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A game over function space. (**)

1. – Introduction.

The present paper is concerned with a specific zero-sum two-person game over function space, which arose in a natural way, and whose investigation appears to possess considerable theoretical interest. We obtain an explicit saddle-point solution to this game; we know of very few other examples (none of which are published) of games over function space for which explicit solutions have been found.

The fundamental facts concerning the theory of games are set forth in [6]. By a *zero-sum two-person-game* is meant a (real-valued) game between two players, A and B, in which, whatever the outcome, the numerical *gain* to A (B) is equal to the numerical *loss* to B (A). Any particular manner of play chosen by A — his total aggregate of « moves » during the game — may be thought of as constituting a « strategy » x selected by A from a given « strategy space » $\mathfrak{X}$. Likewise B selects a strategy y from a space $\mathcal{Y}$. The game is characterized by the so-called « payoff function », a function (or functional) $\pi(x, y)$, which represents the « payment to A by B » (positive, negative, or zero) which results from the adoption of their respective strategies x, y. (It is not necessary to consider separately the payment to B by A, since this is, by assumption, merely the negative of the payment to A.)

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Thus A wishes to maximize $\pi(x, y)$ (but he controls only x), while B wishes to minimize $\pi(x, y)$ (but controls only y). The game is said to have a (saddle-point) *solution* if

$$\max_{x \in \mathscr{X}} \min_{y \in \mathscr{Y}} \pi(x, y) = \min_{y \in \mathscr{Y}} \max_{x \in \mathscr{X}} \pi(x, y) ;$$

strategies $x_0 \in \mathscr{X}$, $y_0 \in \mathscr{Y}$ such that $\pi(x_0, y_0)$ has this saddle-point value are called *optimal* strategies. It should be observed that in general there is no reason to suppose that solutions (or optimal strategies) exist.

The model for the game which concerns us here is a *duel*, between an *attacker*, A, and a *defender*, B: the motive of A is to destroy B; that of B, to survive. Each is equipped with a machine gun, and with given finite amounts of ammunition. With each bullet fired by A from any given distance t to B there is associated a given *probability* that the round will be lethal to B; likewise for the rounds fired by B. It is assumed that the distance between A and B is decreasing during the duel, that is, with increasing time, and that the probabilities in question increase as the distance becomes smaller.

The strategies available to the players are their rate-of-fire functions; it is assumed that these rates cannot exceed certain given bounds.

Various idealizations are assumed in the problem. It is supposed, for example, that destruction, if it occurs, is instantaneous with the release of the lethal round. Etc..

In the mathematical formulation below, x and y represent the strategies, ξ and η the lethality probabilities. The functions Q_x and Q_y which appear represent the (cumulative) survival probabilities; classical probability reasoning will explain their particular exponential form. The parameter t represents range or distance, and thus decreases as the duel proceeds.

$\pi(x, y)$ is then the probability that B is destroyed by A in the course of the duel. A wishes to maximize $\pi(x, y)$ and B wishes to minimize it.

Speaking now purely mathematically, we seek a saddle-point solution to the game determined by the payoff

$$\pi(x, y) = \int_0^\infty Q_y(t)\, \mathrm{d}Q_x(t) ,$$

where we have put

$$Q_x(t) = \exp\left[-\int_t^\infty x(\tau)\xi(\tau)\, \mathrm{d}\tau\right]$$

and

$$Q_y(t) = \exp\left[-\int_t^\infty y(\tau)\eta(\tau)\, \mathrm{d}\tau\right] ,$$

$\xi(t)$ and $\eta(t)$ being fixed strictly decreasing continuous positive functions summable over $(0, \infty)$. The functions $x(t)$ and $y(t)$ will be selected from the strategy classes $\mathcal{X}$ and $\mathcal{Y}$ respectively, where $\mathcal{X}$ is the class of all measurable functions satisfying, for a given positive constant X,

$$0 \leqslant x(t) \leqslant 1 \quad \text{and} \quad \int_0^\infty x(t)\, dt \leqslant X ,$$

and $\mathcal{Y}$ the class of all measurable functions satisfying, for a given positive constant Y,

$$0 \leqslant y(t) \leqslant 1 \quad \text{and} \quad \int_0^\infty y(t)\, dt \leqslant Y .$$

