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Contours of a Fréchet surface. (**)

If S is any parametric continuous surface in the p-space E_3, $p = (x, y, z)$, let $[S]$ denote the set of points covered by S and $L(S)$ the LEBESGUE area of S. Let $f(p)$, $p \in S$, be any real, single-valued function defined on $[S]$ and satisfying the relation $|f(p) - f(p')| \leqslant K|p - p'|$ for all $p, p' \in [S]$. By denoting by $l(t)$ a convenient generalized length associated, for each real t, to the subset S_t of all points $p \in [S]$ with $f(p) = t$, I have proved the following inequality

$$K\,L(S) \geqslant \int_{-\infty}^{+\infty} l(t)\,dt \tag{1}$$

(cf. the abstract [5, (c)] submitted for publication in 1950; a proof of (1) is contained in my book [5, (A)]). This inequality, involving length and area, extends a classical inequality to all continuous surfaces. The generalized length $l(t)$ is essentially the JORDAN length. Indeed, whenever $l(t)$ is finite, S_t is the countable sum of continuous path-curves γ (besides a point-set which is the image of a completely disconnected set), and the generalized length $l(t) = \sum \lambda(\gamma)$ is the sum of the JORDAN lengths $\lambda(\gamma)$ of the curves γ [5, (A)].

In the present paper I recall the various concepts underlying inequality (1) and I prove that *the length $l(t)$ is invariant with respect to Fréchet equivalence.*

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(**) Ricevuto il 4-VIII-1953. Questa Memoria è scritta in inglese in quanto fa parte di un gruppo di lavori che il CESARI ha iniziato in collaborazione con R. E. FULLERTON. Di tali lavori — che questa Rivista ha il piacere di pubblicare — il primo è già comparso: L. CESARI and R. E. FULLERTON, *On regular representations of surfaces*, Rivista Mat. Univ. Parma 2, 279-288 (1951). Altro lavoro del gruppo figura in questo stesso fascicolo della Rivista: R. E. FULLERTON, *On the rectification of contours of a Fréchet surface*, Rivista Mat. Univ. Parma 4, 207-212 (1953).

This result will be utilized by R. E. FULLERTON [8, (a); 8, (b)] in questions concerning the representation problem for surfaces.

In nos. 2 and 3 I recall some points of CARATHÉODORY's theory of ends and prime ends [4] and the recent concept of right and left wing of a prime-end (H. D. URSELL and L. C. YOUNG, [17]). In no. 4 I prove a lemma which I shall utilize in nos. 14 and 15. In nos. 7-10 I recall the definition of generalized length $l(t)$ and its properties. The question of the invariance of $l(t)$ with respect to FRÉCHET-equivalence is discussed in the nos. 11-15.

I mention here that a particular elementary case of (1) was recently pointed out by L. C. YOUNG [19, (a)] and by L. CESARI [5, (a)] in questions of Calculus of Variations, and that an analogous classical case was utilized by H. LEWY [12] in a question of Functional Analysis. The inequality (1) for LEBESGUE area corresponds to an analogous relation for HAUSDORFF measures [6; 7; 1; 19, (b)]. For further recent independent research involving families of curves on a continuous surface and their lengths see also [2, 3, 11, 15, 16]. For the extension of the prime-end theory to the space E_3 see, e.g., [10].

1. – Notations.

Given a point set A in the real Euclidean space E_N we denote by A^o, A^*, $\bar{A}=A+A^*$ the set of the interior points of A, the boundary and the closure of A respectively. A single point p, considered as a set of only one element is denoted by (p). A set $A\subset E_N$ is compact if closed and bounded, a continuum if closed, bounded, and connected. Given any two sets $A\subset B\subset E_N$, A open in B, we denote by $\mathcal{F}(A)$, or frontier of A in B, the set $\mathcal{F}(A)=(\bar{A}-A)B=(A^*-AA^*)B$, thus $\mathcal{F}(A)=\bar{A}-A=A^*-AA^*$ if B is closed. We denote by $|p-q|$ the distance between any two points $p, q\in E_N$, by $\{A, B\}=\operatorname{Inf}|p-q|$ for all $p\in A$, $q\in B$, the distance between two sets A and B, and by diam $A=\operatorname{Sup}|p-q|$ for all $p, q\in A$ the diameter of a set A. If A_n, $n=1, 2,\dots$, is any sequence of sets $A_n\in E_N$, we denote by $A'=\liminf A_n$ the set of all points $p\in E_N$ in each neighborhood of which there are points of all but a finite number of sets A_n, and by $A''=\limsup A_n$ the set of all points $p\in E_N$ in each neighborhood of which there are points of infinitely many A_n. Thus, $A'\subset A''$ and A', A'' are both closed. If all sets A_n are continua and $A'\neq 0$, then A'' is also a continuum (ZORETTI's theorem, [9, p. 38], [18, I, 9.1]). By $\underline{\lim}\, A_n$, $\overline{\lim}\, A_n$ we denote as usual the sets $l=\sum_r(A_rA_{r+1}\dots)$, $L=\Pi_r(A_r+A_{r+1}+\dots)$ respectively. If $l=L$, then $l=L=\lim A_n$.

2. – On the boundary of bounded simply connected open sets (Carathéodory's theory).

