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On regular representations of surfaces. (**)

1. – Let Q be the square $[0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1]$, let S by any FRÉCHET surface in the (x, y, z)-space and $\{ T \}$ the family of all FRÉCHET equivalent single-valued continuous mappings T: $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $(u, v) \in Q$, which represent the surface S on Q. The three projections of S on the coordinate planes are flat surfaces S_1, S_2, S_3 which are represented by the mappings T_1: $y = y(u, v)$, $z = z(u, v)$; T_2: $z = z(u, v)$, $x = x(u, v)$; T_3: $x = x(u, v)$, $y = y(u, v)$; $(u, v) \in Q$.

S is said to be an *open non degenerate* surface if, for any $T \in \{ T \}$, there is in Q no maximal continuum of constancy for T separating Q, or the (u, v) plane. S is said to be *closed non degenerate* if, for any $T \in \{ T \}$, the previous condition is satisfied with exception of one maximal continuum of constancy for T which contains the boundary Q^* of Q, separates the (u, v) plane but does not separate Q [5]. (These conditions are invariant for FRÉCHET equivalence). Any open, or closed non degenerate surface is called here simply *non degenerate.*

A representation $T \in \{ T \}$ of a surface S is called *light* if each maximal continuum of constancy for T is a single point. It is well known (see e.g. [9]) that any open non degenerate surface possesses a light representation; any closed non degenerate surface possesses a representation which is light in the interior of Q and has the boundary Q^* of Q as a maximal continuum of constancy. The proofs which follow will hold in both the open and closed cases since identifications of points on the boundary will make no difference.

In questions concerning surfaces it is frequently convenient to consider surfaces which have regular representations. A representation T is said to be *regular* [2] in case there exists in Q a countable dense set $\{ \eta_\mu \}$ of orizontal

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line segments $[v = v_\mu,\ 0 \leqslant u \leqslant 1]$, and a countable dense set $\{\xi_\nu\}$ of vertical line segments $[u = u_\nu,\ 0 \leqslant v \leqslant 1]$, such that the images of each segment under the mappings T_1, T_2, T_3 are of plane measure zero. The purpose of this paper is to prove that *every non degenerate surface of finite* LEBESGUE *area has a regular representation.*

This statement has been already proved by using methods involving the DIRICHLET integral: namely it has been proved [3, 5, 8] that each open non degenerate surface of finite LEBESGUE area possesses certain generalized conformal representations and the previous statement can be deduced from this result. In the present paper we shall give a proof which depends solely upon geometrical considerations and therefore is closer to the nature of the above statement. Moreover it may be more valuable in generalizations to higher dimensions.

2. – If I, J are sets contained in an euclidean space E_n, we denote by I^* the boundary of I, by $\bar{I} = I + I^*$ the closure of I, by I^0 the set of all interior points of I, by $|I|$ the LEBESGUE measure of I, by $\{I, J\}$ the distance between I and J. We denote by θ the nul set and, if p is a point of E_n we denote by p also the set whose unique element is p.

If T is a mapping, we denote by $T(p)$ the image (x, y, z) of a point $p = (u, v) \in Q$, by $T(I)$ the set of all points (x, y, z) which are images of at least one point p of the set $I \subset Q$, by $T^{-1}(q)$, $T^{-1}(J)$ the sets of all points $(u, v) \in Q$ whose images are the point $q = (x, y, z)$, or any point $q = (x, y, z) \in J$.

Let $O \subset Q$ be any connected open set. For any two points, $p, q \in O$ there exists a polygonal line $t \subset O$ joining p, q. A point $p \in O^*$ is said to be *accessible* from O if there exists a simple arc $t \subset O \cup p$, $t \cap O^* = p$, and it can be supposed that each subarc of t not containing p is a polygonal line. If O is simply connected, O^* is a continuum and points of O^* are accessible from O [5, 7].

Any two mappings, T_1, T_2 are said to be FRÉCHET equivalent if for any $\varepsilon > 0$ there is a homeomorphism h of Q onto itself such that $\{T_1(p), T_2[h(p)]\} \leqslant \varepsilon$ for any $p \in Q$. T_1, T_2 are said to be LEBESGUE equivalent if there exists a homeomorphism h_0 of Q onto itself such that $T_2[h_0(p)] = T_1(p)$ for any $p \in Q$. The LEBESGUE equivalence is a particular case of the FRÉCHET equivalence. If S is a FRÉCHET surface, $[S]$ the set of the points occupied by S in E_3, T_1, T_2 any two FRÉCHET equivalent representation of S, then $[S] = T_1(Q) = T_2(Q)$.