Specifically, then, we seek functions x_0 and y_0, with $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$, such that

$$(\mathrm{S}_1) \qquad \pi(x_0, y) \geqslant \pi(x_0, y_0) \quad \text{for all} \quad y \in \mathcal{Y}$$

and

$$(\mathrm{S}_2) \qquad \pi(x, y_0) \leqslant \pi(x_0, y_0) \quad \text{for all} \quad x \in \mathcal{X} .$$

Under further assumptions on $\xi(t)$, $\eta(t)$, X and Y (stated in Section 4) (which were satisfied in the model which led us to this problem), we are able to give in Section 5 explicit formulas for the pure optimal strategies x_0 and y_0, and incidentally to show that they are unique. Without such further assumptions the complexity of the problem is greatly magnified. However, we are able to prove the existence of a saddle-point without using these assumptions; this we proceed to do in the following section.

2. – Existence of a saddle point.

We shall need a well-known fixed-point theorem. The statement quoted is that given in [4]; a more general theorem had already been given by BEGLE in [7], unbeknownst to the author of [4]. The less general result is adequate for our purposes.

Fixed-point Theorem. *Given a closed point to convex set mapping Φ of a convex compact subset S of a convex Hausdorff linear topological space into itself, there exists a fixed point $z \in \Phi(z)$.*

We introduce a topology into $\mathfrak{X}$ by regarding it as a subset of the space L_∞ of essentially bounded measurable functions on $(0, \infty)$ and employing the weak star topology of L_∞. Thus, recalling that $L_\infty = L_1^*$, the directed set (cf. [2] or [3]) $\{x_\delta\}_{\delta\in\Delta}$ will be said to converge to x_0 provided that for every $u(t) \in L_1$ we have

$$\int_0^\infty x_\delta(t)u(t)\,\mathrm{d}t \to \int_0^\infty x_0(t)u(t)\,\mathrm{d}t\,.$$

$\mathfrak{X}$ is clearly a subset of the unit sphere K of L_∞. Furthermore, it is closed in K, as may be seen from very simple arguments. K itself is compact HAUSDORFF in the weak star topology, according to a theorem of L. ALAOGLU ([1], theorem 1:3). Accordingly, $\mathfrak{X}$ is compact. It is obvious that both $\mathfrak{X}$ and L_∞ are convex. We treat $\mathfrak{Y}$ in the same way. We then take as the set S of the fixed point theorem the product space $\mathfrak{X}\times\mathfrak{Y}$, which is obviously of the type required by the theorem.

Now we shall prove that $\pi(x, y)$ is continuous. Suppose that $x_\delta \to x_0$ and $y_\delta \to y_0$. Employing the summability of ξ and η and the uniform boundedness of $x_\delta(t)$ and $y_\delta(t)$, it is easy to show, e.g., by the classical procedure of ARZELÀ, that the functions $Q_{x_\delta}(t)$ and $Q_{y_\delta}(t)$ converge uniformly in t to $Q_{x_0}(t)$ and $Q_{y_0}(t)$ respectively. Further, $Q_{x_\delta}(t)$ and $Q_{y_\delta}(t)$ converge uniformly in δ to 1 as $t \to \infty$. Hence $\pi(x_\delta, y_\delta) \to \pi(x_0, y_0)$, as required.

