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On identifications in singular homology theory. (**)

§ 1. - Introduction.

1.1. - The purpose of this paper is to study the relevancy of the various identifications that occur in singular homology theory. To introduce our main result, let us consider a general topological space X. Let E_∞ denote HILBERT space (that is, the space of all infinite sequences $r_1, \ldots, r_n, \ldots$ of real numbers such that $r_1^2 + \ldots + r_n^2 + \ldots$ converges, with the usual assignment of distance). For $p \geqq 0$, let $v_0, \ldots, v_p$ be a sequence of $p+1$ points in E_∞ which are not required to be distinct or linearly independent, and let $| v_0, \ldots, v_p |$ be the convex hull of these points. We associate with X an abstract closure-finite complex $R = R(X)$ as follows. For $p \geqq 0$, a p-cell $(v_0, \ldots, v_p, T)^R$ of R is an aggregate consisting of a sequence $v_0, \ldots, v_p$ (of the type just described) and of a continuous mapping T: $| v_0, \ldots, v_p | \to X$. In terms of these p-cells, we introduce in the usual manner the group C_p^R of finite integral p-chains, with the standard convention that for $p < 0$ the group C_p^R reduces to a zero element. We introduce then the boundary homomorphism

$$\partial^R : C_p^R \to C_{p-1}^R,$$

by the conventional formula

$$\partial^R (v_0, \ldots, v_p, T)^R = \sum_{i=0}^{p} (-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_p, T)^R$$

for $p \geqq 1$. For $p \leqq 0$, ∂^R is defined as the trivial zero-homomorphism. Since obviously $\partial^R \partial^R = 0$, our $R = R(X)$ is a manifestly closure-finite abstract complex (in the sense of [1]; numbers in square brackets refer to the bibliography at the end of this paper). To stress the fundamental feature of the definition of R, we state explicitly that two p-cells $(v_0', \ldots, v_p', T')^R$,

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$(v''_0, \ldots, v''_p, T'')^R$ are considered as equal if and only if they are identical, that is, if $v'_0 = v''_0, \ldots, v'_p = v''_p$, $T' = T''$. Accordingly, our approach differs from the classical one in various respects.

(*a*) Suppose that the points $w_0, \ldots, w_p$ form an odd permutation of the linearly independent points $(v_0, \ldots, v_p)$. In the classical approach, one identifies then the singular p-simplex $(w_0, \ldots, w_p, T)$ with $-(v_0, \ldots, v_p, T)$.

(*b*) Suppose that $v'_0, \ldots, v'_p$ are linearly independent points, and $v''_0, \ldots, v''_p$ are also linearly independent. Let $t\colon |v'_0, \ldots, v'_p| \to |v''_0, \ldots, v''_p|$ be a linear transformation such that the points $t(v'_0), \ldots, t(v'_p)$ form an even permutation of the points $v''_0, \ldots, v''_p$, in the indicated order. Let $T'\colon |v'_0, \ldots, v'_p| \to X$, $T''\colon |v''_0, \ldots, v''_p| \to X$ be continuous mappings which satisfy the relation $T' = T''t$ (products of mappings are to be read from the right to the left). In the classical approach, one identifies then the singular simplexes $(v'_0, \ldots, v'_p, T')$ and $(v''_0, \ldots, v''_p, T'')$.

(*c*) In the classical approach, one considers mappings from simplexes located in finite-dimensional Euclidean spaces. In view of the classical identifications (*a*) and (*b*), the use of HILBERT space is not an essential departure from the classical ways, even though it is important for our own purposes, as will be seen later on.

(*d*) In the classical approach, the points $v_0, \ldots, v_p$ (see above) are required to be linearly independent, while we permit them to be linearly dependent and to coincide among themselves arbitrarily. However, inspection of the proof of our main theorem reveals that it applies also, after plausible modifications, if we define our complex R in terms of p-cells $(v_0, \ldots, v_p, T)^R$ where $v_0, \ldots, v_p$ are required to be linearly independent. Actually, by admitting arbitrary systems $v_0, \ldots, v_p$ we make our main result stronger. In any case, the restriction to linearly independent systems $v_0, \ldots, v_p$ could be made, if desired, without affecting the validity of our main result.

1.2. – Our objective is then, basically, to determine what happens if one drops the identifications (*a*) and (*b*) in classical singular homology theory. *Our main result is that the homology groups remain the same.* In the present paper, we restrict ourselves to the case of the integral homology groups, since our main purpose is to develop the formal apparatus needed for this line of work. Further relevant questions will be treated elsewhere. However, we want to make some comments about the motivation of our work. The identifications (*a*) and (*b*), described above, are based partly on *permutations of vertices*, and partly on what may be termed *affine equivalence of mappings*. In an important paper which appeared in 1944, EILENBERG [3] clarified the role of the identification with respect to *permutations of vertices*. The present investigation was motivated by the desire to complete the result of EILENBERG by showing that the *identification with respect to affine equivalence is also irrelevant.*

1.3. – We proceed to state our main theorem. The topological space X will remain fixed throughout the discussion, and will not be displayed in the notations. Superscripts R will be used to refer to the complex R described in 1.1. Thus c_p^R will denote an integral p-chain in R, z_p^R a p-cycle in R, H_p^R the p-th integral homology group of R, and so forth. Following a suggestion of EILENBERG, we shall compare the groups H_p^R with the singular homology groups as defined in the version of the theory which is used in the (as yet unpublished) book [4] of EILENBERG and STEENROD (the writer wishes to express his gratitude for the privilege of having access to the manuscript of this book). In the EILENBERG-STEENROD version, a closure-finite abstract complex $S = S(X)$ is attached to the topological space X as follows. For each dimension $p \geqq 0$, a fixed fundamental p-simplex is selected. For our own purposes, it is convenient to choose in HILBERT space E_∞ the points $d_0 = (1, 0, ...)$, $d_1 = (0, 1, 0, ...)$, $d_2 = (0, 0, 1, 0, ...)$, and so on, and take $d_0, ..., d_p$ as the vertices of the fundamental p-simplex. A p-cell of S is then an aggregate $(d_0, ..., d_p, T)^S$, where $T\colon |d_0, ..., d_p| \to X$ is a continuous mapping. Now let $v_0, ..., v_p$ be arbitrary points in E_∞. There exists then a unique linear transformation $t\colon |d_0, ..., d_p| \to |v_0, ..., v_p|$ such that $t(d_i) = v_i$, $i = 0, ..., p$. Let us denote this transformation by $[v_0, ..., v_p]$. The boundary homomorphism

$$\partial^S\colon C_p^S \to C_{p-1}^S$$

is then defined, for $p \geqq 1$, by the formula

$$\partial^S(d_0, ..., d_p, T)^S = \sum_{i=0}^{p} (-1)^i (d_0, ..., d_{p-1}, T[d_0, ..., \hat{d}_i, ..., d_p])^S.$$