3. – Let S be any FRÉCHET surface and T any representation of S. We denote by $L(S)$ the LEBESGUE area of S. If r_i is a positively oriented closed JORDAN curve in Q and C its image under T_3, we let $O(x, y; C)$ denote the topological index of (x, y) with respect to C $[O(x, y; C) = 0$ for any $(x, y) \in C]$. If $\{r_i\}$ is any finite collection of closed disjoint JORDAN regions in Q, each with boundary r_i^* positively oriented, and if C_i is the image of r_i^* under T_3, let $g(r_i) = \iint_K |O(x, y; C_i)| \, dx \, dy$, where K is a square of the (x, y)-plane containing $T_3(Q)$. Let $G(T_3) = \text{l.u.b.} \sum_i g(r_i)$, were the upper bound is taken with respect to all finite collections $\{r_i\}$ as above. Similar definitions hold for $G(T_1)$, $G(T_2)$. If r_i is as above, let $g(r_i) = [g_1^2(r_i) + g_2^2(r_i) + g_3^2(r_i)]^{1/2}$ and $G(T) = \text{l.u.b.} \sum_i g(r_i)$. It is well known that $G(T)$, as well as $G(T_r)$, $(r = 1, 2, 3)$, are invariant for FRÉCHET equivalence, hence they depend only on the FRÉCHET surface S, or the FRÉCHET flat surfaces S_r, represented by T_r, $(r = 1, 2, 3)$. Therefore, by definition, we have $G(S) = G(T)$; $G(S_r) = G(T_r)$, $(r = 1, 2, 3)$.

To define the total variation of the mappings T_r, $(r = 1, 2, 3)$, we put $\Psi(x, y; T_3) = \text{l.u.b.} \sum_i |O(x, y; C_i)|$, where the l.u.b. is taken over all finite collections $\{r_i\}$ as above. The function $\Psi(x, y; T_3)$ is called the characteristic function of T_3, is not negative, measurable, zero outside $T_3(Q)$. The total variation of T_3 is the LEBESGUE integral $W(T_3) = \iint_K \Psi(x, y; T_3) \, dx \, dy$. Analogously for T_1, T_2.

Through only geometrical considerations the following statements have been proved [1, 2] for all FRÉCHET surfaces S: (a) $L(S) < +\infty$ if and only if $W(T_r) < +\infty$, $(r = 1, 2, 3)$; (b) $W(T_r) \leqslant L(S) \leqslant W(T_1) + W(T_2) + W(T_3)$; (c) $G(S) \leqslant L(S)$; (d) $G(T_r) \leqslant G(S)$; (e) $G(T_r) = W(T_r) = L(S_r)$, $(r = 1, 2, 3)$. Through additional considerations involving the DIRICHLET integral it has also been proved [3, 4]: (f) $G(S) = L(S)$.

4. - If T is any mapping defined on Q, let us call $\Gamma(T)$ the collection of all maximal continua of constancy γ for the mapping T in Q. Analogous definitions hold for the collections $\Gamma(T_r)$, $(r = 1, 2, 3)$. For each point $q = (x, y, z) \in T(Q)$ all components of the closed set $T^{-1}(q) \subset Q$ are continua $\gamma \in \Gamma(T)$; for each point $q = (x, y) \in T_3(Q)$ all components of the closed set $T_3^{-1}(q) \subset Q$ are continua $\gamma \in \Gamma(T_3)$; analogously for T_1, T_2. In addition for each $\gamma \in \Gamma(T)$ and r, $(r = 1, 2, 3)$, there is a $\gamma_r \in \Gamma(T_r)$ such that $\gamma \subset \gamma_r$.

Through only geometrical considerations the following statement has been proved [1, pg. 287]: (A) If $W(T_r) < +\infty$, $(r = 1, 2, 3)$, then for almost all points $q = (x, y) \in T_3(Q)$ all components of the closed set $T_3^{-1}(q)$ are continua $\gamma \in \Gamma(T)$ as well as continua $\gamma \in \Gamma(T_3)$; analogously for T_1, T_2.

5. - Theorem I. *Let* T: $x = x(u, v)$, $y = (u, v)$, $(u, v) \in Q$, *be a continuous light mapping; let* O *be a connected open subset of* Q, *and* p_1, p_2 *be two points of* O^* *accessible from* O. *Then there is a simple arc* s *joining* p_1 *and* p_2, $s \subset O \cup p_1 \cup p_2$, *such that* $|T(s)| = 0$.