Fix an $x \in \mathfrak{X}$. Since $\mathfrak{Y}$ is compact and π continuous, there exists a $y \in \mathfrak{Y}$ such that $\pi(x, y)$ is a minimum. Let y_0, y_1 be two elements of $\mathfrak{Y}$, both of which yield the minimum. Then for any λ, $0 \leqslant \lambda \leqslant 1$, if we write $y_\lambda = (1-\lambda)y_0 + \lambda y_1$ it is evident that $y_\lambda \in \mathfrak{Y}$ and we have

$$\pi(x, y_\lambda) = \int_0^\infty \exp\left[-\int_t^\infty y_\lambda(\tau)\eta(\tau)\,\mathrm{d}\tau\right]\mathrm{d}Q_x(t)\,,$$

$$\leqslant (1-\lambda)\pi(x, y_0) + \lambda\pi(x, y_1)\,,$$

so that y_λ is also minimizing. Similarly, to each $y \in \mathfrak{Y}$ there corresponds a non-empty convex set of x's $\in \mathfrak{X}$ which are maximizing. Denote by $\{x'_y\}$ the set of minimizing y's corresponding to any fixed y, and by $\{y'_x\}$ the set of minimizing y's for fixed x. Consider the point to convex set mapping

$$\Phi(x, y) = (\{x'_y\}, \{y'_x\})$$

of $\mathfrak{X}\times\mathfrak{Y}$ into itself. Since $\pi(x, y)$ is continuous this is a closed mapping;

accordingly, it has a fixed point (x_0, y_0), i.e., such that $x_0 \in \{x'_{y_0}\}$ and $y_0 \in \{y'_{x_0}\}$. But these are precisely the conditions that (x_0, y_0) be a saddle-point.

We observe that it is obvious that $\int_0^\infty x_0(t)\,dt = X$ and $\int_0^\infty y_0(t)\,dt = Y$.

3. – Necessary conditions on the optimal strategies.

In this section we shall derive conditions that x_0 and y_0, as optimal strategies, must satisfy. Define

$$H(t) = \int_0^t Q_{y_0}(\tau)\,dQ_{x_0}(\tau) \tag{1}$$

and

$$K(t) = Q_{x_0}(0)Q_{y_0}(0) + \int_0^t Q_{x_0}(\tau)\,dQ_{y_0}(\tau)\,. \tag{2}$$

We shall show that there exist positive constants h and k such that the two conditions

$$H(t) \begin{cases} \geqslant \dfrac{h}{\eta(t)} & \text{whenever} \quad y_0(t) > 0\,, \\ \leqslant \dfrac{h}{\eta(t)} & \text{whenever} \quad y_0(t) < 1 \end{cases} \tag{C$_1$}$$

and

$$K(t) \begin{cases} \geqslant \dfrac{k}{\xi(t)} & \text{whenever} \quad x_0(t) > 0\,, \\ \leqslant \dfrac{k}{\xi(t)} & \text{whenever} \quad x_0(t) < 1 \end{cases} \tag{C$_2$}$$

are simultaneously satisfied at every point of approximate continuity (for definition see [5], p. 132) of $x_0(t)$ and $y_0(t)$, and hence almost everywhere. It suffices to demonstrate (C_1), the proof for (C_2) being analogous.

Since x_0 and y_0 are optimal we have

$$\pi(x_0, y) \geqslant \pi(x_0, y_0) \qquad \text{for all} \quad y \in \mathcal{Y}\,, \tag{3}$$

(i.e., the saddle-point condition (S_1) of Section 1). Let y_1 be an arbitrary element of the strategy space $\mathcal{Y}$, and for each λ, $0 \leqslant \lambda \leqslant 1$, define $y_\lambda(t) = (1-\lambda)y_0(t) +$

$+\lambda y_1(t)$; then $y_\lambda \in \mathcal{Y}$. A simple argument shows that $\pi(x_0, y_\lambda)$ is everywhere differentiable with respect to λ. From (3) it is easy to see that

$$\frac{\partial}{\partial\lambda}\left[\pi(x_0, y_\lambda)\right]_{\lambda=0} \geqslant 0 \quad \text{for all} \quad y_1 \in \mathcal{Y}. \tag{4}$$