Here C_p^S denotes the group of finite integral p-chains c_p^S in terms of the p-cells $(d_0, ..., d_p, T)^S$. For $p < 0$, C_p^S is defined as consisting of a zero-element alone. For $p \leqq 0$, ∂^S is then a trivial homomorphism. Since clearly $\partial^S\partial^S = 0$, we obtain in this manner a closure-finite abstract complex $S = S(X)$. The integral homology groups of S will be denoted by H_p^S. For each dimension p, we have then obvious homomorphisms

$$\sigma_p\colon C_p^R \to C_p^S, \qquad \tau_p\colon C_p^S \to C_p^R,$$

defined as follows. For $p < 0$, we have of course the trivial zero-homomorphisms. For $p \geqq 0$, we have

$$\tau_p(d_0, ..., d_p, T)^S = (d_0, ..., d_p, T)^R,$$

$$\sigma_p(v_0, ..., v_p, T)^R = (d_0, ..., d_p, T[v_0, ..., v_p])^S.$$

It is easily seen that σ_p is a chain-mapping. Our main result is contained in the following statement.

Theorem. *The induced homomorphism* $\sigma_p : H_p^R \to H_p^S$ *is an isomorphism onto, for every dimension* p.

1.4. – The proof consists in showing that σ_p is a *chain-equivalence in a certain restricted sense* (cf. the comments in 4.6). Accordingly, we shall endeavor to exhibit homomorphisms

$$F_p : C_p^S \to C_p^R ,$$

such that $\sigma_p F_p \backsimeq 1$, $F_p \sigma_p \backsimeq 1$. A first difficulty arises from the obvious and regrettable fact that the most plausible *mate* to σ_p, namely the τ_p introduced in 1.3, is *not* a chain-mapping. However, we shall construct homomorphisms $F_p : C_p^S \to C_p^R$ which are chain-mappings and do satisfy the relation $\sigma_p F_p \backsimeq 1$. A second difficulty arises from the fact that the relation $F_p \sigma_p \backsimeq 1$ does not hold in the standard sense. The crucial issue in the proof is precisely to find a weaker but still adequate homotopy relation, and to construct an appropriate chain homotopy operator, in a properly restricted sense. Fundamentally, the proof is based upon certain properties of the barycentric subdivision. Some of these properties may be of independent interest and may lead to further relevant applications.

§ 2. – The auxiliary complex *K*.

2.1. – Points of HILBERT space E_∞ will be treated as vectors in the usual manner. The origin $(0, 0, 0, \ldots)$ will be denoted by 0, since it plays the role of a zero-vector. A system of $p+1$ points $v_0, \ldots, v_p$ of E_∞, where $p \geqq 0$, is termed linearly dependent if there exist real numbers $\lambda_0, \ldots, \lambda_p$, not all zero, such that $\lambda_0 v_0 + \ldots + \lambda_p v_p = 0$, $\lambda_0 + \ldots + \lambda_p = 0$. Otherwise $v_0, \ldots, v_p$ are termed linearly independent.

2.2. – Let $x_0, \ldots, x_p$, where $p \geqq 0$, be arbitrary points of E_∞, which may be linearly dependent and which are not required to be distinct. Their *linear hull* $L(x_0, \ldots, x_p)$ is the set of all those points x which can be written in the form $x = \lambda_0 x_0 + \ldots + \lambda_p x_p$, where the real numbers $\lambda_0, \ldots, \lambda_p$ satisfy the relation $\lambda_0 + \ldots + \lambda_p = 1$. Then $L(x_0, \ldots, x_p)$ is the smallest linear subspace of E_∞ containing $x_0, \ldots, x_p$.

2.3. – Now let $y_0, \ldots, y_p$ be arbitrary points of E_∞, and let $x_0, \ldots, x_p$ be linearly independent points of E_∞. Then there exists a unique linear transformation $t: L(x_0, \dots, x_p) \to L(y_0, \dots, y_p)$ such that $t(x_i) = y_i$, $i = 0, \dots, p$. Explicitly, t operates as follows. A point $x \in L(x_0, \dots, x_p)$ can be written uniquely in the form $x = \lambda_0 x_0 + \dots + \lambda_p x_p$, where $\lambda_0 + \dots + \lambda_p = 1$. One has then $t(x) = \lambda_0 y_0 + \dots + \lambda_p y_p$. The linear transformation t is continuous.

The following situation will arise in the sequel. Let $d_0, \dots, d_p$, $p \geqq 0$, be the vertices of the fundamental p-simplex (see 1.3), and let $\eta_0, \dots, \eta_q$, $q \geqq 0$, be $q + 1$ points in E_∞ such that the system $d_0, \dots, d_p, \eta_0, \dots, \eta_q$ is linearly independent. Let $x_0, \dots, x_r$, $0 \leqq r \leqq p$, be linearly independent points in $L(d_0, \dots, d_p)$. It is easy to see that the system $x_0, \dots, x_r, \eta_0, \dots, \eta_q$ is linearly independent. Hence, if we take points $y_0, \dots, y_r$ in $L(\eta_0, \dots, \eta_q)$, which need not be distinct or linearly independent, then we have a unique linear transformation

$$t: L(x_0, \dots, x_r, \eta_0, \dots, \eta_q) \to L(y_0, \dots, y_r, \eta_0, \dots, \eta_q) = L(\eta_0, \dots, \eta_q) ,$$

such that $t(x_i) = y_i$, $i = 0, \dots, r$ and $t(\eta_j) = \eta_j$, $j = 0, \dots, q$. This unique t will be denoted by $\alpha(x_0, \dots, x_r, y_0, \dots, y_r)$. Of course, α depends also upon $\eta_0, \dots, \eta_q$, but in the application $\eta_0, \dots, \eta_q$ (and of course $d_0, \dots, d_p$) will be fixed, and hence need not be diplayed in the notation. The following facts are easily verified.

