EDWARD SILVERMAN (*)

An intrinsic property of LEBESGUE area. (**)

1. – Introduction.

H. BUSEMANN gave a definition of intrinsic area in which the area of a surface depends only upon geodesic distances in the surface and not on the way in which the surface is imbedded in its containing space [2] ([1]). It is the purpose of this paper to show that if a FRÉCHET surface admits a representation for which the geodesic distance between pairs of points varies continuously with these points, then the LEBESGUE area is also intrinsic.

The procedure used here will depend upon definitions and results of [6]. In particular, we will use the area defined there which, in fact, agrees with LEBESGUE area for surfaces in Euclidean space.

Let C be the set of continuous functions defined on a square Q to m, the space of bounded sequences [1]. If $x \in C$, denote the area of x by $L(x)$ and the i^{th} component of x by x^i. As usual, we say that x is Lipschitzian if for some $M > 0$, $\|x(p) - x(q)\| \leqq M\|p - q\|$ for all p, q in Q. (The norm, of course, depends upon the space to which it applies). By $D(x, y)$ we understand a modified FRÉCHET distance between x and y defined by

$$D(x, y) = \inf_{h} \{ \max_{p,q \in Q} | \, \| x(p) - x(q)\| - \|y(h(p)) - y(h(q)) \| \, | \}$$

where h is a homeomorphism of domain x onto domain y. We use the correspondingly modified FRÉCHET distance between functions defined on closed intervals of the line. We write $x_n \rightrightarrows x$, for $x_n, x \in C$, $n = 1, 2, \ldots$, if $\sup_{p \in Q} \|x_n(p) - x(p)\| \to 0$.

It will be convenient for our purposes to introduce the following notation. If $x \in C$, let $\tilde{x}$ be defined on $Q \times Q$ to the real numbers by $\tilde{x}(p, q) = \|x(p) - x(q)\|$.

(*) Address: Department of Mathematics, Kenyon College, Gambier, Ohio, U.S.A..
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([1]) Numbers in square brackets refer to the bibliography at the end of this Note.

Clearly $\tilde{x}$ satisfies the properties S:

$$\begin{aligned} &S_1: && \tilde{x}(p, q) = \tilde{x}(q, p) , \\ &S_2: && \tilde{x}(p, p) = 0 , \quad \tilde{x}(p, q) \geqq 0 , \\ &S_3: && \tilde{x}(p, q) + \tilde{x}(q, r) \geqq \tilde{x}(p, r) , \\ &S_4: && \tilde{x} \text{ is continuous.} \end{aligned}$$

That the properties S characterize such functions is a consequence of the following lemma.

Lemma 1: *Let $\tilde{z}$ be defined on $Q \times Q$ and satisfy* S. *Then, if $\{ p_i \}$ is an everywhere dense sequence in Q, the function $z^* \in C$ defined by $z^*(q) = \{ \tilde{z}(p_i, q) \}$ has the property that $\tilde{z}^* = \tilde{z}$.*

Proof: That $\tilde{z}$ is uniformly bounded on $Q \times Q$ follows from its continuity. Hence $z^*(q) \in m$ for $q \in Q$. By S_1, S_2, S_3,

$$| \tilde{z}(p, q) - \tilde{z}(p, r) | = | \tilde{z}(q, p) - \tilde{z}(p, r) | \leqq \tilde{z}(q, r)$$

and by S_4, $\tilde{z}(q, r) \to 0$ as $q \to r$. Hence $z^* \in C$. That $\| z^*(p) - z^*(q) \| = \tilde{z}(p, q)$ for all $(p, q) \in Q \times Q$ follows from the continuity of $\tilde{z}$.

The following theorems are proved in [6].

Theorem 1: *If $x \in C$, $y \in C$, and $\tilde{x} \leqq \tilde{y}$ [i.e., $\tilde{x}(p, q) \leqq \tilde{y}(p, q)$ for $(p, q) \in Q \times Q$], then $L(x) \leqq L(y)$.*

Theorem 2: *If $x \in C$ is Lipschitzian and if Q is subdivided into triangles $\Delta_1, \ldots, \Delta_n$, then $L(x) = \sum_{i=1}^{n} L(x \mid \Delta_i)$.*

Theorem 3: *Let x_n, $n = 0, 1, \ldots$, be in C. If $x_n^i \rightrightarrows x_0^i$ for each i, or if $\lim_{n \to \infty} D(x_n, x_0) = 0$, then $L(x_0) \leqq \liminf_{n \to \infty} L(x_n)$.*

Theorem 4: *If $x \in C$, then there exists a sequence $\{ x_n \}$ of quasilinear functions in C such that $\lim_{n \to \infty} D(x_n, x) = 0$ and $\lim_{n \to \infty} L(x_n) = L(x)$.*

2. - ν-length.

