EARL J. MICKLE (*)

# Some examples in surface area theory. (**)

## 1. - Introduction.

**1.1.** - It is the purpose of this Note to give some examples showing limitations on certain methods and results in surface area theory. For definitions and concepts relating to surface area theory we shall use as a general reference the book *Length and Area* of T. RADÓ. Hereafter this book will be referred to as LA (see the Bibliography [5] at the end of this Note). Throughout this Note the term surface will denote a surface of the type of the 2-cell (LA, II,3.44).

**1.2.** - Let

$$(1.1) \qquad T: \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \qquad (u, v) \in Q,$$

be a continuous mapping from a unit square $Q$: $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$, into Euclidean $xyz$-space. We shall be concerned with the following properties for such a mapping.

*a*) The first partial derivatives $x_u$, $x_v$, $y_u$, $y_v$, $z_u$, $z_v$ exist almost everywhere in $Q$ and are summable with their squares in $Q$.

*b*) $x_u^2 + y_u^2 + z_u^2 = x_v^2 + y_v^2 + z_v^2$, $x_u x_v + y_u y_v + z_u z_v = 0$, almost everywhere in $Q$.

*c*) The functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are ACT (absolutely continuous in the TONELLI sense, see LA, III, 2.64).

*d*) The ordinary Jacobians exist almost everywhere and are summable in $Q$ and the LEBESGUE area $A(T)$ of the surface represented by $T$ is given by the formula

$$A(T) = \int_0^1\int_0^1 \left\{ \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}^2 + \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}^2 + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}^2 \right\}^{1/2} du\, dv .$$

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(*) Associate Professor of Mathematics. Address: Department of Mathematics, Ohio State University, Columbus 10, Ohio, U.S.A..

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**1.3.** – A representation of a surface satisfying conditions *a*), *b*), *c*) of 1.2 is called *generalized conformal* (LA, V, 2.29) and *d*) is satisfied for this representation (LA, V, 2.30). MORREY [4] (see also LA, V, 2.43) has shown that every non-degenerate surface (LA, II, 3.2) of finite LEBESGUE area admits of a generalized conformal representation. CESARI [2] calls a representation of a surface satisfying only conditions *a*) and *b*) of 1.2 *almost conformal* and shows that every surface of finite LEBESGUE area admits of an almost conformal representation for which condition *d*) of 1.2 holds. The question arises as to why CESARI did not use generalized conformal representations, i.e., why was condition *c*) of 1.2 dropped. Indeed, the answer to this question is that *not every surface of finite* LEBESGUE *area admits of a generalized conformal representation.* In § 2 we give an example of a surface of finite LEBESGUE area for which no representation satisfies condition *c*) of 1.2 or even condition *c*) of 1.2 with ACT replaced by BVT (bounded variation in the TONELLI sense, see LA, III, 2.64).

**1.4.** – For a continuous mapping $T$ as given in (1.1) and for $h \neq 0$ let

$$(1.2)\qquad \Delta(u, v; h) = \left\{ \begin{vmatrix} \dfrac{y(u+h, v) - y(u, v)}{h} & \dfrac{z(u+h, v) - z(u, v)}{h} \\ \dfrac{y(u, v+h) - y(u, v)}{h} & \dfrac{z(u, v+h) - z(u, v)}{h} \end{vmatrix}^2 + \begin{vmatrix} \dfrac{z(u+h, v) - z(u, v)}{h} & \dfrac{x(u+h, v) - x(u, v)}{h} \\ \dfrac{z(u, v+h) - z(u, v)}{h} & \dfrac{x(u, v+h) - x(u, v)}{h} \end{vmatrix}^2 + \begin{vmatrix} \dfrac{x(u+h, v) - x(u, v)}{h} & \dfrac{y(u+h, v) - y(u, v)}{h} \\ \dfrac{x(u, v+h) - x(u, v)}{h} & \dfrac{y(u, v+h) - y(u, v)}{h} \end{vmatrix}^2 \right\}^{1/2}.$$