Let α be any open bounded simply connected set of the w-plane E_2, $w=(u,v)$. Then α^* is a continuum. An arc b is said to be an end-cut of α if (1), $b\alpha^*=(w)$, $b\subset\alpha+(w)$, where w is a single point of α^*. An arc b is said to be a cross-cut of α if $b\alpha^*=(w_1)+(w_2)$, $b\subset\alpha+(w_1)+(w_2)$. A point $w_0\in\alpha^*$ is said to be accessible from α if there exists an end-cut b such that $b\alpha^*=(w_0)$. The set of the points of α^* accessible from α form an uncountable collection everywhere dense in α^* [**14**, p. 162].

Two end-cuts b, b' of α are said to define the same *end* η of α if (1) b and b' have the same end-point $w_\eta\in\alpha^*$; (2) either $bb'(U-w_\eta)\neq 0$ for any neighborhood U of w_η, or, there are two subarcs b_1 of b and b_1' of b', and a simple arc c such that $b_1b_1'=(w_\eta)$, $c\subset\alpha$, $cb_1=(w)$, $cb_1'=(w')$, and the open JORDAN region J whose boundary is b_1+c+b_1' is contained in α. If (1) is not satisfied, or (1) is but (2) is not, then b, b' are said to define different ends of α. We shall say also that η is an end of α^* in α. Thus each accessible point $w\in\alpha^*$ is a point w_η relative to at least one end η, but it may happen that for more than one end, namely a finite or countable collection of ends η, we have $w=w_\eta$ [**4**, **9**].

Let $\{\eta\}_\alpha$, or $\{\eta\}$, denote the family of all (different) ends η of α. If η_i, $(i=1,2,3,4)$, are four different ends and b_i, $(i=1,2,3,4)$, any four end-cuts defining the ends η_i, then we can suppose that the arcs b_i have no point in common besides those of the points w_{η_i} which may coincide. Let us connect the end points of b_1 and b_2 which are in α with a single arc $c\subset\alpha$ having no point in common with the arcs b_i, $(i=1,2,3,4)$, besides the end points of b_1 and b_2 in α. Then b_1+c+b_2 is a cross-cut, separates α into two parts [**14**, p. 118], and b_3 and b_4 may be in different parts or in the same part. This property does not depend on the particular arcs b_i, c we have considered, but only on the ends η_i, $(i=1,2,3,4)$. Accordingly we shall say that η_1, η_2 separate η_3, η_4 in $\{\eta\}$ (and then η_3, η_4 separate η_1, η_2 in $\{\eta\}$) [**9**]. Therefore, the collection $\{\eta\}$ can be cyclically ordered and we shall denote by Ω_1', Ω_2' the two fundamental orderings of $\{\eta\}_\alpha$. If we denote by ∞ any one of the ends η, then, given any two distinct ends $\eta_1,\eta_2\in\{\eta\}$, $[\eta_1,\eta_2\neq\infty]$, by the open *interval* (η_1,η_2) is meant the family of all ends η such that η, ∞ separate η_1, η_2 in $\{\eta\}$. Thus by the closed interval $[\eta_1,\eta_2]$ is meant $(\eta_1,\eta_2)+(\eta_1)+(\eta_2)$.

A section $\omega=(A,B)$ in $\{\eta\}_\alpha$ or a *prime end* ω of α is defined by a sequence $[(\eta_n,\eta_n'),\ n=1,2,...]$ of intervals, $[\eta_{n+1},\eta_{n+1}']\subset(\eta_n,\eta_n')$, such that no end η, or at most one, is contained in all intervals (η_n,η_n'). Each end $\eta\in\{\eta\}_\alpha$

defines a prime end ω [say $\omega = \eta$], but there may be prime ends ω which do not correspond to any end η. The family $\{\omega\} = \{\omega\}_\alpha$ of all prime ends can be cyclically ordered in the same two orderings Ω_1', Ω_2' above [4, 9].

The concept of equivalence of two sequences $[(\eta_n, \eta_n'),\ n = 1, 2, \dots]$ defining the same prime end ω can be established as for real numbers.

Let ω be any prime end and $[(\eta_n, \eta_n'),\ n = 1, 2, \dots]$ a sequence defining ω. Let E_ω be the set of all points w which have the following property: there is a sequence $(\eta_k,\ k = 1, 2, \dots)$ of ends such that $\eta_k \in [\eta_{n_k}, \eta_{n_k}']$, $k = 1, 2, \dots$, $w_{\eta_k} \to w$, $n_k \to \infty$ as $k \to \infty$. The set E_ω does not depend upon the particular sequence (η_n, η_n') defining ω and is a subcontinuum of α^* [4, 9].

If ω is a prime end corresponding to an end η, then $w_\eta \in E_\omega$, but E_ω may contain other points (accessible and not accessible from α). Different prime ends ω may have sets E_ω not disjoint, even coincident. The family $\{E_\omega\}$ of all sets E_ω is a covering of α^*.

Note I. Though some authors denote the sets E_ω by prime ends, we prefer to keep the expressive CARATHÉODORY's term prime end for denoting a section ω in the collection of the ends. Indeed while ω determines E_ω, the set E_ω does not determine necessarily the section ω.