(*a*) $\alpha(x_0, \dots, x_r, y_0, \dots, y_r)(y) = y$, for $y \in L(\eta_0, \dots, \eta_q)$. In particular, this applies to $y = y_i$, $i = 0, \dots, r$.

(*b*) $\alpha(x_0, \dots, \hat{x}_i, \dots, x_r, y_0, \dots, \hat{y}_i, \dots, y_r)$ agrees with $\alpha(x_0, \dots, x_r, y_0, \dots, y_r)$ on $L(x_0, \dots, \hat{x}_i, \dots, x_r, \eta_0, \dots, \eta_q)$.

2.4. – We associate with E_∞ a complex K as follows (K is identical with the *formal complex* of a set A, in the sense of [4], for the case when $A = E_\infty$). For $p \geqq 0$, a p-cell of K is a sequence of $p + 1$ points $(v_0, \dots, v_p)$ of E_∞ which need not be distinct, and are not required to be linearly independent. Formally, a p-cell of K may be defined as a mapping from the set of integers $0, \dots, p$ into E_∞. Thus the order of the points $v_0, \dots, v_p$ is essential: $(v'_0, \dots, v'_p) = (v''_0, \dots, v''_p)$ if and only if $v'_i = v''_i$, $i = 0, \dots, p$. In terms of these p-cells, one defines the group C_p of integral chains of K. For $p < 0$, C_p is defined as consisting of a zero-element alone. For $p \geqq 1$, the boundary homomorphism

$$\partial : C_p \to C_{p-1}$$

is defined by the formula

$$\partial(v_0, \dots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p) .$$

For $p \leqq 0$, ∂ is defined as the trivial zero-homomorphism. Clearly $\partial\partial = 0$, and thus K is a manifestly *closure-finite abstract complex*.

If v is an assigned point of E_∞, then for $p \geqq 0$ one defines the *cone-homomorphism*

$$h_p^v : C_p \to C_{p+1}$$

by the formula

$$h_p^v(v_0, \ldots, v_p) = (-1)^{p+1}(v_0, \ldots, v_p, v) .$$

For $p \geqq 1$, one has then the identity

$$\partial h_p^v + h_{p-1}^v \partial = 1 .$$

The *barycentric homomorphism*

$$\beta_p : C_p \to C_p$$

is defined as follows. For $p < 0$, β_p is the trivial zero-homomorphism. For $p = 0$, $\beta_0 = 1$. For $p > 0$, β_p is then defined recursively by the formula

$$\beta_p(v_0, \ldots, v_p) = h_{p-1}^{b(v_0, \ldots, v_p)} \beta_{p-1} \partial(v_0, \ldots, v_p) ,$$

where $b(v_0, \ldots, v_p)$ is the barycenter of the points $v_0, \ldots, v_p$. One verifies the identity

$$\partial \beta_p = \beta_{p-1} \partial .$$

In other words, β_p is a chain-mapping. Furthermore, β_p is chain-homotopic to the identity. The corresponding chain-homotopy operator

$$\varrho_p : C_p \to C_{p+1}$$

is constructed as follows. For $p < 0$, ϱ_p is trivial. For $p = 0$, one sets $\varrho_0 = 0$. For $p \geqq 1$, ϱ_p is defined recursively by the formula

$$\varrho_p(v_0, \ldots, v_p) = h_p^{b(v_0, \ldots, v_p)}(\beta_p - 1 - \varrho_{p-1}\partial)(v_0, \ldots, v_p) .$$

One verifies the relation

$$\partial \varrho_p + \varrho_{p-1} \partial = \beta_p - 1 ,$$

which shows that $\beta_p \simeq 1$.

2.5. – Now let A be a convex subset of E_∞. Whe shall write $(v_0, \ldots, v_p) \subset A$ to state that $v_i \in A$, $i = 0, \ldots, p$ (as a consequence, $| v_0, \ldots, v_p | \subset A$). If c_p is a p-chain of K, then we shall write $c_p \subset A$ to state that every p-cell $(v_0, \ldots, v_p)$ occurring in c_p satisfies the inclusion $(v_0, \ldots, v_p) \subset A$. One verifies the inclusions

$$\beta_p(v_0, \ldots, v_p) \subset | v_0, \ldots, v_p | , \quad \varrho_p(v_0, \ldots, v_p) \subset | v_0, \ldots, v_p | .$$

2.6. – We shall discuss presently in some detail two properties of β_p which are important for our purposes. To state and prove these properties, we introduce homomorphisms

$$t_{pj}: C_p \to C_p\,, \qquad p \geqq 1\,, \qquad 0 \leqq j \leqq p-1\,,$$

by the formula

$$t_{pj}(v_0, \ldots, v_j, v_{j+1}, \ldots, v_p) = (v_0, \ldots, v_{j+1}, v_j, \ldots, v_p)\,.$$

Thus t_{pj} effects a *transposition* of v_j and v_{j+1}. Furthermore, we introduce, for every pair i, p of integers such that $p \geqq 1$, $0 \leqq i \leqq p$, the homomorphism

$$i_p: C_p \to C_{p-1}$$

by the formula

$$i_p(v_0, \ldots, v_p) = (v_0, \ldots, \widehat{v_i}, \ldots, v_p)\,, \qquad\qquad p \geqq 1\,.$$

The following formulas are then obvious.