Let f and g be continuous functions defined on closed intervals $[a, b]$ and $[c, d]$, respectively, with range in a metric space. If $f(b) = g(c)$ then let $f \oplus g$ be defined on $[a + c, b + d]$ by

$$\begin{aligned} (f \oplus g)(t) &= f(t - c) , && a + c \leqq t \leqq b + c , \\ (f \oplus g)(t) &= g(t - b) , && b + c \leqq t \leqq b + d . \end{aligned}$$

Now let Γ be the space of continuous functions defined on bounded closed intervals of the line with range in m.

Definition 1: A real valued function μ on Γ is a ν-length if there exists a constant K such that for $f, g \in \Gamma$,

(a) diam range $f \leqq \mu(f) \leqq K$ diam range f,

(b) $|\mu(f) - \mu(g)| \leqq 2KD(f, g)$,

(c) if f is linear, then $\mu(f) =$ diam range f,

(d) $\mu(f \oplus g) \leqq \mu(f) + \mu(g)$, if $f \oplus g$ is defined.

We now define some functions on Γ which are ν-lengths.

Let $f \in \Gamma$ and suppose that f is defined on $[a, b]$. For each $n \geqq 1$, let $t_1 \leqq ... \leqq t_{n+1}$ be $n + 1$ points in $[a, b]$ and let $(p_1, ..., p_{n+1})$ be the set of corresponding points, under f, in m. Let this set of points be denoted by S_n. Then S_n will be termed an *admissible* set of points for f. Denote the sum of the distances $\|p_{i+1} - p_i\|$, $i = 1, ..., n$, by $d(S_n)$ and let $\mu_n(f)$ be the least upper bound of such numbers for all admissible S_n.

Lemma 2: *μ_n is a ν-length.*

Proof: If we take $K = n$ in Definition 1, then we see that (a), (b), and (c) are evident. To prove (d), use the notation defining $f \oplus g$ and let $a + c \leqq \leqq t_1 \leqq ... \leqq t_{n+1} \leqq b + d$ determine S_n for $f \oplus g$. Let k be the largest integer for which $t_k \leqq b + c$. Now define S'_n and S''_n for f and g, respectively, by $S'_n = (u_1, ..., u_{n+1})$, $S''_n = (v_1, ..., v_{n+1})$, where $u_1 = t_1 - c, ..., u_k = t_k - c$, $u_{k+1} = = ... = u_{n+1} = b$, and $v_1 = ... = v_k = c$, $v_{k+1} = t_{k+1} - b, ..., v_{n+1} = t_{n+1} - b$. Then $d(S_n) \leqq d(S'_n) + d(S''_n)$ and the lemma follows.

The following lemma is obvious.

Lemma 3: $\mu_n(f) \leqq \mu_{n+1}(f)$.

Next, let $\{a_n\}$, $n = 1, 2, ...$, be a sequence of non-negative numbers, with an infinite number of non-zero terms, such that

$$\sum_{n=1}^{\infty} a_n = 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} n a_n = K .$$

Now define ν on Γ by

$$\nu(f) = \sum_{n=1}^{\infty} a_n \mu_n(f) , \qquad f \in \Gamma .$$

Then ν is a ν-length. ν is also a μ-length in the sense of M. MORSE [4].

3. - Functions generated by ν-lengths.

Suppose that μ is a ν-length on Γ and that $x \in C$. Let $p, q \in Q$ and let f be any continuous function defined on an interval $[a, b]$ such that $f(a) = p$, $f(b) = q$, and range f is contained in Q. Then f is *admissible* for p and q.