Let ω be any prime end, let $[(\eta_n', \eta_n''),\ n = 1, 2, \dots]$ be any sequence defining ω and Ω' any one of the orderings Ω_1', Ω_2'. Thus we can enumerate the ends η_n', η_n'' is such a way that $\eta_1' < \eta_2' < \eta_3' < \dots < \eta_3'' < \eta_2'' < \eta_1''$. Therefore each end $\eta \in (\eta_m', \eta_m'')$ belongs to one and only one interval $[\eta_n', \eta_{n+1}']$, or $[\eta_{n+1}'', \eta_n'']$, $n \geqslant m$, with exception of each η_n', η_n'', $n > m$, which belongs to two adjacent intervals and at most one end η_0 which belongs to all intervals (η_n', η_n'') and to no interval $[\eta_n', \eta_{n+1}']$, $[\eta_{n+1}'', \eta_n'']$, $n \geqslant m$.

Let E_ω', E_ω'' be the sets of all points w which have the following property: there is a sequence $(\eta_k,\ k = 1, 2, \dots)$ of ends such that $\eta_k \in [\eta_{n_k}', \eta_{n_{k+1}}']$, or $\eta_k \in [\eta_{n_k+1}'', \eta_{n_k}'']$, $w_{\eta_k} \to w$, $n_k \to \infty$ as $k \to \infty$ [5, (b)]. Thus both E_ω', E_ω'' are subcontinua of α^* and $E_\omega' + E_\omega'' = E_\omega$, $E_\omega' E_\omega'' \neq 0$. E_ω', E_ω'' are said to be the *left* and *right wings* of E_ω.

Note II. The expressions left and right wings have been introduced by H. D. URSELL and L. C. YOUNG [17] using other definitions, whose equivalence with the definitions above is proved in [17, 11.2]. We have preferred the definitions above because they are better suited for our purpose and connected with the proof (L. CESARI [5, (b)]) of the statement below [see also 5, (A)].

(i) Given any two points $w_1, w_2 \in E_\omega'$ there is a sequence $(\eta_k,\ k = 1, 2, \dots)$ of ends such that (1) $w_{\eta_{2k-1}} \to w_1$, $w_{\eta_{2k}} \to w_2$ as $k \to \infty$; (2) $\eta_k \in [\eta_{n_k}', \eta_{n_{k+1}}']$, $n_k + 1 \leqslant n_{k+1}$, $k = 1, 2, \dots$. An analogous statement holds for E_ω'' with $\eta_k \in [\eta_{n_k+1}'', \eta_{n_k}'']$, [5, (A), (b)].

We recall here from [4] and [9] that for every prime end ω there are sequences $[l_n]$ of cross-cuts with the following properties: (1) l_n joins the points w_{η_n}, $w_{\eta'_n}$ relative to ends η_n, η'_n such that the sequence $[(\eta_n, \eta'_n), n = 1, 2,...]$ is a decreasing sequence of open intervals defining ω; (2) l_n divides α into two regions r_n, r'_n [l_n separates r_n from the end $\eta = \infty$ in α]; (3) $\operatorname{diam} l_n < n^{-1}$, $l_n l_m \alpha = 0$, $m \neq n$, $l_{n+1} \subset r_n$, $m, n = 1, 2, \ldots$. A sequence $[l_n]$ satisfying (1), (2), (3) is called a fundamental sequence of cross-cuts relative to ω.

Each region r_n, or any open subset of α containing a set r_n can be thought of as a «neighborhood» of ω in α (or of the continuum E_ω).

Let us observe that if a point w is the limit of a sequence $[w_n]$ of points $w_n \in l_n$, $n = 1, 2,\ldots$, $w_n \to w$ as $n \to \infty$, then, since $\operatorname{diam} l_n < n^{-1}$, w is also the limit of any other sequence $[w'_n]$ of points $w'_n \in l_n$, $n = 1, 2, \ldots$, and $w = \liminf l_n = \limsup l_n$. We say briefly that w is the limit of a fundamental sequence $[l_n]$ of cross-cuts relative to ω.

According to CARATHÉODORY [4], by principal part $E_\omega^{(0)}$ of E_ω is meant the set of all points w which are limit of some fundamental sequence $[l_n]$ of cross-cuts relative to ω.

(ii) $E_\omega^{(0)}$ is a subcontinuum of E_ω and $E_\omega^{(0)} \subset E'_\omega E''_\omega \subset E_\omega$.

The first part of (ii) is proved in [4], the second part is a consequence of the definitions above.

(iii) Given any open, bounded, simply connected set α and the closed unit circle Γ of the w-plane E_2, there is a mapping τ such that: (1) τ is bicontinuous and one-one between α and Γ^0; (2) τ is bicontinuous and one-one between the points $w \in \Gamma^*$ and the prime-ends $\omega \in \{\omega\}$ (or the sets E_ω); (3) τ is continuous in α and Γ, provided the neighborhoods of the prime ends ω [or of the sets E_ω] are chosen as it has been explained above; (4) if $(T, \bar{\alpha})$ is any given continuous mapping from $\bar{\alpha}$ into the p-space E_3 and T is constant on each set E_ω, then the mapping (t, Γ) defined by $t = T\tau$ is single-valued and continuous in Γ.

(1), (2), (3) are proved in [4] and in [9]; (4) is a consequence of (3). See also [5, (A)] and [17, 11.3].