(1) $$\partial(v_0, \ldots, v_p) = \left(\sum_{i=0}^{p} (-1)^i\, i_p\right)(v_0, \ldots, v_p)\,, \quad p \geqq 1\,.$$

(2) $$i_p t_{pj}(v_0, \ldots, v_p) = t_{p-1, j-1} i_p(v_0, \ldots, v_p)\,, \qquad 0 \leqq i < j < p\,.$$

(3) $$i_p t_{pj}(v_0, \ldots, v_p) = t_{p-1, j} i_p(v_0, \ldots, v_p)\,, \qquad 0 \leqq j < i-1 \leqq p-1\,.$$

(4) $$j_p t_{pj}(v_0, \ldots, v_p) = (j+1)_p(v_0, \ldots, v_p)\,, \qquad 0 \leqq j < p\,.$$

(5) $$(j+1)_p t_{pj}(v_0, \ldots, v_p) = j_p(v_0, \ldots, v_p)\,, \qquad 0 \leqq j < p\,.$$

(6) $$i_{p+1} h_p^v(v_0, \ldots, v_p) = -h_{p-1}^v i_p(v_0, \ldots, v_p)\,, \qquad p \geqq 1\,, \quad 0 \leqq i \leqq p\,.$$

For β_p we have then the identity

(7) $$\beta_p t_{pj}(v_0, \ldots, v_p) = -\beta_p(v_0, \ldots, v_p)\,, \qquad 0 \leqq j < p\,,$$

which implies of course, for every p-chain c_p of K, the identity

(8) $$\beta_p t_{pj} c_p = -\beta_p c_p\,, \qquad 0 \leqq j < p\,.$$

To prove (7), one verifies it directly for $p = 1$ (the only admissible value of j is then $j = 0$). For $p > 1$, (7) follows then readily by induction, using the identities (1)-(6) and the recursive definition of β_p. Incidentally, the identities (1)-(6) hold for every p-chain c_p of K, by virtue of the linearity of homomorphisms. Finally, we have the identities

(9) $$i_p \beta_p c_p = 0\,, \qquad 0 \leqq i < p\,,$$

(10) $$p_p \beta_p c_p = (-1)^p \partial \beta_p c_p\,, \qquad p \geqq 1\,,$$

for every p-chain c_p of K. Of course, (9) implies (10), in view of (1). The formulas (9) and (10) are verified directly for $p = 1$, and for $p > 1$ one obtains them readily by simultaneous induction, using the previous identities and the recursive definition of β_p. As a corollary, we obtain the formulas

(11) $$i_{p-1}\beta_{p-1}\partial c_p = 0\,, \qquad 0 \leqq i < p-1\,,$$

(12) $$(p-1)_{p-1}\beta_{p-1}\partial c_p = 0\,, \qquad p \geqq 2\,.$$

The formula (11) is a direct consequence of (9), while (12) is obtained as follows:

$$(p-1)_{p-1}\beta_{p-1}\partial c_p = (-1)^{p-1}\partial\beta_{p-1}\partial c_p\,,$$

by (10). As $\partial\beta_{p-1} = \beta_{p-2}\partial$, (12) follows since $\partial\partial = 0$.

§ 3. – The complex $R = R(X)$.

3.1. – The following device, employed in [4], will be useful. Let A be a convex subset of E_∞, and let C_p^A denote the subgroup of C_p generated by those p-cells $(v_0, \ldots, v_p)$ of K which satisfy the relation $(v_0, \ldots, v_p) \subset A$. For $p < 0$, C_p^A is defined as consisting of a zero-element alone. Let $T: A \to X$ be a continuous mapping. We can then define homomorphisms

$$T_p: C_p^A \to C_p^R\ .$$

by the formula

$$T_p(v_0, \ldots, v_p) = (v_0, \ldots, v_p, T)^R, \qquad p \geqq 0\,, \qquad (v_0, \ldots, v_p) \in C_p^A\,.$$

For $p < 0$, T_p is the trivial zero-homomorphism. For $c_p \in C_p^A$, $T_p c_p$ is obtained by linearity, since the p-cells $(v_0, \ldots, v_p) \subset A$ form a base of C_p^A. It will be convenient to use the symbol $(c_p, T)^R$ to denote $T_p c_p$, where of course it is assumed that $c_p \in C_p^A$. The following statements are then obvious.

(*a*) If c_p is the zero p-chain of C_p^A, then $(c_p, T)^R$ is the zero p-chain of C_p^R. In symbols: $(0, T)^R = 0$.

(*b*) $\partial^R(c_p, T)^R = (\partial c_p, T)^R$, where $c_p \in C_p^A$.

3.2. – In terms of the preceding notations, we define now homomorphisms

$$\beta_p^R: C_p^R \to C_p^R\,,$$
$$\varrho_p^R: C_p^R \to C_{p+1}^R\,,$$

by the formulas

$$\beta_p^R(v_0, \ldots, v_p, T)^R = \big(\beta_p(v_0, \ldots, v_p), T\big)^R, \qquad p \geqq 0\,,$$
$$\varrho_p^R(v_0, \ldots, v_p, T)^R = \big(\varrho_p(v_0, \ldots, v_p), T\big)^R, \qquad p \geqq 0\,.$$

For $p < 0$, β_p^R and ϱ_p^R are of course defined as the trivial zero-homomorphisms.

Since $\beta_p(v_0, \ldots, v_p) \subset | v_0, \ldots, v_p |$, $\varrho_p(v_0, \ldots, v_p) \subset | v_0, \ldots, v_p |$ by 2.5, the homomorphisms β_p^R, ϱ_p^R are well defined. The identities stated in 2.4 for β_p, ϱ_p yield then easily, by means of the technique described in 3.1, the identities

$$\partial^R \beta^R = \beta_{r-1}^R \partial^R , \tag{1}$$

$$\partial^R \varrho_p^R + \varrho_{p-1}^R \partial^R = \beta_p^R - 1 . \tag{2}$$

The identity (1) states that β_p^R is a chain-mapping, while (2) states that $\beta_p^R \backsimeq 1$.

3.3. – In terms of the homomorphisms t_{pj}, i_p introduced in 2.6, we now define homomorphisms

$$t_{pj}^R: \quad C_p^R \to C_p^R , \qquad p \geqq 1 , \quad 0 \leqq j \leqq p-1 ,$$
$$i_p^R: \quad C_p^R \to C_{p-1}^R , \qquad p \geqq 1 , \quad 0 \leqq i \leqq p ,$$

by the formulas (cf. 3.1)

$$t_{pj}^R(v_0, \ldots, v_p, T)^R = \big(t_{pj}(v_0, \ldots, v_p), T\big)^R ,$$
$$i_p^R(v_0, \ldots, v_p, T)^R = \big(i_p(v_0, \ldots, v_p), T\big)^R .$$