The expression (1.2) has been used to obtain far reaching results in the GEÖCZE problem for surfaces with a non-parametric representation by HUSKEY [3] and for general surfaces by RADÓ [6]. Let

$$A_*(T) = \inf \liminf_{h \to 0} \int_0^1 \int_0^1 \Delta(u, v; h)\, du\, dv\,,$$

$$A^*(T) = \sup \limsup_{h \to 0} \int_0^1 \int_0^1 \Delta(u, v; h)\, du\, dv\,.$$

It follows from the methods of HUSKEY [3] and RADÓ [6] that we have the inequality

$$A_*(T) \geqslant A(T), \tag{1.3}$$

where $A(T)$ is the LEBESGUE area of the surface represented by $T$. YOUNG [7] has shown that in the non-parametric case we have the inequality

$$A^*(T) \leqslant \sqrt{2}\, A(T). \tag{1.4}$$

As can be seen from the definition of $\Delta(u, v; h)$ in (1.2) an inequality such as the inequality of YOUNG in (1.4) in the general case combined with the inequality (1.3) would be invaluable in determining relationships between the area of a surface and the areas of its projections on the three coordinate planes. For such a relationship see CESARI [1]. Unfortunately, *no such inequality as* (1.4) *exists in the general case.* In § 3 we give an example of a surface of LEBESGUE area zero with a representation $T$ as in (1.1) for which $A_*(T) = +\infty$. The writer is indebted to WILLIAM SCOTT for the particular function defined in 3.1 to satisfy the conditions needed for the surface defined in 3.4.

**1.5.** – In view of certain objections to current definitions of area YOUNG [8] introduces the following definition of area. For a continuous mapping $T$ as given in (1.1) let $M(x, y, z)$ denote the number (possibly $+\infty$) of components in the inverse set $T^{-1}(x, y, z)$. For a CARATHÉODORY-HAUSDORFF 2-dimensional measure $\mu$ let

$$A_Y(T) = \sum_n \mu(E_n),$$

where $E_n$ is the set where $M(x, y, z) \geqslant n > 0$.

It is of course to be expected that two different definitions of area should lead to different properties. However, certain simple properties of the LEBESGUE area are not possessed by $A_Y(T)$. First, $A(T)$ is *lower semi-continuous* (LA, V, 2.6) whereas simple examples show that $A_Y(T)$ is not lower semi-continuous. Second, T. RADÓ and the writer noted that whereas $A(T)$ is *monotone* (LA, V, 1.10) $A_Y(T)$ is not monotone. That is to say, if $R$ is a subrectangle of $Q$ and $T^*$ is the continuous mapping $T$ considered only on $R$ then $A(T^*) \leqslant A(T)$, whereas $A_Y(T^*)$ is not necessarily less than or equal to $A_Y(T)$. A simple example to show this is given in § 4.

## 2. – A surface of finite area with no generalized conformal representation.

**2.1.** – For $0 \leqslant r \leqslant 1$ we set

$$\varphi(r) = \begin{cases} (1/n)\sin[(r-1+1/2^{n-1})2^n\pi], & 1-1/2^{n-1} \leqslant r \leqslant 1-1/2^n \quad (n=1, 2, \dots), \\ 0, \quad r = 1. \end{cases}$$

In terms of this function $\varphi(r)$ (using complex numbers $w = u + iv$) we set

$$x = f(w) = \varphi(|w|), \qquad |w| \leqslant 1. \tag{2.1}$$

Then

$$f(w) = \begin{cases} 0 & \text{for} \quad |w| = 1, \quad |w| = 1 - 1/2^{n-1}, \qquad (n = 1, 2, \dots), \\ 1/n & \text{for} \quad |w| = 1 - 1/2^{n-1} + 1/2^{n+1}, \qquad (n = 1, 2, \dots). \end{cases} \tag{2.2}$$

The function $f(w)$ defined in (2.1) is a real-valued, continuous function for $|w| \leqslant 1$.

**2.2.** – We will denote by $Z$ the class of continuous complex-valued functions $w = w(t)$, $0 \leqslant t \leqslant 1$, which satisfy the conditions: *a*) $|w(t)| \leqslant 1$. *b*) $|w(0)| = |w(1)| = 1$. *c*) There is a $t$ for which $|w(t)| = 1/2$.