Proof. Let K be the unit square in the (x, y)-plane. It can be assumed without loss of generality, that $B = T(Q) \subset K$. By the use of a suitably constructed CANTOR sets it is possible to construct a closed, totally disconnected subset of K with plane measure arbitrarily near unity. Let H_1 be such a set $H_1 \subset K$, $|H_1| > 3/4$, $T(p_i)$ not in H_1, $(i = 1, 2)$. Since T is light, $\mathcal{H}_1 = T^{-1}(H_1 \cap B)$ is also closed and totally disconnected. Let $G_1 = O - (\mathcal{H}_1 \cap O)$. G_1 is open and connected and, because p_1, p_2 are not in $\mathcal{H}_1$, there are two circles $C_{1i} = C(p_i, \delta_1)$ of center p_i, radius $0 < \delta_1 \leqslant 1/4$ containing no points of $\mathcal{H}_1$, $(i=1, 2)$. Because the points $p_i \in O^*$ are accessible from O, there exist simple arcs b_{1i} contained in $G_1 \cup p_1 \cup p_2$ which join p_i to points of $G_1 \cap C_{1i}$ such that each subarc of b_{1i} not containing p_i is a finite polygonal line and there exists a polygonal line $s'_1 \subset G_1$, such that $b_{11} \cup b_{12} \cup s'_1$, is a connected set joining p_1 and p_2. $b_{11} \cup b_{12} \cup s'_1$ is not necessarily a simple arc but, if p'_{1i} is the intersection point of b_{1i} and s'_1, which is first encountered on leaving p_i, $(i=1, 2)$ and b'_{1i} is the subarc of b_{1i} joining p_i to p_{1i}, if we perform the same operation on s'_1, then $s_1 = b'_{11} \cup b'_{12} \cup s'_1$ is a simple arc joining p_1 and p_2, $s_1 \subset G_1 \cup p_1 \cup p_2$, $T(s_1) \subset K - H_1$, $|T(s_1)| < 1/4$, and each subarc of s_1, not containing the end-points p_1, p_2 is a simple polygonal line. If $w_1 = T(s_1)$, we have $w_1 \cap H_1 = \theta$, hence $\{w_1, H_1\} = 2\eta_1 > 0$. Let $W_1 = (w_1)_{\eta_1}$ be the set of all points $(x, y) \in K$ whose distance from w_1 is $\leqslant \eta_1$. Then W_1 is a closed set, $W_1 \supset w_1$, $W_1 \cap H_1 = \theta$, hence $|W_1| < 1/4$. Divide the polygonal line $s'_1 = \widehat{p'_{11}p'_{12}}$ (subarc of s_1) in $k_1 - 2$ segments, say $t_{12}, t_{13}, \ldots, t_{1,k_1-1}$, each of length $\leqslant 1/4$, and call $q_{11} = p'_{11}$, $q_{12}, \ldots, q_{1,k_1-1} = p'_{12}$ the points of subdivision. Let $t_{11} = b'_{11}$, $t_{1,k_1} = b'_{12}$, $q_{10} = p_1$, $q_{1k_1} = p_2$ In such a way $s_1 = \bigcup_i t_{1i}$, $(i = 1, 2, \ldots, k_1)$, is divided in k_1 arcs t_{1i}, of which t_{11}, t_{1k_1}, have diameter $\leqslant 1/4$ and all others are segments of length $\leqslant 1/4$. Let each segment t_{1i}, $(i = 2, 3, \ldots, k_1 - 1)$, be now included in a rhombus R_{1i} with t_{1i} as one of the diagonals and the other diagonal chosen sufficiently small to insure that no two of the R_{1i} intersect except at the end points of the t_{1i} and no R_{1i} contains p_1 or p_2. Let us observe that $T^{-1}(W_1)$ is a closed set which contains all points of the segments t_{1i} as interior points, hence we can suppose that the second diagonal of R_{1i} is chosen so small that the rhombus R_{1i} is completely interior to $T^{-1}(W_1)$, $(i=2, 3, \ldots, k_1 - 1)$. This implies $t_{1i} \subset R_{1i} \subset T^{-1}(W_1)$, $t_{11} \subset C_{11}$, $t_{1k_1} \subset C_{12}$, hence, if $M_1 = W_1 \cup T[(C_{11} \cup C_{12}) \cup Q]$, $L_1 = (\bigcup_i R_{1i}) \cup C_{11} \cup C_{12}$, we have $T(L_1) \subset M_1$.

Let H_2 be a closed totally disconnected set $H_2 \subset K$, $|H_2| > 7/8$, $T(q_{1i})$ not in H_2, $(i = 0, 1, \ldots, k_1)$. If $\mathcal{H}_2 = T^{-1}(H_2 \cap B)$ then q_{1i} is not in $\mathcal{H}_2$, and $G_2 = G_1 - (\mathcal{H}_2 \cap G_1)$ is an open connected set. Because $p_1 = q_{10}$, $p_2 = q_{1,k_1}$ are not in $\mathcal{H}_2$, there are two circles $C_{2i} = C(p_i, \delta_2)$, of centers p_1, p_2, radius $\delta_2 \leqslant \delta_1$, $\delta_2 \leqslant 1/8$, free of points of $\mathcal{H}_2$. Let $b_{2i} = \widehat{p_i p_{2i}}$, $(i = 1, 2)$, be a subarc of b'_{1i} $[b'_{11} = t_{11}, b'_{12} = t_{1k_1}]$, completely contained in C_{2i}. Then b_{2i},