3.4. – The homomorphisms considered so far in the complex R are of a simple and familiar character. However, we shall need a set of further and more complicated homomorphisms, related to the well-known *prism construction*. Let $p \geqq 0$, $q \geqq 0$ be given integers. Let $d_0, \ldots, d_p$ be the vertices of the fundamental p-simplex, and let $\eta_0, \ldots, \eta_q$ be $q+1$ points in E_∞ such that the points $d_0, \ldots, d_p, \eta_0, \ldots, \eta_q$ are linearly independent. The integers p, q and the points $d_0, \ldots, d_p, \eta_0, \ldots, \eta_q$ are cansidered as fixed, and will not be displayed in the notations, unless needed for special reasons. For every integer $r \geqq 0$, let Γ_r denote the subgroup of C_r^R generated by those r-cells $(y_0, \ldots, y_r, T)^R$ which satisfy the condition $(y_0, \ldots, y_r) \subset L(\eta_0, \ldots, \eta_q)$. Now let $x_0, \ldots, x_r$ be linearly independent points in $L(d_0, \ldots, d_p)$; thus $0 \leqq r \leqq p$. Finally, let k be an integer such that $0 \leqq k \leqq r$. In terms of $x_0, \ldots, x_r, k$ (and of the fixed points $d_0, \ldots, d_p, \eta_0, \ldots, \eta_q$), we define then a homomorphism

$$\prod\nolimits_k^{x_0, \ldots, x_r} : \Gamma_r \to C_{r+1}^R$$

by the formula

$$\prod\nolimits_k^{x_0, \ldots, x_r}(y_0, \ldots, y_r, T)^R =$$
$$= \sum_{j=0}^{k} (-1)^j \big(x_0, \ldots, x_j, y_j, \ldots, y_r, T\alpha(x_0, \ldots, x_r, y_0, \ldots, y_r)\big)^R ,$$

where $\alpha(x_0, \ldots, x_r, y_0, \ldots, y_r)$ is the linear trasnformation described in 2.3, and of course $(y_0, \dots, y_r, T)^R \in \Gamma_r$. Since $\alpha(x_0, \dots, x_r, y_0, \dots, y_r)$ carries $x_0, \dots, x_j, y_j, \dots, y_r$ into $y_0, \dots, y_j, y_j, \dots, y_r$ respectively, the convex hull of $x_0, \dots, x_j, y_j, \dots, y_r$ is mapped by α into $|y_0, \dots, y_r|$, and thus $T\alpha$: $|x_0, \dots, x_j, y_j, \dots, y_r| \to X$ is a well-defined continuous mapping. Thus the homomorphisms $\prod$ are well defined.

3.5. – The proof of our main theorem will depend upon a set of identities involving the various homomorphisms introduced above. These identities will be listed presently for convenient reference.

I. 1. $\quad \partial^R \beta_p^R = \beta_{p-1}^R \partial^R$.

I. 2. $\quad \partial^R \varrho_p^R + \varrho_{p-1}^R \partial^R = \beta_p^R - 1$.

I. 3. $\quad \beta_p^R t_{pj}^R = -\beta_p^R$, $\quad 0 \leqq j < p$.

I. 4. $\quad i_p^R \beta_p^R = 0$, $\quad 0 \leqq i < p$.

I. 5. $\quad p_p^R \beta_p^R = (-1)^p \partial^R \beta_p^R$, $\quad p \geqq 1$.

I. 6. $\quad i_{p-1}^R \beta_{p-1}^R \partial^R = 0$, $\quad 0 \leqq i < p-1$.

I. 7. $\quad (p-1)_{p-1}^R \beta_{p-1}^R \partial^R = 0$, $\quad p \geqq 2$,

II. 1. $\quad \sigma_p \tau_p = 1$.

II. 2. $\quad \partial^S = \sigma_{p-1} \partial^R \tau_p$.

II. 3. $\quad \partial^R = \sum_{i=0}^{p} (-1)^i i_p^R$, $\quad p \geqq 1$.

II. 4. $\quad \sigma_{p-1} \partial^R \tau_p \sigma_p = \sigma_{p-1} \partial^R$.

III. 1. $\quad \sigma_p \beta_p^R \tau_p \sigma_p = \sigma_p \beta_p^R$.

III. 2. $\quad \sigma_{p+1} \varrho_p^R \tau_p \sigma_p = \sigma_{p+1} \varrho_p^R$.

III. 3. $\quad \tau_{p-1} \sigma_{p-1} \partial^R \beta_p^R = \partial^R \tau_p \sigma_p \beta_p^R$.

IV. Let $d_0, \dots, d_p, \eta_0, \dots, \eta_q, \Gamma_p$ have the meanings assigned to these symbols in 3.4. Then, for $p \geqq 1$, and for every p-chain $c_p^R \in \Gamma_p$, we have the identity

$$\partial^R \prod_p^{d_0, \dots, d_p} = 1 - \tau_p \sigma_p - \sum_{i=0}^{p} (-1)^i \prod_{p-1}^{d_0, \dots, \hat{d_i}, \dots, d_p} i_p^R .$$

3.6. – The experienced reader will have little difficulty in verifying these identities, and we restrict ourselves to brief hints about the proofs. The identities in group I follow readily from the analogous identities, noted in § 2 for β_p, ϱ_p, t_{pj}, i_p, by means of the technique described in 3.1. The identities in group II are immediate consequences of the definitions of the symbols involved. The identities III. 1, III. 2 are best verified first for $p = 0$, and then established by induction on p, using the recursive definition of β_p and ϱ_p. The identity III. 3 follows readily by using on its right-hand side the expression for ∂^R in terms of the homomorphisms i_p^R, as given in II. 3. The identity IV is manifestly closely related to the familiar homotopy identity concerning the *prism construction*, but since the present writer had some little trouble initially in first thinking of and then dealing with the homomorphisms $\prod$, a few comments will be made about the proof. It seems best, to avoid trivial complications, to verify IV directly for $p = 1$. For $p \geqq 2$, the following procedure may be employed. Using for ∂^R the expression II. 3, one finds that the task consists in finding convenient explicit expressions for terms of the general form

$$i_{p+1}^R(-1)^j(d_0, \ldots, d_j, y_j, \ldots, y_p, T\alpha(d_0, \ldots, d_p, y_0, \ldots, y_p))^R,$$

where $p \geqq 2$, $0 \leqq i \leqq p+1$, $0 \leqq j \leqq p$, and $(y_0, \ldots, y_p, T)^R$ is a p-cell belonging to the group Γ_p. If we set

$$A = i_{p+1}^R \prod\nolimits_p^{d_0, \ldots, d_p}(y_0, \ldots, y_p, T)^R,$$

then straightforward calculation yields the following formulas for the various values of i (in deriving these formulas, the reader will have to use the remarks (*a*) and (*b*) in 2.3.).