As above let $\eta = \infty$ denote any one of the ends η and Ω' one of the orderings of $\{\eta\}_\alpha$, $\{\omega\}_\alpha$. Then, given any two distinct prime ends ω_1, ω_2 (and distinct from $\eta = \infty$), the set of all prime ends ω such that $\infty < \omega_1 < \omega < < \omega_2 < \infty$ (in Ω'), is the open interval (ω_1, ω_2), while $(\omega_1, \omega_2) + (\omega_1) + (\omega_2)$ is the closed interval $[\omega_1, \omega_2]$. We shall denote by $I(\omega_1, \omega_2)$ the set $E''_{\omega_1} + \sum E_\omega + E'_{\omega_2}$, where $\sum$ is extended over all $\omega \in (\omega_1, \omega_2)$. It easily proved that $I(\omega_1, \omega_2)$ is a subcontinuum of α^* (for a proof see, e. g., [5, (d)]). In particular, if $\omega_1 = \eta_1$, $\omega_2 = \eta_2$ are prime ends corresponding to ends η_1, η_2,

$I(\omega_1, \omega_2)$ shall be denoted by $I(\eta_1, \eta_2)$. Because of (i) the set $E''_{\omega_1}+\sum E_\omega+E'_{\omega_2}$ is the closure of the set $\sum E_\omega$. If we denote by $I^*(\omega_1, \omega_2)$ the set $E_{\omega_1}+\sum E_\omega+E_{\omega_2}$, then also $I^*(\omega_1, \omega_2)$ is a continnum and $I\subset I^*$. Whenever $\omega_1=\omega_2=\omega$ we set $I(\omega, \omega)=E^{(0)}_\omega$, $I^*(\omega, \omega)=E_\omega$.

3. – *f*-systems relative to a prime end ω.

Let α be any open, bounded, simply connected set, and $\{\eta\}$, $\{\omega\}$ be the ordered collections of all ends and prime ends.

For each $\omega\in\{\omega\}$ let $[l_n]$ be a fundamental sequence (no. 2) of cross-cuts l_n relative to ω and let r_n be the one of the two regions in which l_n separates α which contains all l_p with $p>n$. For each n let $\varrho_n=r_n-\bar r_{n+1}$. Then r_n, ϱ_n are open, bounded, simply connected sets and $r_n^*=l_n+I(\eta_n, \eta'_n)$, $\varrho_n^*=l_n+$ $+l_{n+1}+I(\eta_n, \eta_{n+1})+I(\eta'_{n+1}, \eta'_n)$.

For any n let w_n be a point of αl_n and let b_n be a simple arc, $b_n\subset\varrho_n+(w_n)+$ $+(w_{n+1})$. Such an arc exists because $w_n\in l_n$, $w_{n+1}\in l_{n+1}$, where l_n, l_{n+1} are simple arcs and therefore both w_n, w_{n+1} are accessible from ϱ_n. Let b be the set $b=b_1+b_2+b_3+\dots$. The set b is an «indefinite arc», i.e., the homeomorphic image of a half-closed, half-open interval, $0<u\leqslant 1$. Indeed we have only to represent each b_n on the closed interval $(n+1)^{-1}\leqslant u\leqslant n^{-1}$, $n=1, 2,\dots$. Any fundamental sequence $[l_n]$ relative to ω and any corresponding sequence $[b_n]$ of arcs as above are said to constitute an *f*-system $[l_n, b_n]$ relative to ω.

(i) For any *f*-system $[l_n, b_n]$ relative to ω the set $h=\limsup b_n$ is a continuum and $E^{(0)}_\omega\subset h\subset E_\omega$. There exist particular *f*-systems for which $E^{(0)}_\omega=h$ and other ones for which $E_\omega=h$ [**4**, **9**].

4. – Sequences of continua.

Let us use the same notations as in no. **3**. Let $[\beta_k, k=1, 2,\dots]$ be any sequence of continua satisfying the following condition: (Q) $\beta_k\subset\alpha$, $\beta_k\beta_{k+1}\neq 0$, $k=1, 2,\dots$, $\limsup\beta_k\subset\alpha^*$. Set $\beta=\beta_1+\beta_2+\dots$. Obviously all sequences $[b_m, m=1, 2,\dots]$ of no. **3** satisfy condition (Q).

Let $\omega\in\{\omega\}$ and $[l_n]$ be any fundamental sequence of cross-cuts relative to ω. We shall say that a prime end ω is reached by $[\beta_k]$ if $l_n\beta\neq 0$ for all n large enough, i.e., there is an $\bar n$ such that each l_n with $n\geqslant\bar n$ has a non-empty intersection with some β_k. Obviously the property of a prime end ω to be reached by $[\beta_k]$ is independent of the particular fundamental sequence $[l_n]$.

(i) Given a sequence $[\beta_k, k=1, 2,\dots]$ of continua β_k satisfying (Q), then the collection I of all prime-ends ω reached by $[\beta_k]$ is either a single prime end ω, or a closed interval $[\omega', \omega'']$. If $H = \limsup \beta_k$, then H is a continuum and $I(\omega', \omega'') \subset H \subset I^*(\omega', \omega'')$.

Proof. Let us prove first that the collection I of the prime ends ω reached by $[\beta_k]$ is not empty. Though not necessary, it is easier to transform $\bar{\alpha}$ onto the closed unit circle Γ as in no. 2, (iii) by a continuous mapping τ. Then $\tau(\beta_k) = \beta'_k$, $(k = 1, 2, ...)$, are continua in Γ^0 and $\limsup \beta'_k \subset \Gamma^*$. Each prime end ω and relative set E_ω is mapped by τ into a point $w \in \Gamma^*$ and any f-system $[l_n, b_n]$ relative to ω is mapped by τ into an f-system $[l'_n, b'_n]$ relative to w. In addition $b' = (b'_1 + b'_2 + ...) + (w)$ is a simple arc in Γ with $b\Gamma^* = (w)$.