$(i=1,2)$, are both free of points of $\mathcal{H}_2$. On the other hand the points p_{2i}, p'_{1i} $[p'_{11}=q_{11},\ p'_{12}=q_{1,k_1-1}]$ are in G_1 and also in a connected component F_{1i} of $G_1 \cap C_{1i} - \cup R_{1i}$ because the arc $\widehat{p_{2i}p'_{1i}}$ joins them in $G_1 \cap C_{1i}$. Because $\mathcal{H}_2$ is totally disconnected, $F_{1i}-(F_{1i}\cap \mathcal{H}_2)$ is connected and hence ther e is a polygonal line t'_{11}, or t'_{1k_1} joining p_{2i}, p'_{1i} in $F_{1i}-(F_{1i}\cap\mathcal{H}_2)$, hence in $G_2 \cap C_{1i}$. In each rhombus R_{1i} $(i=2,3,\ldots,k_1-1)$, the vertices $q_{1,i-1}$, q_{1i} do not belong to $\mathcal{H}_2$, hence there is a polygonal line $t'_{1i}\subset R_{1i}\cap G_2$, joining $q_{1,i-1}$, q_{1i}. Now $s'_2=\bigcup_i t'_{1i}$ is a simple polygonal line, $s'_2\subset G_2$. The curve $b_{21}\cup b_{22}\cup s'_2$ is not necessarily simple. If we denote with $b'_{2i}=\widehat{p_i p'_{2i}}$ the subarc of b_{2i}, where p'_{2i} is the intersection point of b_{2i} with s'_2 first encountered on b_{2i} leaving p_i, $(i=1,2)$, then $s_2=b'_{21}\cup b'_{22}\cup s'_2$ is a simple arc joining p_1 and p_2, $s_2\subset G_2\cup p_1\cup p_2$, $T(s_2)\subset K-H_2$, $T(s_2)\subset W_1$, $|T(s_2)|<1/8$, and any subarc of s_2 not containing the end points p_1, p_2 is a simple, polygonal line. If $w_2=T(s_2)$ we have $w_2\cap H_2=\theta$, $w_2\subset (W_1)^0$, $\{w_2,\ H_2\cup W_1^*\}=2\eta_2>0$. Let $W_2=(w_2)_{\eta_2}$ be the set of all points $(x,y)\in K$ whose distance from w_2 is $\leqslant\eta_2$. Then W_2 is a closed set, $w_2\subset(W_2)^0$, $W_2\subset(W_1)^0$, $W_2\cap H_2=\theta$, hence $|W_2|<1/8$. Divide the polygonal line $s'_2=\widehat{p'_{21}p'_{22}}$ (subarc of s_2) in h_2-2 segments, say $t_{22}, t_{23},\ldots, t_{2,h_2-1}$, each of length $\leqslant 1/8$, where $h_1=k_1$, $h_2=k_1k_2$, k_1, k_2 integers $\geqslant 1$, and call $q_{21}=p'_{21}$, $q_{22},\ldots, q_{2,h_1-2}=p'_{22}$, the points of subdivision, in such a way that each of the arcs t'_{11}, t'_{1,k_1} is divided into k_2-1 parts, and each of the other arcs $t'_{12},\ldots, t'_{1,k_1-1}$ is divided in k_2 parts. Let $t_{21}=b'_{21}$, $t_{2,h_2}=b'_{22}$, $q_{20}=p_1$, $q_{2,h_2}=p_2$. In such a way $s_2=\bigcup_i t_{2i}$, $(i=1,2,\ldots,h_2)$ is divided in $h_2=k_1k_2$ arcs t_{2i}, of which t_{21}, t_{2,h_2} have diameter $\leqslant 1/8$ and all others are segments of length $\leqslant 1/8$. Let each segment t_{2i}, $(i=2,3,\ldots,h_2-1)$, be now included in a rhombus R_{2i}, as above, and we can also suppose each rhombus R_{2i} completely contained in one rhombus R_{1j}, or in C_{11}, or C_{12}. We have $t_{2i}\subset R_{2i}\subset T^{-1}(W_2)$, $t_{21}\subset C_{21}$, $t_{2,h_2}\subset C_{22}$, hence if $M_2=W_2\cup$ $\cup\, T[(C_{21}\cup C_{22})\cap Q]$, $L_2=(\bigcup_i R_{2i})\cup C_{21}\cup C_{22}$, also $T(L_2)\subset M_2$, $L_2\subset L_1$, $M_2\subset M_1$.