On rewriting (4) we obtain, after inverting the order of integration,

$$\int_0^\infty H(t)\eta(t)\left[y_0(t) - y_1(t)\right] dt \geqslant 0 \quad \text{for all} \quad y_1 \in \mathcal{Y}. \tag{5}$$

To obtain (C_1) from (5) we use an elementary variational procedure. Write for convenience $W(t) = H(t)\eta(t)$. It will suffice to prove that if y_0 is approximately continuous at two points t_1 and t_2, satisfying $0 < y_0(t_1) \leqslant 1$ and $0 \leqslant y_0(t_2) < 1$ respectively, then $W(t_1) \geqslant W(t_2)$. Let ε be any positive number less than $y_0(t_1)$ and such that $y_0(t_2) < 1 - 2\varepsilon$. In view of the approximate continuity there exist disjoint measurable sets E_1 and E_2 containing t_1 and t_2 respectively and satisfying $0 < \mathrm{m}(E_1) < 2\mathrm{m}(E_2)$, such that

1) on E_1, $y_0(t) > \varepsilon$ and $W(t) < W(t_1) + \varepsilon$,
2) on E_2, $y_0(t) < 1 - 2\varepsilon$ and $W(t) > W(t_2) - \varepsilon$.

Now put

$$y_1(t) = \begin{cases} y_0(t) - \varepsilon, & t \in E_1, \\ y_0(t) + \varepsilon\dfrac{\mathrm{m}(E_1)}{\mathrm{m}(E_2)}, & t \in E_2, \\ y_0(t), & t \notin (E_1 + E_2). \end{cases}$$

Clearly $y_1 \in \mathcal{Y}$ and so it satisfies (5). Hence applying this we get

$$W(t_1) \geqslant W(t_2) - 2\varepsilon, \tag{6}$$

which in view of the arbitrary choice of ε is the result desired. Thus (C_1) holds, as we set out to prove. To see that k is positive, it suffices to observe the form of $K(t)$ and that $x_0(t)$ is less than unity except on a set of measure at most X. To see that h is positive it suffices to note that if t is sufficiently large, then $x_0(t)$ will have been positive on a set of positive measure to the left of t and so $H(t)$ positive, and then to take account of the fact that $y_0(t)$ is less than unity except on a set of measure at most Y.

At this stage it is convenient to note some properties of H and K. First, observe that, for all t,

$$H(t) + K(t) = Q(t) \,, \tag{7}$$

where we have written

$$Q(t) = Q_{x_0}(t) Q_{y_0}(t) \,. \tag{8}$$

Next, H, K, Q_{x_0} and Q_{y_0} are all differentiable at any point at which $x_0(t)$ and $y_0(t)$ are both approximately continuous. They are absolutely continuous and non-decreasing in t. Finally, as $Q(t)$ is obviously $\leqslant 1$, then so a fortiori both $H(t)$ and $K(t)$ are $\leqslant 1$.

4. – Additional assumptions.

Let $\bar{t}$ be a positive number satisfying the inequalities

$$\left\{ \begin{array}{l} \bar{t} > X \,, \\ \xi(\bar{t}) < \xi(0) \exp\left[- X\xi(0) - X\eta(0) - Y\eta(0)\right] . \end{array} \right. \tag{9}$$

We shall assume that throughout the interval $(0, \bar{t})$ both

$$\frac{-\xi'(t)}{\xi(t)\eta(t)} < 1 \tag{A$_1$}$$

and

$$\frac{-\eta'(t)}{\xi(t)\eta(t)} < 1 \tag{A$_2$}$$

are satisfied, except for a set of measure zero. Observe that we may *not* assume either (A_1) or (A_2) to be valid in the infinite interval. The reader will find the reason for the choice of $\bar{t}$ in the proof of Lemma 3 in the next section.