If (ϱ, θ) are polar coordinates and θ_{ni} are the real numbers $\theta_{ni} = 2^{-n}(2\pi i)$, $(i = 0, 1, ..., 2^n)$, let σ_{ni} be the sets $\sigma_{ni} = [1 - n^{-1} \leqslant \varrho \leqslant 1,\ \theta_{ni} \leqslant \theta \leqslant \theta_{n,i+1}]$. For each $n = 1, 2, ...$, we can successively determine an index $i = i(n)$, such that $\sigma_{ni}\beta_k \neq 0$ for infinitely many integers k, and $\sigma_{ni} \supset \sigma_{n+1,i'}$, $i = i(n)$, $i' = i(n+1)$. Thus the arcs $I_n = [\theta_{ni}, \theta_{n,i+1}]$, $I_{n+1} \subset I_n$, determine a point $w_0 \in \Gamma^*$. We can replace the arcs I_n by somewhat larger arcs $I'_n = (\theta_n, \theta'_n)$ such that $\prod I'_n = (w_0)$, $I'_{n+1} \subset (I'_n)^0$ and also such that both θ_n, θ'_n are images under τ of ends η_n, $\eta'_n \in \{\eta\}$ for every n. Then $[(\eta_n, \eta'_n),\ n = 1, 2...]$ is a sequence defining a prime end ω and ω is certainly reached as it is immediately verified. Thus, the collection I is not empty.

We have now to prove that, if ω_1, ω_2 are any two prime ends both reached by $[\beta_k]$, then all prime ends ω of one (at least) of the two complementary intervals (ω_1, ω_2), (ω_2, ω_1) of $\{\omega\}$ (in the given ordering Ω') are reached by $[\beta_k]$. Obviously, if all prime ends $\omega \in \{\omega\}$ are reached the statement is proved. In the contrary case there is at least one prime end, say $\omega_3 \in (\omega_2, \omega_1)$, which is not reached.

Let ω_4 be any prime-end $\omega_4 \in (\omega_1, \omega_2)$. Let $[l_{in}, b_{in}]$ be any f-system relative to ω_i, $i = 1, 2, 3, 4$. We can also define a continuum $B = b' + b_3 + b_4$, $b_3 = \sum b_{3n} + E_{\omega_3}$, $b_4 = \sum b_{4n} + E_{\omega_4}$, where b' is a simple arc joining the first end-points of b_{31} and b_{41}, and not having other points in common with b_{31} and b_{41}. The continuum B separates the prime ends ω_1 and ω_2 in α (i.e., the systems $[l_{1n}, b_{1n}]$, $[l_{2n}, b_{2n}]$ from both of which finitely many arcs l_{in}, b_{in}, $i = 1, 2$, may be suppressed). Since ω_1, ω_2 are both reached, we have $l_{2n}\beta \neq 0$, $l_{3n}\beta \neq 0$ for all n (here too finitely many arcs l, b may be suppressed). Hence $l_{2n}\beta_{k'_n} \neq 0$, $l_{3n}\beta_{k''_n} \neq 0$, for each n and some k'_n, k''_n, and we have k'_n, $k''_n \to \infty$ as $n \to \infty$, since for each k we have $\{M_k, \alpha^*\} > 0$, $M_k = \beta_1 + ... + \beta_k$. As a consequence, if $\underline{m}_n$, $\overline{m}_n$ denote the min and max of k'_n, k''_n, and S_n denotes the set $S_n = \sum \beta_t$ where the sum ranges over all $\underline{m}_n \leqslant t \leqslant \overline{m}_n$, then $S_n B \neq 0$.

Let w_n be any point of $S_n B$, thus $w_n \in \beta_{m_n}$, $\underline{m}_n \leqslant m_n \leqslant \overline{m}_n$, $\underline{m}_n \to \infty$ as $n \to \infty$, and each point w of accumulation of the sequence $[w_n]$ belongs to H. Since each set $M'_m = b' + b_{31} + \ldots + b_{3m} + b_{41} + \ldots + b_{4m}$ is interior to α and $\{M'_m, \alpha\} > 0$, each point w_n must belong to an arc b_{3m}, or b_{4m}, with $m = m(n) \to \infty$ as $n \to \infty$. Let $\bar{m}$ be large enough so that all arcs $l_{j, \bar{m}+r}$, $r = 1, 2, \ldots$, are separated by $l_{j\bar{m}}$ from β_1, $j = 3, 4$. Suppose, if possible, that the first case $w_n \in b_{3m}$, $m = m(n)$, happens for infinitely many n. Then for the same integers n (with $m(n) > \bar{m}$) we have $w_n \in b_{3m}$, $b_{3m}\beta \neq 0$, and finally $l_{3n}\beta \neq 0$ for all $\bar{m} \leqslant n \leqslant m(n) - 1$. In conclusion $l_{3n}\beta \neq 0$ for all $n \geqslant \bar{m}$, a contradiction, because ω_3 is not reached. Therefore, for all n large enough, $w_n \in b_{4m}$, $m = m(n)$, $\beta b_{4m} \neq 0$, and also $l_{4n}\beta \neq 0$ for all $m \leqslant n \leqslant m(n) - 1$. In conclusion $l_{4,n}\beta \neq 0$ for all $n \geqslant \bar{m}$ and ω_4 is reached. In addition each point of accumulation w of the sequence $[w_n]$ belongs to E_{ω_1}. This proves that all $\omega \in (\omega_1, \omega_2)$ are reached. Thus it is also proved that I is either an open, or a closed interval, or a single element.