**2.3.** – Lemma. Let $f(w)$ be the function defined in (2.1). For each number $a > 0$ there is an $\varepsilon(a) > 0$ such that if a real-valued continuous function $f^*(w)$, $|w| \leqslant 1$, satisfies the inequality

$$|f^*(w) - f(w)| < \varepsilon(a), \qquad |w| \leqslant 1, \tag{2.3}$$

then, for any function $w = w(t)$, $0 \leqslant t \leqslant 1$, in $Z$ (see 2.2) the length $L$ of the curve

$$x = f^*[w(t)], \qquad 0 \leqslant t \leqslant 1, \tag{2.4}$$

satisfies the inequality $L \geqslant a$.

Proof. Let $n_0$ be an integer such that

$$2a < 1/2 + 1/3 + \dots + 1/n_0. \tag{2.5}$$

Set

$$\varepsilon(a) = a/2n_0. \tag{2.6}$$

Let $f^*(w)$, $|w| \leqslant 1$, be a real-valued continuous function which satisfies (2.3) and let $w = w(t)$, $0 \leqslant t \leqslant 1$, be a function in $Z$. For each integer $n = 2, 3, \dots$, let $t_n$ be the largest value of $t$ for which $|w(t)| = 1 - 1/2^{n-1}$.

Then $0 < t_2 < \ldots < t_n < \ldots < 1$. Since $w(t)$ is continuous, there are numbers $t'_n$, $t_n < t'_n < t_{n+1}$, $(n = 2, 3, \ldots)$, such that $|w(t'_n)| = 1 - 1/2^{n-1} + 1/2^{n+1}$. By (2.2) we thus have the relations: $f[w(t_n)] = 0$, $f[w(t'_n)] = 1/n$, $(n = 2, 3, \ldots)$. By (2.3) we thus have the inequalities

(2.7) $$|f^*[w(t_n)]| < \varepsilon(a), \quad |f^*[w(t'_n)]| > 1/n - \varepsilon(a), \quad (n = 2, 3, \ldots).$$

From (2.7), (2.5), (2.6), the length $L$ of the curve (2.4) satisfies the inequality

$$L \geqslant \sum_{n=2}^{n_0} | f^*[w(t_n)] - f^*[w(t'_n)] | \geqslant \sum_{n=2}^{n_0} [1/n - \varepsilon(a) - \varepsilon(a)] > 2a - a/2 - a/2 = a .$$

**2.4.** – Let $K$ be the unit disc $|w| \leqslant 1$, let $K^*$ be the unit disc $|w^*| \leqslant 1$, in the $w^* = u^* + iv^*$ plane, and let $I$ be the unit interval $0 \leqslant x \leqslant 1$. Then for the function $f(w)$ defined in (2.1)

(2.8) $$T: \quad x = f(w), \qquad |w| \leqslant 1,$$

is a continuous mapping from $K$ onto $I$. Let

(2.9) $$T^*: \quad x = g(w^*), \qquad |w^*| \leqslant 1,$$

be a continuous mapping from $K^*$ onto $I$. Set

$$E_0^* = \underset{w^*}{E}[0 \leqslant g(w^*) \leqslant 1/4], \qquad E_1^* = \underset{w^*}{E}[3/4 \leqslant g(w^*) \leqslant 1],$$

(2.10) $$\delta = (\text{distance between } E_0^* \text{ and } E_1^*) > 0,$$

(2.11) $$K_0 = \underset{w}{E}[\,|w| \leqslant 1/2].$$

**2.5.** – L e m m a. Assume that the mappings $T$ and $T^*$ given in (2.8) and (2.9) are FRÉCHET equivalent (LA, II, 1.25). For $0 < \varepsilon < 1/4$ let $w^* = h_\varepsilon(w)$, $w \in K$, be a homeomorphism from $K$ onto $K^*$ such that $|f(w) - g[h_\varepsilon(w)]| < \varepsilon$, $w \in K$. Then the diameter of $h_\varepsilon(K_0)$ (see (2.11)) is greater than or equal to $\delta$ (see (2.10)).

P r o o f. For the points $w_0 = 0$, $w_1 = 1/4$ in $K_0$, $f(w_0) = 0$, $f(w_1) = 1$. Hence $h_\varepsilon(w_0) \in E_0^*$ and $h_\varepsilon(w_1) \in E_1^*$. Thus the diameter of $h_\varepsilon(K_0)$ is greater than or equal to $\delta$.