Let successive steps be carried out in the same manner, each step obtaining a simple arc $s_n=t_{n1}\cup t_{n2}\cup\ldots\cup t_{n,h_n}$, $h_n=k_1k_2\ldots k_n$, $s_n\subset O\cup p_1\cup p_2$, joining p_1 and p_2, each subarc of s_n not containing p_1 and p_2 being a polygonal line. Here t_{n1}, t_{n,h_n} are simple arcs contained in the circles $C_{ni}=C(p_i,\delta_n)$, $(i=1,2)$, $\delta_n\leqslant\delta_{n-1}$, $\delta_n\leqslant 1/2^{n+1}$; each t_{ni}, $(i=2,3,\ldots,h_n-1)$, is a segment whose length is $\leqslant 1/2^{n+1}$, and one of the diagonals of a rhombus R_{ni}. If $w_n=T(s_n)$, there is an $\eta_n>0$ such that $W_n=(w_n)_{\eta_n}$ and $T[\bigcup_i R_{ni}]\subset(W_{n-1})_0$, $W_n\subset$ $\subset(W_{n-1})_0$. If $M_n=W_n\cup T[(C_{n1}\cup C_{n2})\cap Q]$, $L_n=(\bigcup_i R_{ni})\cup C_{n1}\cup C_{n2}$, then $T(L_n)\subset M_n$, $L_n\subset L_{n-1}$, $M_n\subset M_{n-1}$; $|W_n|<1/2^{n+1}$. The sets M_n are closed, $M_n\subset M_{n-1}$, hence $M=\bigcap_n M_n$, is a closed non empty set. On the other hand $T[(C_{n1}\cup C_{n2})\cap Q]$ is the union of two closed sets contained in two circles of center $T(p_1)$, $T(p_2)$ and radius ϱ_n approaching zero as $n\to\infty$. Therefore $|M_n|<1/2^{n+1}+2\pi\varrho_n^2$, $|M|=0$. Let us observe that the sets L_n

are continua joining p_1 and p_2 in $O \cup p_1 \cup p_2$, and $L_n \subset L_{n-1}$, hence $L = \bigcap_n L_n$ is also a continuum joining p_1 and p_2 in O. We have only to prove that L is a simple arc. Let $I \equiv (0 \leqslant \alpha \leqslant 1)$ and define a mapping τ of I into L as follows. Divide I into $h_n = k_1 k_2 \dots k_n$ equal parts $\delta_{ni} = (\alpha_{n,i-1}, \ \alpha_{ni})$, $(i = 1, 2, \dots, h_n)$, where α_{ni}, $(i = 0, 1, \dots, h_n)$, are the points of subdivision. Then by putting $\tau(\alpha_{ni}) = q_{ni}$, $(i = 0, 1, \dots, h_n)$, we have a consistent definition of τ at the points $\alpha_{ni} \in I$. Let us observe that $\tau(0) = q_{n0} = p_1$, $\tau(1) = q_{n,h_n} = p_2$, that the points $\alpha_{n,i-1}$, α_{ni}, end-points of δ_{ni} are mapped into the points $q_{n,i-1}$, q_{ni}, vertices of the rhombus R_{ni}, and that the points α_{mj}, $\alpha_{n,i-1} \leqslant \alpha_{mj} \leqslant \alpha_{ni}$, $m > n$, are mapped in points $q_{mj} \in R_{ni}$. For any $0 < \alpha < 1$ there is a sequence of nested intervals δ_{ni} whose lengths approach zero as $n \to \infty$ and the points q_{ni} images of the end-points of δ_{ni} are the vertices of a sequence of nested rhombuses R_{ni} whose diameters do not exceed $1/2^n$, hence approach zero as $n \to \infty$. Therefore a unique point $q = \tau(\alpha)$ is defined and $q \in L$. In such a way the mapping $q = \tau(\alpha)$ is defined in I and $\tau(I) \subset L$. On the other hand, if q is any point of L, then q is determined by a sequence of nested rhombuses R_{ni} and, if δ_{ni} are the corresponding intervals, a point $\alpha \in L$ is determined such that $\tau(\alpha) = q$. Hence $\tau(I) = L$. Any point $0 < \alpha < 1$ belongs to at most two intervals δ_{ni}, $(i = 1, 2, \dots, h_n)$, and for each α' the points $\tau(\alpha)$, $\tau(\alpha')$ belong to the same rhombus R_{ni}, or to two adjacent rhombuses, R_{ni} hence $\{\tau(\alpha), \ \tau(\alpha')\} < 1/2^n$. This proves that $\tau(\alpha)$ is a continuous mapping. (Analogous reasoning holds for $\alpha = 0$ and $\alpha = 1$.) Finally let us observe that, if $\alpha < \alpha'$, are any two points in I, there exists an n such that $\alpha_{n,i-1} \leqslant \alpha \leqslant \alpha_{ni} < \alpha_{n,j-1} \leqslant \alpha' \leqslant \alpha_{nj}$, $i < j - 1$, hence $\tau(\alpha)$ and $\tau(\alpha')$ belong to two rhombuses R_{ni}, R_{nj} completely disjoint, hence $\tau(\alpha) \neq \tau(\alpha')$. This proves that $q = \tau(\alpha)$, $\alpha \in I$, is a homeomorphism between I and L, i.e., L is a simple arc.