These assumptions are valid in many physically practicable cases.

5. – The optimal strategies.

We shall be able to write down the explicit solution of the game after proving a series of lemmas.

Lemma 1. *There exists a non-degenerate maximal interval* $(0, t_1)$ *on which* $y_0(t) = 0$ *almost everywhere.*

Proof. It is sufficient to prove the existence of a non-degenerate interval $(0, a)$ with this property. Suppose no such interval exists. Then for any a there is a set of points t of positive measure, lying in $(0, a)$, for which $H(t) \geqslant h/\eta(t)$. It follows that $H(0) \geqslant h/\eta(0) > 0$, which is obviously impossible.

Lemma 2. *On $(0, t_1)$ we have $x_0(t) = 1$ almost everywhere.*

Proof. For any $\varepsilon > 0$ there is by Lemma 1 a set of points t of positive measure, and lying in $(t_1, t_1+\varepsilon)$, for which $y_0(t) > 0$. Accordingly, we have $H(t_1) \geqslant h/\eta(t_1)$. Now $K(t)$ is obviously constant over $(0, t_1)$. On comparing $K(t)$ with $k/\xi(t)$ it is clear that there is a t_2 satisfying $0 \leqslant t_2 \leqslant t_1$ such that on $(0, t_2)$ we have $x_0(t) = 1$ almost everywhere and on (t_2, t_1) we have $x_0(t) = 0$ almost everywhere. Suppose now that $t_2 < t_1$. Then as t_2 is a limit point of points at which $y_0(t)$ is zero, we would have $H(t_2) \leqslant h/\eta(t_2)$. But we would also have $H(t_2) = H(t_1)$, so that $H(t_1) < h/\eta(t_1)$, a contradiction.

Lemma 3. *There is a smallest number t_0, satisfying $t_1 \leqslant t_0 < \bar{t}$, such that $x_0(t) = 0$ almost everywhere on (t_0, ∞).*

Proof. Let $(0, t_2)$ be a maximal interval on which $x_0(t) = 1$ almost everywhere. Obviously $t_2 < \bar{t}$. Then for any $\varepsilon > 0$ there is on $(t_2, t_2+\varepsilon)$ a set of points t of positive measure on which $x_0(t) < 1$. Accordingly $K(t_2) \leqslant k/\xi(t_2)$. We rewrite this as

$$k \geqslant \xi(t_2)K(t_2) \; ; \tag{10}$$

we shall prove that $k > \xi(\bar{t})$. Now by hypothesis (A_1) we have, on $(0, t_2)$,

$$-\frac{\xi'(t)}{\xi(t)} < \eta(t) \tag{11}$$

almost everywhere. On integrating we obtain

$$\xi(t_2) > \xi(0) \exp\left[-\int_0^{t_2} \eta(t)\, dt\right] > \xi(0) \exp\left[-X\eta(0)\right], \tag{12}$$

where we have taken account of the fact that $t_2 < X$. Turning to $K(t_2)$, we obtain easily

$$K(t_2) \geqslant Q(0) > \exp\left[-X\xi(0) - Y\eta(0)\right]. \tag{13}$$

Accordingly,

$$k > \xi(0) \exp\left[-X\xi(0) - X\eta(0) - Y\eta(0)\right] > \xi(\bar{t}) \,. \tag{14}$$

Now, since $x_0(t) = 1$ over $(0, t_2)$, we have $K(t_2) \geqslant k/\xi(t_2)$. Hence $\xi(t_2) > k$. It follows that there is a number t_3 on the interval $(t_2, \bar{t})$ such that $k = \xi(t_3)$. Let $t_4 > t_3$ be a point of approximate continuity for $x_0(t)$. Then if $x_0(t_4)$ were positive, we would have $K(t_4) \geqslant k/\xi(t_4) > 1$, an impossibility. Consequently, $x_0(t) = 0$ almost everywhere on (t_3, ∞). Let t_0 be the smallest number with the property that $x_0(t) = 0$ almost everywhere on (t_0, ∞). Evidently $t_1 \leqslant \leqslant t_0 < t_3 < \bar{t}$, as desired.