**2.6.** – Assume that the mappings $T$ and $T^*$ given in (2.8) and (2.9) are FRÉCHET equivalent. Let $M > 0$ be given and let $a = M/\delta$, where $\delta$ has the value given in (2.10). Let $\varepsilon = \varepsilon(a)$, be the number determined in the lemma in 2.3. Let $w^* = h(w)$ be a homeomorphism from $K$ onto $K^*$ such that

$$|f(w) - g[h(w)]| < \varepsilon, \qquad w \in K.$$

By the lemma in 2.5 there are points $w_1^*$, $w_2^* \in h(K_0)$ (see (2.11)) such that $|w_1^* - w_2^*| \geq \delta$. Let us assume for simplicity that $w_1^* = u_1^* + iv_0^*$, $w_2^* = u_2^* + iv_0^*$ and let us further assume that $u_1^* < u_2^*$. Let $\lambda(u^*)$ be the line segment through $u^*$ perpendicular to the $u^*$-axis and extending across $K^*$ from boundary to boundary. For $u_0^*$, $u_1^* \leq u_0^* \leq u_2^*$, let $w = w^*(t)$, $0 \leq t \leq 1$, be a topological representation of the line segment $\lambda(u_0^*)$. It is easily verified that the continuous function $w = h^{-1}[w^*(t)]$, $0 \leq t \leq 1$, is in $Z$ (see 2.2). Hence by the lemma in 2.3 the length $L$ of the curve

$$x = g[hh^{-1}(w^*(t))] = g[w^*(t)], \qquad 0 \leq t \leq 1,$$

satisfies the inequality

$$L \geq a. \tag{2.12}$$

Consider the curves

$$C(u^*): \quad x = g(w^*), \qquad w^* \in \lambda(u^*).$$

Let $L[C(u^*)]$ denote the length of the curve $C(u^*)$. By (2.12) $L[C(u^*)] \geq a$ for $u_1^* \leq u^* \leq u_2^*$. Hence

$$\int_{-1}^{1} L[C(u^*)]\, du^* \geq a\delta = M.$$

Since $M > 0$ is arbitrary and since BVT with respect to one set of axes implies BVT with respect to any set of axes, we have the following result.

Lemma. No continuous mapping (2.9) which is FRÉCHET equivalent to the continuous mapping (2.8) can be BVT.

**2.7.** – Now let $S$ be the surface given by the representation

$$T: \quad x = f(w), \quad y = f(w), \quad z = f(w), \quad |w| \leq 1, \tag{2.13}$$

where $f(w)$ is the function defined in (2.1). Then the LEBESGUE area of $S$ is zero. If

$$T^*: \quad x = x^*(u^*, v^*), \quad y = y^*(u^*, v^*), \quad z = z^*(u^*, v^*), \quad u^{*2} + v^{*2} \leq 1,$$

is another representation of $S$ then $x^*(u^*, v^*) \equiv y^*(u^*, v^*) \equiv z^*(u^*, v^*)$ and $T$ and $T^*$ are FRÉCHET equivalent. Hence the continuous mappings $x = f(w)$, $|w| \leq 1$, and $x = x^*(u^*, v^*)$, $u^{*2} + v^{*2} \leq 1$, are FRÉCHET equivalent. By the lemma in 2.6 $x^*(u^*, v^*)$ is not BVT.

Therefore (2.13) is a surface of finite LEBESGUE area for which there is no representation in which any one of the coordinates functions is BVT. Hence the surface given in (2.13) does not admit of a generalized conformal representation. It should be perhaps noted that the representation is given on a unit disc. This is done merely for convenience of notation. The representation can be equally well be given on a unit square.

## 3. – The inequality of Young.

**3.1.** – For each integer $n = 1, 2, \ldots,$ and integer $i = 0, 1, \ldots, 2^{2n+1} - 1$, we define a function $\varphi_{n,i}(u)$ on the interval

(3.1) $$I_{n,i}:\ 1 - 1/2^{n-1} + i/2^{3n+1} \leqslant u \leqslant 1 - 1/2^{n-1} + (i+1)/2^{3n+1},$$

as follows. $\varphi_{n,i}(u)$ is zero at the end points, $n/2^n$ at the mid point, linear from the left end point to the mid point and linear from the mid point to the right end point.