$$A = (y_0, \ldots, y_p, T)^R - \prod\nolimits_{p-1}^{d_1, \ldots, d} 0_p^R(y_0, \ldots, y_p, T)^R \quad \text{for} \quad i = 0.$$

$$A = \left(\prod\nolimits_0^{d_0, d_2, \ldots, d_p} - \prod\nolimits_{p-1}^{d_0, d_2, \ldots, d_p}\right) 1_p^R(y_0, \ldots, y_p, T)^R \quad \text{for} \quad i = 1.$$

$$A = \prod\nolimits_{p-2}^{d_0, \ldots, d_{p-2}, d_p}(p-1)_p^R(y_0, \ldots, y_p, T)^R \quad \text{for} \quad i = p.$$

$$A = \left(\prod\nolimits_{p-1}^{d_0, \ldots, d_{p-1}} p_p^R + (-1)^p \tau_p \sigma_p\right)(y_0, \ldots, y_p, T)^R \quad \text{for} \quad i = p+1.$$

$$A = \left(\prod\nolimits_{i-2}^{d_0, \ldots, \widehat{d_{i-1}}, \ldots, d_p}(i-1)_p^R + \prod\nolimits_{i-1}^{d_0, \ldots, \widehat{d_i}, \ldots, d_p} i_p^R - \right.$$
$$\left. - \prod\nolimits_{p-1}^{d_0, \ldots, \widehat{d_i}, \ldots, d_p} i_p^R\right)(y_0, \ldots, y_p, T)^R \quad \text{for} \quad 2 \leqq i \leqq p-1.$$

In view of the identity II. 3, summation yields then the desired result.

3.7. – Those of the identities in 3.5 which contain σ_p, τ_p show a common feature which may be worth pointing out. Let us first recall that EILENBERG and STEENROD, in [3], introduce for their complex $S = S(X)$ a barycentric homomorphism β_p^S which in our terminology is given by the formula

$$\beta_p^S = \sigma_p \beta_p^R \tau_p.$$

One finds readily that the formula $\partial^S \beta_p^S = \beta_{p-1}^S \partial^S$, stating that β_p^S is a chain-mapping, follows immediately from our identity III. 3. In a similar manner, one finds that the identities in 3.5 represent, in a sense, *antecedents* of various well-known formulas in algebraic topology.

§ 4. – Proof of the main theorem.

4.1. – Let us first note that the homomorphism $\sigma_p : C_p^R \to C_p^S$ is a chain-mapping. Indeed, the identities II. 2, II. 4 in 3.5 yield directly $\partial^S \sigma_p = \sigma_{p-1}\partial^R$. To show that σ_p is a *chain-equivalence in a certain weak sense*, we introduce now a « *mate* » F_p to σ_p. The homomorphisms

$$F_p : C_p^S \to C_p^R$$

are defined by the formula

$$F_p = \tau_p \sigma_p \beta_p^R \tau_p .$$

We first verify that F_p is a chain-mapping, that is,

(1) $$\partial^R F_p = F_{p-1}\partial^S .$$

Using the identity III. 3 in 3.5, we find

(2) $$\partial^R F_p = \partial^R \tau_p \sigma_p \beta_p^R \tau_p = \tau_{p-1}\sigma_{p-1}\partial^R \beta_p^R \tau_p .$$

On the other hand, the identities III. 1, I. 1 in 3.5 yield

(3) $$F_{p-1}\partial^S = \tau_{p-1}\sigma_{p-1}\beta_{p-1}^R \tau_{p-1}\sigma_{p-1}\partial^R \tau_p = \tau_{p-1}\sigma_{p-1}\beta_{p-1}^R \partial^R \tau_p = \tau_{p-1}\sigma_{p-1}\partial^R \beta_p^R \tau_p .$$

The formulas (2), (3) imply (1).

4.2. – Next we verify that $\sigma_p F_p \backsimeq 1$, by establishing the identity

(1) $$\partial^S \sigma_{p+1}\varrho_p^R \tau_p + \sigma_p \varrho_{p-1}^R \tau_{p-1}\partial^S = \sigma_p F_p - 1 .$$

We find, by using the identities II.2, II.4, III.2, I.2, II.1 in 3.5,

$$\partial^S \sigma_{p+1}\varrho_p^R \tau_p + \sigma_p \varrho_{p-1}^R \tau_{p-1}\partial^S = \sigma_p \partial^R \tau_{p+1}\sigma_{p+1}\varrho_p^R \tau_p + \sigma_p \varrho_{p-1}^R \tau_{p-1}\sigma_{p-1}\partial^R \tau_p =$$
$$= \sigma_p \partial^R \varrho_p^R \tau_p + \sigma_p \varrho_{p-1}^R \partial^R \tau_p = \sigma_p(\beta_p^R - 1)\tau_p = \sigma_p \tau_p \sigma_p \beta_p^R \tau_p - \sigma_p \tau_p = \sigma_p F_p - 1 .$$

Let us note that Eilenberg and Steenrod, in [3], introduce a barycentric homomorphism β_p^S and a corresponding homotopy operator ϱ_p^S by the formulas (in our terminology)

$$\beta_p^S = \sigma_p \beta_p^R \tau_p , \qquad \varrho_p^S = \sigma_{p+1}\varrho_p^R \tau_p ,$$

and note the homotopy identity

(2) $$\partial^S \varrho_p^S + \varrho_{p-1}^S \partial^S = \beta_p^S - 1 .$$

In view of the identity $\sigma_p\tau_p = 1$, clearly (1) and (2) are equivalent. In other words, the relation $\sigma_p F_p \backsimeq 1$ is equivalent to the relation $\beta_p^S \backsimeq 1$. On the other hand, inspection reveals that the relation $\partial^R F_p = F_{p-1}\partial^S$ is stronger than the corresponding relation $\partial^S \beta_p^S = \beta_{p-1}^S \partial^S$ in the EILENBERG-STEENROD theory.