Let us prove that H is a continuum. Let w_0 be any point of H and $[w_n]$ any sequence such that $w_n \to w_0$, $w_n \in \beta_{m(n)}$, $m = m(n) \to \infty$ as $n \to \infty$. We can also suppose $m(n) \leqslant m(n+1)$. Let $H_n = \sum \beta$, where $\sum$ ranges over all $m(n) \leqslant t \leqslant m(n+1)$. Then $w_0 \in \liminf H_n$, $H = \limsup H_n = \limsup \beta_n$. As a consequence, by ZORETTI's theorem, H is a continuum.

Let us observe that, if ω is reached, then each point $w \in E^{(0)}_\omega$ belongs to H, i.e. $E^{(0)}_\omega \subset H$. Indeed, by no. **2**, there is at least one fundamental sequence $[l_n]$ such that w_n is limit point of $[l_n]$ and, since diam $l_n < n^{-1}$, $l_n\beta_k \neq 0$, then $w \in \limsup \beta_k$, i.e. $w \in H$. As a consequence, for each end $\eta \in I$, i.e. reached by $[\beta_k]$, we have $w_\eta \in H$. This last result implies that, if ω is reached and ω is interior to the interval I, then $E_\omega \subset H$. Indeed each point $w \in E_\omega$ is a point of accumulation of points w_η relative to ends η of a neighborhood of ω [in $\{\omega\}$] and therefore $w_\eta \in H$ and $w \in H$. Analogously if $\omega'[\omega'']$ is the first [second] end-element of I, then $E''_{\omega'} \subset H$, $E'_{\omega''} \subset H$. Thus $I(\omega', \omega'') \subset H$.

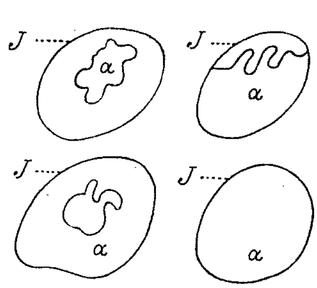
Finally I is a closed interval (or a single element). Indeed, if $[l_n]$ is a fundamental sequence relative to $\omega'[\omega'']$, then there is an $\bar{n}$ such that $r_{\bar{n}}\beta_1 = 0$. Hence, for all $n \geqslant \bar{n}$, l_n separates β_1 from all ends $\eta \in (\eta_n, \eta'_n)$, and, since all ends η'_p, $p > n[\eta_p, p > n]$, are reached by $[\beta_k]$, then $l_n\beta \neq 0$. Thus $l_n\beta \neq 0$ for all $n \geqslant \bar{n}$, i.e. $\omega'[\omega'']$ is reached.

Let us observe that, if a point $w \in H$, then, by repeating the reasoning used at the beginning, we can prove that w belongs to a set E_ω relative to a prime end ω reached by $[\beta_k]$. Thus $I(\omega', \omega'') \subset H \subset I^*(\omega', \omega'')$. Thereby (i) is proved.

5. – A first extension to sets α open in a simple Jordan region.

Let J be a closed simple JORDAN region of the w-plane E_2, $w = (u, v)$, and let α be any connected set open in J, $\alpha \subset J$. Then α, $\alpha^* \subset J$, $\alpha\alpha^* \subset J^*$, and the set $F = \mathcal{F}(\alpha) = \alpha^* - \alpha^*\alpha = \bar{\alpha} - \alpha$ (boundary of α in J) is a closed, bounded set; hence the components of F are continua.

Let γ be a component of F. We will suppose in this section that γ is the only component of F. Thus the definitions of end-cut of α relative to a point $w \subset \gamma$, of cross-cut of α relative to two points $w_1, w_2 \in \gamma$, of point $w \in \gamma$ accessible from α, of *end* η *of* γ *in* α, of *prime end* ω *of* γ *in* α, remain unchanged as in no. 2; the statements that there are points $w \in \gamma$ accessible from α and that they are everywhere dense in γ remain unmodified, as well as the property of separation of four ends η of γ in α; only it may happen here that there is a cross-cut, $c = b_1 + c_0 + b_1'$ in α, relative to two points w_1, $w_2 \in \gamma$ dividing α in two parts one of which contains no end-cut b relative to points $w \in \gamma$. In this case we can suppose b_1, b_2 to be subarcs of J^* defining two ends η_1, η_2 of γ in α and the ends η and prime ends ω of γ in α can be linearly ordered (as the points of a closed interval, namely the interval $[\eta_1, \eta_2]$). Let us denote by Ω_1, Ω_2 the corresponding two linear orderings of the collections $\{\eta\}_{\gamma,\alpha}$, $\{\omega\}_{\gamma,\alpha}$ of the ends η and prime ends ω of γ in α. [In the first and third illustration $\{\omega\}_{\gamma,\alpha}$ can be cyclically ordered (orderings Ω_1', Ω_2'); in the second illustration $\{\omega\}_{\gamma,\alpha}$ can be linearly ordered (orderings Ω_1, Ω_2). The case where γ is a single point (fourth illustration) is exceptional and trivial, because the collections $\{\eta\}_{\gamma,\alpha}$, $\{\omega\}_{\gamma,\alpha}$ contain each only one single element.]