6. – Theorem II. *If T: $x = x(u, v)$, $y = y(u, v)$, $(u, v) \in Q$ is any light mapping, there is a light regular mapping T': $x = x'(u, v)$, $y = y'(u, v)$, $(u, v) \in Q'$, LEBESGUE-equivalent to T, coinciding with T on Q^*.*

Proof. Let $k_1 = 4$ and $\{Q_1\}$ the subdivision of Q into $k_1^2 = 16$ equal squares Q_1 (each of diameter $< 1/2$) by equally spaced horizontal and vertical line segments. For each Q_1 let four arcs s be constructed from the mid point of each side to the center, such that $|T(s)| = 0$, such that no two of the four arcs have points in common except the center and such that each arcs s has no point in common with Q_1^* except the mid point of the corresponding side of Q_1. These arcs s can be constructed as follows. By application of Theorem I to the two rectangles in which Q_1 is divided by a parallel to one of the sides we construct two arcs s joining the center of Q_1 with the mid point of two opposite sides of Q_1. Then these two arcs s give a subdivision of Q_1 into two JORDAN regions and by application of the same Theorem I we obtain two new arcs s joining the center of Q_1 with the midpoints of the two remaining sides of Q_1. The four arcs s divide each square Q_1 into four JORDAN regions. Hence we have a final subdivision $\{U\}$ of Q into $4k_1^2 = 64$ JORDAN regions U_k each of diameter $< 1/2$. The arcs s form a collection $\{\sigma\}_1$ of $2k_1$ arcs of which k_1 join two opposite sides of Q, another k_1 the two remaining sides of Q, and $|T(\sigma)| = 0$ for any $\sigma \in \{\sigma\}_1$. Let us consider all segments s' joining the center of the squares Q_1 with the mid points of the sides of Q_1. Let φ_1^{-1} be a homeomorphism of Q onto itself, identical on Q^* as well as on each Q_1^*, mapping each arc s onto the segment s' having the same end points. Hence each of the four JORDAN regions U, contained in each Q_1 is mapped into one of the four quadrants V_1 in which the segments s' divide Q_1. Let $\{V_1\}$ be the collection of all squares V_1 so obtained. The homeomorphism φ_1 lets correspond a segment σ' to each $\sigma \in \{\sigma\}_1$ and σ' is parallel to u, or v-axis and joins two opposite sides of Q. Let $\{\sigma'\}_1$ be the collection of the $2k_1$ segments so obtained. If T_1 is the new mapping $T_1 = T\varphi_1$, we have $|T_1(\sigma')| = 0$ for any $\sigma' \in \{\sigma'\}_1$.

Let $k_2 \geqslant 4$ be an integer and divide each square V_1 into k_2^2 equal squares $\{Q_2\}$ by equally spaced horizontal and vertical line segments. The mapping T_1 maps the squares Q_2 into JORDAN regions contained in the U_1 and we can suppose that k_2 is the smallest integer, $k_2 \geqslant 4$, such that all these JORDAN regions have diameter $< 1/2^2$: We can also construct in each square Q_2 four arcs s_0 joining the center with the mid points of the sides and such that $|T_1(s_0)| = 0$. These arcs s_0 divide each square Q_2 into four JORDAN regions V_{20} and each square V_1 into $4k_2^2$ regions V_{20}, to which Q_1 lets correspond in U_1 as many JORDAN regions U_2. Let $\{U_2\}$ be the collection of all regions U_2 so obtained in Q, all of diameter $< 1/2^2$. The arcs s_0 form a collection $\{\sigma_0\}_2$ od $2k_1k_2$ arcs σ_0 joining two opposite sides of Q^* and $|T_1(\sigma_0)| = 0$ for any $\sigma_0 \in \{\sigma_0\}$. Let $\{\sigma\}_2$ be the collection of all arcs σ, images of σ_0 under φ_1. We have $|T(\sigma)| = 0$ for any $\sigma \in \{\sigma\}_2$. Let s' be the segments having the same end points as the arcs s_0, let V_2 be each of the four squares in which the squares Q_2 are divided, let φ_2^{-1} be a homeomorphism of Q into itself, identical on Q^* as well as on each Q_2^*, mapping each s_0 into a segment s'. Let $\{V_2\}$ be the collection of all squares so obtained and $\{\sigma'\}_2$ the collection of all segments σ' in which φ_2 maps all $\sigma_0 \in \{\sigma_0\}_2$. If $T_2 = T_1\varphi_2 = T\varphi_1\varphi_2$ we have $|T_2(\sigma')| = |T_1(\sigma_0)| = |T(\sigma)| = 0$ for any $\sigma' \in \{\sigma'\}_2$.