4.3. – The relation $\sigma_p F_p \backsimeq 1$ implies, by a familiar argument, that the induced homomorphism

$$\sigma_p : H_p^R \rightarrow H_p^S$$

is *onto*. We complete the proof of our main theorem by showing that this induced homomorphism is also an *isomorphism into*. This would be certainly so if we could show that $F_p\sigma_p \backsimeq 1$. Actually, we shall only show that a certain weak version of this relation holds, or rather, this will be the leading idea in the concluding portion of our proof (cf. 4.6). Let z_p^R be a p-cycle in the complex R, and let us suppose that

(1) $$\sigma_p z_p^R \in B_p^S .$$

We have to show that

(2) $$z_p^R \in B_p^R ,$$

where B_p^S, B_p^R denote the groups of p-boundaries in the complexes S, R respectively. The proof will be made in several steps.

4.4. – To avoid trivial discussions, we first consider the case $p = 0$. We shall make use of the following facts.

(*a*) $\tau_0\sigma_0 - 1 \sim 0$ (in the complex R). Indeed, let d_0 be the fundamental 0-simplex (see 1.3) and let $(v, T)^R$ be a 0-cell of R. On $|v, d_0|$, define $T^*: |v, d_0| \rightarrow X$ by setting $T^*(y) = T(v)$ for all $y \in |v, d_0|$. Then clearly

$$\partial^R(v, d_0, T^*)^R = (d_0, T^*)^R - (v, T^*)^R = \tau_0\sigma_0(v, T)^R - (v, T)^R ,$$

and our assertion is proved.

(*b*) $\tau_0\partial^S \sim 0$ (in the complex R). Indeed, let $(d_0, d_1, T)^S$ be a 1-cell of the complex S (see 1.3). By II.2 in 3.5 we have then

$$\tau_0\partial^S(d_0, d_1, T)^S = \tau_0\sigma_0\partial^R\tau_1(d_0, d_1, T)^S = \tau_0\sigma_0(d_1, T)^R - \tau_0\sigma_0(d_0, T)^R .$$

Hence, by the preceding remark (*a*):

$$\tau_0\partial^S(d_0, d_1, T)^S \sim (d_1, T)^R - (d_0, T)^R = \partial^R(d_0, d_1, T)^R \sim 0 .$$

Now suppose that for a certain z_0^R we have the relation $\sigma_0 z_0^R = \partial^S c_1^S$. By (*a*) and (b) above we infer that

$$z_0^R \sim \tau_0\sigma_0 z_0^R = \tau_0 \partial^S c_1^S \sim 0\,,$$

and thus the case $p = 0$ is settled.

4.5. – Now let us consider the situation described in 4.3 for $p \geqq 1$. Since the z_p^R occurring in 4.3 (1) is a *finite* chain, we can select in E_∞ a finite number of linearly independent points $\eta_0, \ldots, \eta_q$ such that $z_p^R \subset L(\eta_0, \ldots, \eta_q)$. We have to distinguish two cases, depending upon the relative position of the points $\eta_0, \ldots, \eta_q$ and of the vertices $d_0, \ldots, d_p$ of the fundamental p-simplex.

Case 1. The points $d_0, \ldots, d_p, \eta_0, \ldots, \eta_q$ are linearly independent. Then we are in the situation described in 3.4. The assumption $z_p^R \subset L(\eta_0, \ldots, \eta_q)$ means, in the terminology of 3.4, that $z_p^R \in \Gamma_p$. Hence, by 2.5, we have also $\beta_p^R z_p^R \in \Gamma_p$, and hence the identity IV in 3.5 applies to $\beta_p^R z_p^R$, yielding

$$\text{(1)} \qquad \partial^R \prod_p^{d_0, \ldots, d_p} \beta_p^R z_p^R = \beta_p^R z_p^R - \tau_p\sigma_p\beta_p^R z_p^R - \sum_{i=0}^{p} (-1)^i \prod_{p-1}^{d_0, \ldots, \hat{d_i}, \ldots, d_p} i_p^R \beta_p^R z_p^R\,.$$

In view of I.4 in 3.5, the summation on the right reduces to its last term, and by I.5 in 3.5 this last term is equal to

$$\prod_{p-1}^{d_0, \ldots, d_{p-1}} \partial^R \beta_p^R z_p^R\,.$$

On the other hand, by I.1 in 3.5 we have $\partial^R \beta_p^R z_p^R = \beta_{p-1}^R \partial^R z_p^R = 0$, since z_p^R is a cycle. Thus the summation on the right-hand side of (1) vanishes. Thus (1) implies that

$$\text{(2)} \qquad \beta_p^R z_p^R \sim \tau_p\sigma_p\beta_p^R z_p^R\,.$$

From III.1 in 3.5 we infer that

$$\text{(3)} \qquad F_p\sigma_p z_p^R = \tau_p\sigma_p\beta_p^R\tau_p\sigma_p z_p^R = \tau_p\sigma_p\beta_p^R z_p^R\,.$$

From (2) and (3) it follows that

$$\text{(4)} \qquad \beta_p^R z_p^R \sim F_p\sigma_p z_p^R\,.$$

Now $\sigma_p z_p^R \sim 0$, and F_p is a chain-mapping (see 4.1). Hence $F_p\sigma_p z_p^R \sim 0$, and thus (4) yields $\beta_p^R z_p^R \sim 0$. But $\beta_p^R \backsimeq 1$ (see 3.2), and hence it follows that $z_p^R \sim \beta_p^R z_p^R \sim 0$.

