate sup. we conclude that

$$P^{(\nu)}[J^{(p-\nu)}), \; Q^{(\nu)}] \leqq B .$$

Thus $J^{(p-\nu)}$ satisfies $\widehat{F}$ over $I_0^{(p-\nu)}$.

THEOREM 6.2. *In case* $0 < \nu < p$, *for fixed* i, j *with* $i + j + \nu = p$ *the coefficient* (6.2) *satisfies the relation*

$$|A_k| \leqq \frac{\left(\frac{2}{\pi}\right)^{p-\nu} P^{(p-\nu)}[J^{(p-\nu)}, I_0^{(p-\nu)}]}{\prod_{\alpha,\beta} M(m_\alpha) N(n_\beta)} \qquad \left\{ \begin{matrix} \alpha = 1, \ldots, i \\ \beta = 1, \ldots, j \end{matrix} \right\} . \tag{6.6}$$

Referring to (6.3) and to the definition of the integral $J^{p-\nu}$ we find that $\pi^{p-\nu} A_k$ is given by the $(p - \nu)$-fold integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} J^{p-\nu}[s^{(m_1)}, \ldots, s^{(m_i)}; \; s^{(n_1)}, \ldots, s^{(n_j)}] \, \varphi_{MN}(s) \, ds^{(m_1)} \ldots ds^{(m_i)} ds^{(n_1)} \ldots ds^{(n_j)} \tag{6.7}$$

$(i + j + \nu = p)$. Apart from the notation for the variables of integration, the integral (6.7) has the form of the integral in (6.3) with k replaced by $J^{p-\nu}$ and p replaced by $p - \nu$. Reasoning as previously we infer that the integral (6.7) is at most in absolute value

$$2^{p-\nu} \frac{P^{(p-\nu)}[J^{(p-\nu)}, I_0^{(p-\nu)}]}{\prod_{\alpha,\beta} M(m_\alpha) N(m_\beta)} \qquad \left\{ \begin{matrix} \alpha = 1, \ldots, i \\ \beta = 1, \ldots, j \end{matrix} \right\} .$$

Relation (6.6) follows.

In case $\nu = p$ we naturally suppose that $\varphi_{MN}(s) \equiv 1$ and that A_k is the constant term in the FOURIER series. Its value $J^{(0)}$ is given by (6.5) with $p = \nu$. If one understands that $P^{(0)}[J^{(0)}, I^{(0)}] = |J^0|$, (6.6) holds even in this extreme case. If on the other hand one understands that $J^{(p)} = k$, (6.6) holds even when $\nu = 0$. Theorems 6.1 and 6.2 can thus be combined.

THEOREM 6.3. *If the values of* $J^{(p-\nu)}$ *are given by* (6.5) *when* $0 < \nu \leqq p$ *and if* $J^{(p)} = k$, *then for fixed integers* $i \geqq 0$, $j \geqq 0$ *with* $i + j + \nu = p$ *the value of the* FOURIER *coefficient* (6.2) (*taken as the constant term when* $i = j = 0$) *satisfies* (6.6).

The case $p = 2$. For simplicity suppose that k has the values $k(s, t)$.

In case $\nu = 0$ the coefficient of any one of the products $\cos ms \sin nt$, $\cos ms \cos nt$, $\sin ns \sin mt$, $\sin ms \cos nt$ has an absolute value at most

$$\left(\frac{2}{\pi}\right)^2 \frac{P^{(2)}[k, I_0^{(2)}]}{mn} \tag{6.8}$$

in accord with Theorem 6.1. When $m > 0$ the coefficient of $\cos ms$, or $\sin ms$ has an absolute value at most

$$\frac{2}{\pi} \frac{V(J)}{m} \qquad (m > 0) \quad \text{[from (6.6)]} \tag{6.9}$$

where from (6.5)

$$J(t) = \frac{1}{2\pi} \int_0^{2\pi} k(s, t)\, dt$$

and $V(J)$ is the JORDAN variation of J over the interval $(0, 2\pi)$. The special limit (6.9) also results from the classical theory as a bound for $| a_m |$ or $| b_m |$ in the FOURIER expansion of J. One can naturally interchange the roles of s and t.

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