The extension of all considerations of nos. 2, 3, 4 to each component γ of the set $\mathcal{F}(\alpha)$ does not offer difficulties provided $\mathcal{F}(\alpha)$ has only finitely many components and α is connected and open in a closed simple JORDAN region J, $\alpha \subset J$.

6. – The general case.

All the considerations above hold even in cases where F has more than one component γ, more exactly they hold, e.g., for each component γ of F such that $\{\gamma, F - \gamma\} = \delta > 0$ (if any): in particular for each component γ of F if F has only a finite collection of components.

Let us now suppose α to be any bounded connected set $\alpha \subset J$, open in a simple, closed JORDAN region J. Then the collection $\{\gamma\}_\alpha$ of all distinct components γ of F may contain infinitely many elements, even uncountably many. Necessarily there are points $w \in F$ which are accessible from α and these points are everywhere dense in F, but there may be some components γ whose points are all inaccessible from α [**14**, p. 162, second exercise].

Let us recall here the following properties of separation of the components γ of F, where by l is meant either a sinple closed curve, or a sinple arc whose end points are in J^* [**9**].

(i) Given any two distinct γ, $\gamma' \in \{\gamma\}_\alpha$ (if $\{\gamma\}_\alpha$ contains more than one element), then there exists $l \subset \alpha$ separating γ and γ' in J, and, given $\varepsilon > 0$, we can also suppose $\delta < \varepsilon$, where $\delta = \max\{(w), \gamma\}$ for all $w \in l$ [or $\delta = \max\{(w), \gamma'\}$ for all $w \in l$].

(ii) Given any $\gamma \in \{\gamma\}_\alpha$ there exists a sequence l_n, $n = 1, 2, \ldots$, such that: (1) $\delta_n \to 0$ where $\delta_n = \max\{(w), \gamma\}$, $w \in l_n$; (2) l_{n+1} separates l_n and γ in J; (3) for any $\gamma' \in \{\gamma\}_\alpha$, $\gamma' \neq \gamma$ (if any), there is an $\bar{n}$ such that l_n separates γ and γ' in J for all $n \geqslant \bar{n}$.

(iii) There is a sequence $[l]_n$, $n = 1, 2, \ldots$, of finite systems $[l]_n$ of disjoint l such that: (1) $[l]_n \subset [l]_{n+1}$; (2) given any $\gamma \in \{\gamma\}_\alpha$ there is a sequence l_n, $n = 1, 2, \ldots$, $l_n \in [l]_n$, satisfying (ii).

Let γ be any component of F, i.e. $\gamma \in \{\gamma\}_\alpha$. We shall define a new set $A = A(\alpha, \gamma)$ open in J, as follows. If γ is the only component of F, let $A = \alpha$. Otherwise, for each $\gamma' \in \{\gamma\}_\alpha$, $\gamma' \neq \gamma$, let us denote by $\beta' = \beta'(\gamma', \gamma)$ the set of all points $w \in J$ which are separated by γ' from γ in J. The set β' may be empty and, if not, is certainly open in J and not necessarily connected. Let

$$A = A(\alpha, \gamma) = \alpha + \sum (\gamma' + \beta'),$$

where the sum is extended over all elements $\gamma' \in \{\gamma\}_\alpha$, $\gamma' \neq \gamma$. The following statement is essentially known.

(iv) The set $A = A(\alpha, \gamma) \subset J$ is open in J, is connected and $AA^* \subset Q^*$, $\mathcal{F}(A) = \bar{A} - A = A^* - AA^* = \gamma$ [**9**].

Now the set A is open in J and its boundary $\mathcal{F}(A)$ in J has only one component, the component γ. Therefore, we can define, as in no. **4**, the collections $\{\eta\}_{\gamma,A}$, $\{\omega\}_{\gamma,A}$ of the ends η and prime ends ω of γ in the set A [not in α]. In such a way, for each component γ of the set $F = \bar{\alpha} - \alpha$, we have defined a complete collection of ends η and prime ends ω, which we may denote for the sake of brevity, «with respect to α», but which are in the larger set $A = A(\alpha, \gamma)$. And this is done for each γ, even for those whose points are all inaccessible from α. We recall here the further statement:

(v) If J is any closed Jordan region, $\alpha \subset J$ a connected set open in α, F the boundary $F = \bar{\alpha} - \alpha$ of α in J, $\gamma \in \{\gamma\}_\alpha$ any component of F, w_0 any point of γ accessible from the set $A = A(\alpha, \gamma)$, $b \subset A + (w_0)$, $b\gamma = (w_0)$, any arc defining an end η of γ in A, $\eta \in \{\eta\}_{\gamma, A}$. Then: (1) $b\alpha$ is a non-empty set, open in b and dense in w_0; (2) if l_n, $(n = 1, 2, \ldots)$, is any sequence relative to γ as in (ii), then there exists an n_0 such that $l_n b \neq 0$ for all $n \geqslant n_0$; (3) For any sequence $[w_n]$ of points $w_n \in l_n b$, $n \geqslant n_0$, we have $w_n \to w_0$ as $n \to \infty$ [**5**, (A)].