On the interval $0 \leqslant u \leqslant 1$ we define the function

(3.2) $$\varphi(u) = \begin{cases} \varphi_{n,i}(u), & u \in I_{n,i}, \\ 0, & u = 1. \end{cases}$$

Since $\lim_{n\to\infty} n/2^n = 0$, $\varphi(u)$ is a continuous function.

**3.2.** – For a number $h$, $0 < |h| < 1/64$, there is a unique integer such that

(3.3) $$1/2^{3n+6} < |h| \leqslant 1/2^{3n+3}.$$

Let $\varphi(u)$ be the function defined in (3.2), let $h$, $0 < |h| < 1/64$, be given, and let $n$ be the unique integer determined by (3.3).

Lemma. Under the above conditions

$$\int_0^1 \frac{[\varphi(u+h) - \varphi(u)]^2}{|h|}\, du \geqslant n^2/16.$$

Proof. Case 1. $h > 0$. For each point in the first quarter of the intervals $I_{n,i}$ (see (3.1)) we have

$$\left[\frac{\varphi(u+h) - \varphi(u)}{h}\right]^2 = \left(\frac{n \cdot 2^{3n+2}}{2^n}\right)^2 = n^2 2^{4n+4}.$$

Since $1/4$ of the sum of the lengths of the intervals $I_{n,i}$, $(i = 0, 1, \ldots, 2^{2n+1} - 1)$ is $1/2^{n+2}$ we have the inequality

$$\int_0^1 \left[\frac{\varphi(u+h) - \varphi(u)}{h}\right]^2 |h|\, du \geqslant (1/2^{n+2}) n^2 \cdot 2^{4n+4} (1/2^{3n+6}) = n^2/16.$$

Case 2. $h < 0$. The statement of the lemma follows in the same manner by considering the points in the last quarter of the intervals $I_{n,i}$ of (3.1).

**3.3.** – Lemma. For the function $\varphi(u)$ defined in (3.2)

$$\lim_{h\to 0}\int_0^1 \frac{[\varphi(u+h)-\varphi(u)]^2}{|h|}\, du = +\infty .$$

Proof. This follows from the lemma in 3.2.

**3.4.** – For the function $\varphi(u)$ defined in (3.2) consider the surface given by the representation

$$(3.4)\qquad T:\quad x=\cos[\varphi(u)+v],\quad y=\sin[\varphi(u)+v],\quad z=0,\quad 0\leqslant u\leqslant 1,\quad 0\leqslant v\leqslant 1.$$

Since the point set determined by the surface is a subarc of a circle, the area of the surface is zero. Set

$$A:\ \varphi(u)+v\,,\qquad B:\ \varphi(u+h)+v\,,\qquad C:\ \varphi(u)+v+h\,,$$

and, for $h \neq 0$ (see (1.2)),

$$\Delta(u, v; h) = \frac{1}{h^2}\begin{vmatrix} \cos B - \cos A & \sin B - \sin A \\ \cos C - \cos A & \sin C - \sin A \end{vmatrix} =$$

$$= -\frac{4}{h^2}\begin{vmatrix} \sin\dfrac{B+A}{2}\sin\dfrac{B-A}{2} & \cos\dfrac{B+A}{2}\sin\dfrac{B-A}{2} \\ \sin\dfrac{C+A}{2}\sin\dfrac{C-A}{2} & \cos\dfrac{C+A}{2}\sin\dfrac{C-A}{2} \end{vmatrix} =$$

$$= -\frac{4}{h^2}\sin\frac{B-A}{2}\sin\frac{C-A}{2}\sin\frac{B-C}{2}\,.$$

For $|t|$ small enough $|\sin t| > (1/2)|t|$ and hence for $|h|$ small enough

$$(3.5)\qquad |\Delta(u, v; h)| \geqslant \frac{1}{16h^2}|B-A||C-A||B-C| =$$

$$= \frac{1}{16|h|}|\varphi(u+h)-\varphi(u)||\varphi(u+h)-\varphi(u)-h| \geqslant$$

$$\geqslant \frac{1}{16|h|}[\varphi(u+h)-\varphi(u)]^2 - \frac{1}{16}|\varphi(u+h)-\varphi(u)|\,.$$