By repeating this procedure n times, we get (a) a subdivision $\{U_n\}$ of Q into $4k_1^2 \cdot 4k_2^2 \ldots 4k_n^2$ JORDAN regions U_n, all of diameter $< 1/2^n$, and also a subdivision $\{V_n\}$ of Q into as many equal squares V_n; (b) a mapping $T_n = T_{n-1}\varphi_n = T_{n-2}\varphi_{n-1}\varphi_n = \ldots = T\varphi_1\varphi_2 \ldots \varphi_n$, where φ_i, $(i = 1, 2, \ldots, n)$, are homeomorphisms of Q onto itself, identical on Q^* as well as on each V_j^*,

$(j=1,2,\ldots,i-1)$, mapping each V_j onto U_j, $(j=1,2,\ldots,i)$; (c) n collections $\{\sigma\}_i$, $(i=1,2,\ldots,n)$, of arcs joining two opposite sides of Q, each σ being a union of arcs of U_i^*, and n collections $\{\sigma'\}_i$ of segments unions of segments of V_n^*, all parallel to the u, or v-axis, and equally spaced, such that $|T(\sigma)|=|T_i(\sigma_i)|=|T_n(\sigma')|=0$.

Let $\psi_n=\varphi_1\varphi_2\ldots\varphi_n$, and let us prove that $\psi=\lim\psi_n$ as $n\to\infty$, exists and is a homeomorphism. First let us observe that $\varphi_{n+1}, \varphi_{n+2}, \ldots$, are identical on V_n^* hence $\psi_{n+r}(p)=\psi_n(p)$, $(r=1,2,\ldots)$, $\psi(p)=\psi_n(p)$, for any $p\in V_n^*$. Let p be any point of Q, and p not in V_n^*, for any n. Then p is contained in a sequence $V_1, V_2, \ldots$, of nested squares V_n, diameter $V_n\to 0$, and, if $U_1, U_2, \ldots, U_n, \ldots$ is the corresponding sequence of $U_n=\psi_n(V_n)$, then diameter $U_n\to 0$ and there is a unique point q contained in all U_n, $q=\lim\varphi_n(p)=\psi(p)$. On the other hand for any point $q\in Q$ there is a point $p\in Q$ such that $q=\psi(p)$, as we prove by the same reasoning. If two points $p, p'\in Q$, $p\neq p'$, then there is an n and two different regions U_n, U_n', $p\in U_n$, $p'\in U_n'$, $U_n\cap U_n'=\theta$; hence $q\in V_n$, $q'\in V_n'$, $V_n\cap V_n'=\theta$, $q\neq q'$. Viceversa, if $q\neq q'$, also $p\neq p'$, by an analogous reasoning. If $p\in Q$, if N_n is the neighborhood of p formed by the JORDAN regions U_n (one, or two, or four of them) which contain p as an interior or a boundary point, if p' is any point $p'\in N_n$, then $q=\psi(p)$, $q'=\psi(p')$, are both contained in the neighborhood N_n' of q constituted by the squares V_n images of the U_n under ψ_n, and diameter $N_n'\to 0$. This assures that ψ is a continuous mapping in Q and also that ψ is a homeomorphism. If $T'=T\psi$, then T' is LEBESGUE equivalent to T and is a light mapping as well as T. All φ_n are identical on Q^* hence ψ is also an identity on Q^*. Finally let us observe that for all segments $\{\sigma'\}_n$, $(n=1,2,\ldots)$, we have $T'(\sigma')=T\psi(\sigma')=T\varphi_1\varphi_2\ldots\varphi_n(\sigma')=T_n(\sigma')$, $|T'(\sigma')|=|T_n(\sigma')|=0$, and the segments $\{\sigma'\}_n$, form two collections of segments $[\xi_r]$, $[\eta_\mu]$, parallel to the axes u and v respectively, everywhere dense in Q, hence T' is regular.

7. – Theorem III. *Every non degenerate* FRÉCHET *surface S of finite* LEBESGUE *area has a light regular representation.*

Proof. Let the surface S be represented by a light mapping T: $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$, $(u,v)\in Q$, and consider the three projections T_1, T_2, T_3 (§ 1). We can suppose without loss of generality that $[S]$ is contained in the cube $K\equiv[0\leqslant x,y,z\leqslant 1]$. Let K_1, K_2, K_3 be the squares $(0,1;\,0,1)$ projections of K into the three coordinate planes (y,z), (z,x), (x,y). The plane mappings T_1, T_2, T_3 are *not* necessarily light. Because $L(S)<\infty$, by § 3, (b) we get $W(T_r)<\infty$, $(r=1,2,3)$, and by § 4, in each coordinate plane the union I_r of all points whose inverse images under T_r are not points is of plane measure zero, $I_r\subset K_r$, $|I_r|=0$, $(r=1,2,3)$.