Lemma 4. *$y_0(t) = 0$ almost everywhere on (t_0, ∞).*

Proof. As $H(t)$ is constant on (t_0, ∞), it will suffice to prove that $H(t_0) \leqslant h/\eta(t_0)$. Suppose the contrary. On comparing $H(t)$ with $h/\eta(t)$ and taking account of (C_1) it is clear that for some interval (t_0, t_2) we have $y_0(t) = 1$ almost everywhere. We may take $t_2 \leqslant \bar{t}$. Now since almost everywhere on (t_0, t_2) we have from (A_1) the inequality

$$-\frac{\xi'(t)}{\xi(t)} < \eta(t)\,, \tag{15}$$

it follows that

$$\begin{aligned} K(t_2) &= K(t_0) + \int_{t_0}^{t_2} Q_{x_0}\, dQ_{y_0} \\ &= K(t_0) + Q(t_0)\left\{\exp\left[\int_{t_0}^{t_2} \eta(t)\, dt\right] - 1\right\} \\ &> K(t_0) + Q(t_0)\left[\frac{\xi(t_0)}{\xi(t_2)} - 1\right] \\ &> K(t_0)\,\frac{\xi(t_0)}{\xi(t_2)} \\ &= \frac{k}{\xi(t_2)}\,, \end{aligned}$$

a contradiction, since $K(t_2) \leqslant k/\xi(t_2)$ from (C_2).

Remark. *It follows trivially that $t_1 < t_0$.*

Lemma 5. *Every non-degenerate subinterval of (t_1, t_0) contains a set of points of positive measure on which $y_0(t) > 0$.*

Proof. If this were false there would be a maximal subinterval (t_2, t_3) of (t_1, t_0) on which $y_0(t) = 0$ almost everywhere. It is obvious (from the specification of t_1 in Lemma 1) that $t_2 > t_1$. Now by an argument we have used before we have $H(t_2) = h/\eta(t_2)$ and $H(t_3) \leqslant h/\eta(t_3)$, with equality holding in the latter if $t_3 < t_0$. As in the proof of Lemma 2, we see that in either event $x_0(t) = 1$ almost everywhere on (t_2, t_3). Then, employing an argument similar to that applied to $K(t)$ in the proof of Lemma 4, we see [with the aid of (A_2)] that $H(t_3) > h/\eta(t_3)$, a contradiction.

Lemma 6. *Every non-degenerate subinterval of* (t_1, t_0) *contains a set of points of positive measure on which* $x_0(t) > 0$.

Proof. Analogous to that for Lemma 5.

Lemma 7. *Every non-degenerate subinterval of* (t_1, t_0) *contains a set of points of positive measure on which* $y_0(t) < 1$.

Proof. We suppose this false, and obtain a maximal interval (t_2, t_3) on which $y_0(t) = 1$ almost everywhere. Clearly $H(t_2) = h/\eta(t_2)$ and $H(t_3) = h/\eta(t_3)$. It follows by an argument now familiar that $x_0(t)$ cannot be equal to unity almost everywhere on (t_2, t_3); hence there is a subset E of (t_2, t_3), of positive measure, on which $x_0(t) < 1$. Let E' denote the set of points of E which are points of density for E, at which $\xi'(t)$ exists, and at which $x_0(t)$ and $y_0(t)$ are approximately continuous. Making use of Lemma 6 it is easy to see that we have

$$K(t) = \frac{k}{\xi(t)}, \tag{16}$$

everywhere on E'. Also, of course, $H(t) \geqslant h/\eta(t)$. Now let $t \in E'$ and take the derivatives of both sides of (16) through points of E', i.e.,

$$\lim_{\substack{t' \to t \\ t' \in E'}} \frac{K(t') - K(t)}{t' - t} = \lim_{\substack{t' \to t \\ t' \in E'}} \frac{\dfrac{k}{\xi(t')} - \dfrac{k}{\xi(t)}}{t' - t}.$$