Case 2. We now drop the assumption that the points $d_0, \ldots, d_p, \eta_0, \ldots, \eta_q$ are linearly independent. Since $d_0, \ldots, d_p$ are linearly independent, we can choose $q+1$ points $\eta_0^*, \ldots, \eta_q^*$ in E_∞ such that the points $d_0, \ldots, d_p, \eta_0^*, \ldots, \eta_q^*$ are linearly independent. Consider now in E_∞ the linear subspaces

$L = L(\eta_0, \ldots, \eta_q)$, $L^* = L(\eta_0^*, \ldots, \eta_q^*)$. Since the systems $\eta_0, \ldots, \eta_q$ and $\eta_0^*, \ldots, \eta_q^*$ are linearly independent individually, we have unique linear transformations $t : L \to L^*$, $t^* : L^* \to L$, such that $t(\eta_i) = \eta_i^*$, $t^*(\eta_i^*) = \eta_i$, $i = 0, \ldots, q$, and $tt^* = 1$, $t^*t = 1$. We denote, for each dimension r, by Γ_r the subgroup of C_r^R generated by those r-cells $(y_0, \ldots, y_r, T)^R$ of R which satisfy the condition $(y_0, \ldots, y_r) \subset L$, and we assign a similar meaning to Γ_r^* relative to L^*. We have then homomorphisms

$$t_r : \Gamma_r \to \Gamma_r^* ,$$

defined by $t_r(y_0, \ldots, y_r, T)^R = \big(t(y_0), \ldots, t(y_r)\, Tt^*\big)^R$. Similarly, we define the homomorphisms $t_r^* : \Gamma^*{}_r \to \Gamma_r$. Whe shall need the following simple facts.

(*a*) Obviously $\partial^R t_r = t_{r-1}\partial^R$, $r \geqq 1$. Hence, if we take a cycle $z_r^R \in \Gamma_r$, the $t_r z_r^R$ is a cycle contained in Γ_r^*, and viceversa.

(*b*) Take a cycle z_r^R, $r \geqq 1$, in Γ_r and suppose that $t_r z_r^R \sim 0$. We assert that $z_r^R \sim 0$. Indeed, by assumption we have an $(r+1)$-chain c_{p+1}^R such that $t_r z_r^R = \partial^R c_{r+1}^R$ (note that c_{r+1}^R may *not* be contained in Γ_{r+1}^*). Since c_{r+1}^R is a *finite* chain, we can choose in E_∞ a finite number of further points $\eta_{q+1}^*, \ldots, \eta_n^*$ such that (i) the points $\eta_0^*, \ldots, \eta_n^*$ are linearly independent, and (ii) $c_{r+1}^R \subset L(\eta_0^*, \ldots, \eta_n^*)$. We then choose points $\eta_{q+1}, \ldots, \eta_n$ such that the system $\eta_0, \ldots, \eta_n$ is linearly independent. Then the linear transformations t, t^* can be extended uniquely to the linear subspaces $L(\eta_0, \ldots, \eta_n)$, $L(\eta_0^*, \ldots, \eta_n^*)$ so that η_j, η_j^* are mated points, $j = 0, \ldots, n$. We extend similarly the groups Γ_r, Γ_r^*, as well as the homomorphisms t_r, t_r^*, preserving notations for simplicity. The assumption $t_r z_r^R = \partial^R c_{r+1}^R$ yields then

$$z_r^R = t_r^* t_r z_r^R = t_r^* \partial^R c_{r+1}^R = \partial^R t_{r+1}^* c_{r+1}^R .$$

Thus $z_r^R \sim 0$, as asserted.

(*c*) Obviously, for every r-chain $c_r^R \in \Gamma_r$ we have the relation

$$\sigma_r c_r^R = \sigma_r t_r c_r^R .$$

Now let us return to the cycle z_p^R of 4.3, (1). By assumption there exists a $(p+1)$-chain c_{p+1}^S such that $\sigma_p z_p^R = \partial^S c_{p+1}^S$. By the remark (*c*), this implies that

$$\sigma_p t_p z_p^R = \partial^S c_{p+1}^S . \tag{1}$$

Now, by the remark (*a*), $t_p z_p^R$ is a p-cycle in the complex R, and $t_p z_p^R \subset L(\eta_0^*, \ldots, \eta_q^*)$. Since the points $d_0, \ldots, d_p, \eta_0^*, \ldots, \eta_q^*$ are linearly independent, the cycle $t_p z_p^R$ comes under Case 1 above, and hence we can infer from (1) that $t_p z_p^R \sim 0$. By remark (*b*), it follows finally that $z_p^R \sim 0$, and the proof of our main theorem is complete.

4.6. – In the course of the preceding discussion, we made repeatedly statements to the effect that σ_p is a *chain equivalence in a certain weak sense*, as far as shown by our method of proof. In this connection, the writer is indebted to S. Eilenberg for a reference to a remarkable general theorem occurring in [4]. Let M, N be Mayer complexes such that the corresponding chain-groups C_p^M, C_p^N are free Abelian groups. Let f_p: $C_p^M \to C_p^N$ be a chain-mapping such that the induced homomorphisms $f_p : H_p^M \to H_p^N$ for the homology groups are *isomorphisms onto* in each dimension p. Then f_p is a chain equivalence in the strict sense. That is, there exists a chain-mapping $h_p : C_p^N \to C_p^M$ such that $f_p h_p \backsimeq 1$ and $h_p f_p \backsimeq 1$ in each dimension p. In view of our main theorem stated in 1.3, application of this remarkable result of Eilenberg and Steenrod yields the fact that there exists a « mate » $h_p : C_p^S \to C_p^R$ to σ_p such that $\sigma_p h_p \backsimeq 1$, $h_p \sigma_p \backsimeq 1$ in the strict sense. It would be interesting to determine whether the proof of the Eilenberg-Steenrod theorem could be used to obtain explicitly such a mate h_p to σ_p. In particular, it would be interesting to determine whether the « mate » F_p to σ_p used by us (see 4.1) satisfies the relation $F_p \sigma_p \backsimeq 1$ in the strict sense (note that the relation $\sigma_p F_p \backsimeq 1$ does hold in the strict sense by 4.2).

4.7. – As regards further literature on *unessential identifications* in singular homology theory, let us mention (beyond the work of Eilenberg already referred to) a paper by Tucker [5] who showed, in fact, that the identification of the so-called *degenerate chains* with zero does not affect the homology groups. A systematic study of unessential identifications is contained in a paper of the present writer, to be published elsewhere.

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Biblography.

[1] S. Lefschetz, *Algebraic Topology*. Amer. Math. Soc. Colloquium Publications, Vol. 27, 1942.

[2] L. Seifert und W. Threlfall, *Lehrbuch der Topologie*. Teubner, 1934.

[3] S. Eilenberg, *Singular homology theory*. Ann. of Math. **45**, 407-447 (1944).

[4] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology* (unpublished).

[5] D. G. Tucker, *Degenerate cycles bound*. Rec. Math. (Moscow) **3**, 287-289 (1938).