Define

$$\tilde{x}_\mu(p, q) = \inf_f \mu(x(f))$$

for all admissible f, and

$$\tilde{x}_l(p, q) = \lim_{n\to\infty} \tilde{x}_{\mu_n}(p, q).$$

Lemma 4: *If p, q, and r are three points in Q, the $|\tilde{x}_\mu(p, q) - \tilde{x}_\mu(q, r)| \leqq \leqq \tilde{x}_\mu(p, r)$. If $\tilde{x}_l(s, t)$ is uniformly bounded for all $s, t \in Q$, then $|\tilde{x}_l(p, q) - \tilde{x}_l(q, r)| \leqq \tilde{x}_l(p, r)$.*

Proof: Let f and g be admissible for q and p, and p and r, respectively. Since μ is a ν-length

$$\mu(x(f \oplus g)) = \mu(x(f) \oplus x(g)) \leqq \mu(x(f)) + \mu(x(g)),$$

and so

$$\tilde{x}_\mu(q, r) \leqq \tilde{x}_\mu(p, q) + \tilde{x}_\mu(p, r).$$

Similarly,

$$\tilde{x}_\mu(p, q) \leqq \tilde{x}_\mu(p, r) + \tilde{x}_\mu(q, r).$$

Finally,

$$|\tilde{x}_l(p, q) - \tilde{x}_l(q, r)| = |\lim_{n\to\infty} \tilde{x}_{\mu_n}(p, q) - \lim_{n\to\infty} \tilde{x}_{\mu_n}(q, r)| =$$

$$= \lim_{n\to\infty} |\tilde{x}_{\mu_n}(p, q) - \tilde{x}_{\mu_n}(q, r)| \leqq \liminf_{n\to\infty} \tilde{x}_{\mu_n}(p, r) = \tilde{x}_l(p, r).$$

If $\tilde{x}$ satisfies conditions S, then we can always define a function $x^* \in C$ such that $\tilde{x}^* = \tilde{x}$ exactly as we did in Lemma 1.

Lemma 5: $x^*_\mu \in C$.

Proof: It is merely necessary to verify that $\tilde{x}_\mu$ satisfies conditions S.

Lemma 6: *If $x, y \in C$, then $D(x^*_\mu, y^*_\mu) \leqq 2KD(x, y)$.*

Proof: Let f be admissible for $r, s \in Q$. If h is a homeomorphism of Q onto Q, then

$$|\mu(x(f)) - \mu(y(h(f)))| \leqq 2KD(x(f), y(h(f))) \leqq$$

$$\leqq 2K \max_{t,u \in \text{domain } f} |\,\|x(f(t)) - x(f(u))\| - \|y(h(f(t))) - y(h(f(u)))\|\,| \leqq$$

$$\leqq 2K \max_{p,q \in Q} |\,\|x(p) - x(q)\| - \|y(h(p)) - y(h(q))\|\,| = 2KA(h)$$

where the equality defines $A(h)$.

We then have, as in Lemma 4, $|\tilde{x}(r, s) - \tilde{y}(h(r), h(s))| \leqq 2KA(h)$ and so $D(x^*_\mu, y^*_\mu) \leqq \inf_h 2K\,A(h) = 2K \inf_h A(h) = 2KD(x, y)$.

Lemma 7: *If $x \in C$, then $\tilde{x}_\mu \geqq \tilde{x}$.*

Proof: That $\tilde{x}_\mu(p, q) \geqq \tilde{x}(p, q)$ is an immediate consequence of Definition 1(a).

Theorem 5: *x^*_μ is the monotone factor in a monotone-light factorization of x.*

Proof: If $y \in C$, let $F\{y\}$ be the components of $y^{-1}(t)$ for $t \in$ range y. We must show that $F\{x\} = F\{x^*_\mu\}$ and that x^*_μ is monotone. Suppose $g \in F\{x\}$ and $\varepsilon > 0$. Take $\delta > 0$ so that $\tilde{x}(r, s) < \varepsilon$ if $\|r - s\| < \delta$ and put $h = \{p \mid \inf_{q \in g} \|p - q\| < \delta\}$. It is not hard to see that h is connected, open (relative to Q), and that diam $x(h) < 2\varepsilon$. Consequently h is arcwise connected and $\tilde{x}_\mu(p, q) < 2K\varepsilon$ for $p, q \in g \subset h$. Thus x^*_μ is constant on g. If $k \in F\{x^*_\mu\}$ then x is constant on k since $\tilde{x} \leqq \tilde{x}_\mu$. Therefore $F\{x\} = F\{\tilde{x}_\mu\}$.