Since the derivatives at t exist and are the same for *any* approach, we obtain

$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} = -\frac{k\xi'(t)}{\xi^2(t)}.$$

On writing out the left side and then dividing we see that

$$y_0(t) = -\frac{k\xi'(t)}{\xi^2(t)\eta(t)Q(t)}.$$

But [recalling (7)]

$$Q(t) \geqslant \frac{h}{\eta(t)} + \frac{k}{\xi(t)}.$$

Accordingly,

$$y_0(t) \leqslant \frac{-k\xi'(t)}{\xi(t)[h\xi(t) + k\eta(t)]} < \frac{-\xi'(t)}{\xi(t)\eta(t)} < 1$$

throughout E'. This is the contradiction desired.

Lemma 8. *$x_0(t) < 1$ almost everywhere on (t_1, t_0).*

Proof. From (C_2) and Lemma 6 we have $K(t) \geqslant k/\xi(t)$ throughout. Now as each subinterval of (t_1, t_0) contains a set of points of positive measure on which $y_0(t) > 0$ and a set of points of positive measure on which $y_0(t) < 1$ we have $H(t) = h/\eta(t)$ identically on (t_0, t_1). On differentiating this identity we obtain the result, in a manner analogous to but somewhat simpler than the argument used in the last part of the proof of Lemma 7.

Clearly, from Lemmas 6 and 8, we have aslo $K(t) = k/\xi(t)$ identically on (t_1, t_0); hence we may differentiate, and replace inequalities there by equalities. *Thus we have our result:*

Theorem. *There exists a t_0, a t_1, and an $m > 0$, such that $0 < t_1 < t_0 < \bar{t}$, and such that*

1) *on* $(0, t_1)$, $x_0(t) = 1$ *and* $y_0(t) = 0$ *almost everywhere,*

2) *on* (t_1, t_0), $x_0(t) = -\dfrac{m\eta'(t)}{\eta(t)[m\xi(t) + \eta(t)]}$ *almost everywhere,*

$y_0(t) = -\dfrac{\xi'(t)}{\xi(t)[m\xi(t) + \eta(t)]}$ *almost everywhere,*

and

3) *on* (t_0, ∞), $x_0(t) = y_0(t) = 0$ *almost everywhere.*

The existence of the quantities t_0, t_1 and m $(= h/k)$ is of course assured by the existence proof of Section 2. It is well to state how these quantities may be found. Applying (C_1) and (C_2) at t_1, and then dividing, we obtain

$$m = \frac{\eta(t_1)}{\xi(t_1)} \left\{ \exp\left[\int_0^{t_1} \xi(t)\,dt \right] - 1 \right\}.$$

There remain two equations for determining the three quantities. We apply merely the conditions that $\int_0^\infty x_0(t)\,dt = X$ and $\int_0^\infty y_0(t)\,dt = Y$. It is an elementary exercise to show that the solution to these three equations is unique, so that the saddle-point of the theorem is the unique solution to the game.

The value of the game now may easily be determined in terms of m and t_0. Apply conditions (C_1) and (C_2) at the point $t = t_0$. We get

$$\int_0^{t_0} Q_{y_0}\,dQ_{x_0} = \frac{h}{\eta(t_0)}$$

and

$$1 - \int_0^{t_0} Q_{y_0}\,dQ_{x_0} = \frac{k}{\xi(t_0)}.$$

From these it follows easily that

$$\pi(x_0, y_0) = \int_0^{t_0} Q_{y_0}\,dQ_{x_0} = \frac{m\xi(t_0)}{m\xi(t_0) + \eta(t_0)}.$$

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