From (3.5) and the lemma in 3.3 it thus follows that

$$\lim_{h\to 0}\int_0^1\int_0^1 |\Delta(u, v; h)|\, du\, dv = +\infty - 0 = +\infty\,.$$

## 4. - The functional $A_Y(T)$.

**4.1.** - Let

$$z = \varphi(r) , \qquad 1/4 \leqslant r \leqslant 1/2 , \tag{4.1}$$

be a topological mapping of the interval $1/4 \leqslant r \leqslant 1/2$ upon an OSGOOD curve $C$ of 2-dimensional LEBESGUE measure 1 in the $z = x + iy$ plane.

**4.2.** - Let $Q$ be the unit square, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$, in the $w = u + iv$ plane. Set $w_0 = 1/2 + i(1/2)$. Then, for the function $\varphi(r)$ defined in (4.1) let

$$T: \quad z = f(w) = \begin{cases} \varphi(|w - w_0|) & \text{for} \quad 1/4 \leqslant |w - w_0| \leqslant 1/2 , \\ \varphi(1/4) & \text{for} \quad |w - w_0| < 1/4 , \\ \varphi(1/2) & \text{for} \quad |w - w_0| > 1/2 , \quad w \in Q . \end{cases} \tag{4.2}$$

Then $T$ is a continuous mapping from the unit square $Q$ onto the OSGOOD curve $C$. If $M(z)$ denotes the number of components of $T^{-1}(z)$ then

$$M(z) = \begin{cases} 1 & \text{for} \quad z \in C , \\ 0 & \text{for} \quad z \notin C . \end{cases}$$

Hence, (see 1.5) $E_1 = C$ and $E_n = 0$ for $n \geqslant 2$. Since $\mu$ reduces to the LEBESGUE plane measure in the $z$-plane

$$A_Y(T) = 1 . \tag{4.3}$$

**4.3.** - Let $R$ be the rectangle $0 \leqslant u \leqslant 1$, $1/4 \leqslant v \leqslant 3/4$. Then

$$T^*: \qquad z = f(w) , \qquad w \in R , \tag{4.4}$$

is a continuous mapping from $R$ onto the OSGOOD curve $C$. If $M^*(z)$ denotes the number of components in $T^{*^{-1}}(z)$ then

$$M^*(z) = \begin{cases} 1 & \text{for} \quad z = \varphi(1/4) , \\ 2 & \text{for} \quad z \in C - \varphi(1/4) , \\ 0 & \text{for} \quad z \notin C . \end{cases}$$

Then $E_1^* = C$, $E_2^* = C - \varphi(1/4)$, $E_n^* = 0$ for $n \geqslant 3$. Thus

$$A_Y(T^*) = 2 . \tag{4.5}$$

Hence, by (4.3) and (4.5), for this subrectangle $R$ of $Q$

$$A_Y(T^*) > A_Y(T) .$$

## Bibliography.

[1]. L. CESARI, *Caratterizzazione analitica delle superficie continue di area finita secondo* LEBESGUE, Annali Scuola Norm. Sup. Pisa (2) **10**, 253-294 (1941) and **11**, 1-42 (1942).

[2]. L. CESARI, *On the representation of surfaces*, Amer. Journal of Math. (1950).

[3]. H. D. HUSKEY, *Further contributions to the problem of* GEÖCZE, Duke Math. Journal, 11, 333-339 (1944).

[4]. C. B. MORREY Jr., *An analytic characterization of surfaces of finite* LEBESGUE *area*, part I, Amer. Journal Math. **57**, 692-702 (1935); part II, ibid., **58**, 313-322 (1936).

[5]. T. RADÓ, *Length and Area*, American Math. Society Colloquium Publications, vol. XXX, 1948.

[6]. T. RADÓ, *On the problem of* GEÖCZE, Annali Scuola Norm. Sup. Pisa (2) **14**, 21-30 (1948).

[7]. L. C. YOUNG, *An expression connected with the area of a surface* $z = F(x, y)$, Duke Math. Journal **11**, 43-57 (1944).

[8]. L. C. YOUNG, *On area and length*, Fundamenta Math. **35**, 275-302 (1948).