If f is admissible for p and q, then diam range $x^*_\mu(f) \leqq K$ diam range $x(f)$ and so $\mu(x^*_\mu(f)) \leqq K$ diam range $x^*_\mu(f) \leqq K^2$ diam range $x(f)$ or $\|(x^*_\mu)^*_\mu(p) - (x^*_\mu)^*_\mu(q)\| \leqq K^2 \|x^*_\mu(p) - x^*_\mu(q)\|$. Thus if $t \in$ range x^*_μ, $(x^*_\mu)^*_\mu$ is constant on $g = x^{*-1}_\mu(t)$. Suppose there are two components, h and k, of g. Let $G_n = \{p \mid \|x^*_\mu(p) - t\| < 1/n\}$ and $F_n = \{p \mid \|x^*_\mu(p) - t\| \leqq 1/n\}$ for each positive integer n. Since $g = \bigcap_n G_n$ we may suppose that for some n the open set G_n has distinct components A and B such that $h \subset A$ and $k \subset B$. Let $H_{n+1} = G_{n+1} \cap A$ and $K_{n+1} = G_{n+1} \cap B$. Then $\overline{H}_{n+1} \cap \overline{K}_{n+1} \subset \overline{G}_{n+1} \cap \overline{A} \cap \overline{B} \subset F_{n+1} \cap \overline{A} \cap \overline{B} \subset G_n \cap \overline{A} \cap \overline{B} = A \cap B = 0$ since $G_n \cap \overline{A}$ and $G_n \cap \overline{B}$ are components of G_n containing the components A and B. If $p \in h \cap H_{n+1} = h$ and $q \in k \cap K_{n+1} = k$, and if f is admissible for p and q, then range f intersects the complement of G_{n+1} since otherwise range f would be contained in G_n and would connect the distinct components A and B. Therefore diam range $x^*_\mu(f)$ cannot be less than $1/(n+1)$. This contradicts the fact that $(x^*_\mu)^*_\mu$ is constant on g. Therefore g has but a single component and x^*_μ is monotone.

4. - Lebesgue area intrinsic with respect to ν-length.

We are now able to show that LEBESGUE area in intrinsic in the sense that the ν-length of a curve is intrinsic.

Lemma 8: *Let $x \in C$ be quasilinear. Then $L(x) = L(x^*_\mu)$.*

Proof: Because of Theorems 1 and 2 and Lemma 7, it is sufficient to show that $\tilde{x}_\mu(p, q) \leqq \tilde{x}(p, q)$ for all p and q in a triangle of linearity of x, but this is evident since there exists a (linear) admissible function f for p and q for which $\mu(x(f)) = \tilde{x}(p, q)$.

Theorem 6: *If $x \in C$, then $L(x^*_\mu) = L(x)$.*

Proof: By Theorem 1 and Lemma 7, it is sufficient to show that $L(x^*_\mu) \leqq L(x)$. By Theorem 4, there exists a sequence $\{x_n\}$ of quasilinear functions in C such that $D(x_n, x) \to 0$ and $L(x_n) \to L(x)$. Then $D(x^*_{n\mu}, x^*_\mu) \to 0$ and so $L(x^*_\mu) \leqq \liminf_{n \to \infty} L(x^*_{n\mu}) = \liminf_{n \to \infty} L(x_n) = L(x)$.

Theorem 7: *If $x_\iota^* \in C$, then $L(x_\iota^*) = L(x)$.*

Proof: As in the preceding theorem, it is sufficient to show that $L(x_\iota^*) \leqq L(x)$. It is easy to verify that $\{ x_{\mu n}^{*\,i} \}$ is a monotonically non-decreasing sequence of continuous functions converging pointwise to the continuous function x_ι^{*i}. Hence the convergence is uniform and the theorem results by the application of Theorems 3 and 6.

5. - A geodesic property of Lebesgue area.

In this section we state and prove the theorem indicated in the introduction.

According to FRÉCHET, the curves of a set E are *uniformly divisible* if for each $\varepsilon > 0$ there exists an integer n such that each curve of E may be decomposed into n consecutive arcs in each of which the oscillation is less than ε.

For brevity, let us say that a curve g is *in* a point set H if the range of a representation of g is contained in H.

Theorem 8: *Let E be a set of curves each in a point set H. Then a necessary and sufficient condition that the curves of E be compact is that they be uniformly divisible and that H be compact.*

Now define ${}_1\tilde{x}(p, q) = \tilde{x}_\iota(p, q)$ and ${}_{n+1}\tilde{x}(p, q) = ({}_n\tilde{x}^*)_\iota(p, q)$ for each positive integer n and $(p, q) \in Q \times Q$. Since ${}_n\tilde{x}(p, q) \leqq {}_{n+1}\tilde{x}(p, q)$, we can define $\tilde{X}$ on $Q \times Q$ by $\tilde{X}(p, q) = \lim_{n\to\infty} {}_n\tilde{x}(p, q)$. Observe that $\tilde{X}(p, q) \leqq (\tilde{X}^*)_{\mu_n}(p, q) \leqq \leqq \tilde{X}_\iota(p, q) = \tilde{X}(p, q)$ and so the equality holds throughout.