Let us observe that in each K_r there are closed totally disconnected sets $H_r \subset K_r$, $H_r \cap I_r = \theta$, whose plane measure $|H_r|$ is as close to 1 as we want. Indeed, let $\eta > 0$ be any arbitrary number, let $h \subset [0 \leqslant x \leqslant 1]$, $h' \subset [0 \leqslant y \leqslant 1]$ be two CANTOR linear closed sets whose linear measures $|h|$, $|h'|$ are greater than $1 - \eta$, let H' be the plane closed totally disconnected set, $H' = h \times h'$, $H' \subset K_3$, of all points (x, y) such that $x \in h$, $y \in h'$. Then we have $|H'| = |h| \cdot |h'| > (1-\eta)^2 > 1 - 2\eta$. Let A be any open set covering I_3 and, because $|I_3| = 0$, we can suppose $|A| < \eta$. Then the set $H = H' - A \cap K_3$ is closed, totally disconnected, and $|H| > |H'| - |A| > 1 - 2\eta - \eta = 1 - 3\eta$.

We can now repeat all the reasoning of § 5, where we consider at each step three closed totally disconnected sets H_{rn} with $H_{rn} \subset K_r$, $H_{rn} \cap I_r = \theta$, $|H_{rn}| > 1 - 1/2^{n+1}$, $(r = 1, 2, 3)$. Then we prove that, given any open connected set $O \subset Q$, and two points p_1, $p_2 \in O^*$, accessible from O, there is a simple arc s joining p_1 and p_2, $s \subset O \cup p_1 \cup p_2$, such that $|T_r(s)| = 0$, $(r=1, 2, 3)$. Finally, by repeating the arguments of § 6 with obvious modifications, we obtain a mapping T' which is light, regular and LEBESGUE equivalent to T.

8. – *The previous Theorem III holds under the hypothesis* $G(S) < +\infty$.

This statement is obvious utilizing no. 3, (f), i.e., the equality $G(S) = L(S)$ proved through considerations involving the DIRICHLET integral. The following proof involves only geometrical considerations. By $G(S) < +\infty$ and no. 3, (d) we have $G(T_r) < +\infty$, by § 3, (e) we have $W(T_r) = G(T_r) < +\infty$, $(r = 1, 2, 3)$, and this is all we need for applying § 4, (A), as in § 7.

9. – *Example of non degenerate* FRÉCHET *surface* S *with* $L(S) = +\infty$, *all of whose representations on* Q *are not regular.*

Let S be the surface defined by T: $x = \varphi(u)$, $y = \psi(u)$, $z = v$, $(u, v) \in Q$, where C: $x = \varphi(u)$, $y = \psi(u)$, $0 \leqslant u \leqslant 1$, is a light representation of a simple arc C, each sub arc of which has positive plane measure. S is a cylinder with diretrix C and generatrices parallel to the z-axis. T is light, therefore S is open non degenerate. Let us suppose, if possible, that S has a regular representation T'. Let $[\xi_\nu]$, $[\eta_\mu]$ be the corresponding sets of segments (no. 1) everywhere dense in Q. For any ξ_ν let us consider $T'_3(\xi_\nu)$. If this set contains more than one point, then ξ_ν is mapped by T'_3 into a subarc λ of C joining two different points of C, and therefore $|\lambda| > 0$, which is impossible because $\lambda = T'_3(\xi_\nu)$, $|T'_3(\xi_\nu)| = 0$. From this it follows that T'_3 is constant on each segment ξ_ν, and, because these segments are everywhere dense in Q, it follows that T'_3 is constant with respect to v. For the same reasons T'_3 is constant with respect to u, i.e., $T'_3(Q)$ is a single point, which is impossible because $T'_3(Q)$ is the projection of $[S]$ on the (x, y)-plane and this projection is C. Thus we have proved that S has no regular representation on Q.

10. – *Example of a degenerate* FRÉCHET *surface S with $L(S) = 0$ all of whose representations on Q are not regular.*

With the notations of § 9, let S be defined by T: $x = \varphi(u)$, $y = \psi(u)$, $z = 0$, $(u, v) \in Q$. The surface S is reduced to the single curve C, therefore $L(S) = 0$. If T', as in § 9, were a regular representation of S, then we would have T': $x = x'(u, v)$, $y = y'(u, v)$, $z = z'(u, v)$, $(u, v) \in Q$, and $z'(u, v) = 0$ in Q. By the same reasoning as in § 9, we have T_3 constant in Q, that is $x'(u, v)$, $y'(u, v)$ constant on Q. That is $[S]$ would be a single point, and not C. Thus we have proved that S has no regular representation.

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