If we put $G(x(f))$ equal to the ordinary length of the curve (represented by) $x(f)$ and $\tilde{x}_G(p, q) = \inf G(x(f))$ for all admissible f, then $\tilde{x}_G(p, q)$ is the geodesic distance between $x(p)$ and $x(q)$.

Theorem 9: $\tilde{X} = \tilde{x}_G$.

Proof: Fix $(p, q) \in Q \times Q$. We observe that $\tilde{X}(p, q) \leqq \tilde{x}_G(p, q)$ and hence we assume that $\tilde{X}(p, q) < +\infty$. Let f_n be an admissible curve for p and q such that $\mu_n(X^*(f_n)) < (\tilde{X}^*)_{\mu_n}(p, q) + 1/n = \tilde{X}(p, q) + 1/n$, $n = 1, 2, \ldots$. Now choose $\varepsilon > 0$ and let $N = [(\tilde{X}(p, q) + 1)/\varepsilon]$. Then, for $n > N$, $\mu_N(X^*(f_n)) \leqq \leqq \mu_n(X^*(f_n)) < \tilde{X}(p, q) + 1/n$. Consequently $\{ X^*(f_n) \}$ is a uniformly divisible sequence and there exists a continuous function g defined on an interval $[a, b]$ of the real line such that $D(X^*(f_{n_k}), g) \to 0$ for some subsequence $\{ n_k \}$ of the integers. Thus $\mu_m(g) = \lim_{k\to\infty} \mu_m(X^*(f_{n_k})) \leqq \liminf_{k\to\infty} \mu_{n_k}(X^*(f_{n_k})) \leqq \leqq \liminf_{k\to\infty} (\tilde{X}(p, q) + 1/n_k) = \tilde{X}(p, q)$. But $\operatorname{diam} g \geqq \tilde{X}(p, q)$, and so $\mu_m(g) = = \tilde{X}(p, q)$ for each integer m.

Now suppose that x is light. Then X^* is light as well as monotone and so is topological. Define g^* on $[a, b]$ by $g^*(t) = X^{*-1}(g(t))$. Then g^* is admissible for p and q and we have that $\mu_m(g) = \mu_m(X^*(g^*))$ for all m. Thus $\tilde{X}(p, q) \geqq (\tilde{X}^*)_G(p, q)$. That $\tilde{X}(p, q) = \tilde{x}_G(p, q)$ results from the trivial observation that $\tilde{x}_G(p, q) \leqq (\tilde{X}^*)_G(p, q)$.

The theorem results from well known properties of the middle space of x [5].

Lemma 9: *If $x_G^* \in C$, then ${}_n x^* \in C$ for each positive integer n.*

Proof: It is sufficient to show that $x_\iota^* \in C$. For this purpose we examine $\tilde{x}_\iota$ and observe that it satisfies S_1, S_2, and S_3. Hence it only remains to show that $\tilde{x}_\iota$ is continuous if $\tilde{x}_G$ is. But

$$| \tilde{x}_\iota(p, q) - \tilde{x}_\iota(p', q') | \leqq | \tilde{x}_\iota(p, q) - \tilde{x}_\iota(p', q) | + | \tilde{x}_\iota(p', q) - \tilde{x}_\iota(p', q') | \leqq$$
$$\leqq \tilde{x}_\iota(p, p') + \tilde{x}_\iota(q, q') \leqq \tilde{x}_G(p, p') + \tilde{x}_G(q, q') .$$

The lemma follows.

Theorem 10: *If $x_G^* \in C$, then $L(x) = L(x_G^*)$.*

Proof: Since ${}_n x^*$ is continuous for each n, we have $L({}_n x^*) = L(x)$; since $\{ {}_n\tilde{x} \}$ is a monotically non-decreasing sequence converging pointwise to a continuous function, $\tilde{x}_G$, we know that the convergence is uniform. The theorem follows by the application of Theorems 3 and 1.

For simplicity the set C was restricted to functions defined on Q with range in m. Actually, we could permit the continuous functions to be defined on JORDAN regions with ranges in metric spaces.

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