7. – Contours of a path surface.

Let (T, Q): $p = T(w)$, $w \in J$, be any continuous mapping (surface) from the simple closed JORDAN region J of the w-plane E_2, $w = (u, v)$, into the p-space E_3, $p = (x, y, z)$. Let $f(p)$, $p \in E_3$, be any real single-valued continuous function in E_3. For any real t, $-\infty < t < +\infty$, let $C = C(t)$, $D^+ = D^+(t)$, $D^- = D^-(t)$, be the sets of all points $w \in J$, where $f[T(w)] = t$, or $> t$, or $< t$, respectively. Since $f[T(w)]$, $w \in J$, is continuous in J, C is closed, and D^+, D^- are open in J (or empty). In any case we have $C \supset \mathcal{F}(D^+) = \bar{D}^+ - D^+$, $C \supset \mathcal{F}(D^-) = \bar{D}^- - D^-$. If $M(t)$ is the subset of E_3 where $f(p) = t$, then we have also $T(C) \subset M$.

Note I. In elementary cases C is simply the contour (a single line, or finite system of lines) corresponding to the value t (level). In the general conditions above it may happen that the two sets $\mathcal{F}(D^+)$, $\mathcal{F}(D^-)$ do not coincide and that C has interior points, besides the customary complications of the boundaries of open sets. Thus $T(C)$ is a general closed set of E_3. As we approach C from D^-, or from D^+, we can say that we approach the «lower border, or the upper border of the contour C» but we will not attempt to introduce a terminology which is only suggested as an help to the reader.

Note II. It is not restrictive to suppose that f is defined only on the compact set $[S]$, even in the hypothesis, we shall consider later (no. **10**) that f satisfies a LIPSCHITZ condition $|f(p) - f(p')| \leqslant K|p - p'|$. Indeed it is possible to extend the domain of definition of f to the whole space E_3 in such a way that continuity holds in E_3, or the LIPSCHITZ condition above holds in E_3 [E. J. MCSHANE, **13**].

8. – The generalized length.

In the conditions of no. **7** let $\{\alpha\} = \{\alpha\}_t$ be the collection of all components α of D^-; hence each α is a bounded connected subset of J, open in J. For each $\alpha \in \{\alpha\}$, let $\{\gamma\}_\alpha$ be the collection of all components γ of the boundary

$\mathcal{F}(\alpha) = \bar{\alpha} - \alpha$ of α in J; hence each γ is a subcontinuum of J. For each $\alpha \in \{\alpha\}$ and $\gamma \in \{\gamma\}_\alpha$, let $A = A(\alpha, \gamma)$ be the set defined at no. **6**, and let $\{\eta\}_{\gamma,A}$, $\{\omega\}_{\gamma,A}$ be the collection of all ends, and prime-ends of γ in A. We shall suppose that an ordering Ω has been chosen for the collections $\{\eta\}_{\gamma,A}$, $\{\omega\}_{\gamma,A}$, [$\Omega = \Omega_1, \Omega_2$, or $= \Omega_1', \Omega_2'$].

Let $[\eta] = [\eta_1, \eta_2, \dots, \eta_n]$ be any finite subfamily of ends $\eta \in \{\eta\}_{\gamma,A}$ ordered as they are in $\{\eta\}_{\gamma,A}$, let us consider the corresponding set $[w_\eta] = [w_1, w_2, \dots, w_n]$, $w_i = w_{\eta_i} \in \gamma$, $(i = 1, 2, \dots, n)$, of points of γ, each w_i being accessible from A [not necessarily from α], and let S be the sum $S = \sum |T(w_i) - T(w_{i+1})|$, where $\sum$ is extended over all the values $i = 1, 2, \dots, n-1$, if Ω is one of the orderings Ω_1, Ω_2, and $i = 1, 2, \dots, n$, $w_{n+1} = w_1$, if Ω is one of the (cyclic) orderings Ω_1', Ω_2'. Finally let $\lambda = \operatorname{Sup} S$, where Sup is taken for all finite ordered subfamilies $[\eta] \subset \{\eta\}_{\gamma,A}$. We shall denote λ also by the more complete notation $\lambda(\gamma, \alpha)$. We have $0 \leqslant \lambda \leqslant +\infty$. The number

$$l(t) = l(t; T, J) = \sum_{\alpha \in \{\alpha\}_t} \sum_{\gamma \in \{\gamma\}_\alpha} \lambda(\gamma, \alpha) \tag{2}$$

shall be denoted by the *generalized length* of the image of $\mathcal{F}(D^-)$ under T. Analogous definition holds for $\mathcal{F}(D^+)$. The number $l(t)$ could be denoted as the (generalized) length of the lower (upper) border of the image of the contour $T(C)$ of level t.

Let us observe explicitely that in (2) the sum with respect to γ may be uncountable. It can also be observed that if a component $\gamma \in \{\gamma\}_\alpha$ is a single point, then the collection $\{\eta\}_{\gamma,A}$ contains a single element and, according to the definition above, we have $\lambda = 0$. Thus all components γ which are single points have no influence on the value of $l(t)$. The following statements are proved in [**5**, (A)].

(i) $\lambda(\gamma, \alpha) = 0$ if and only if γ is a continuum of constancy for T in J. In particular, $\lambda = 0$ for all γ which are single points.

(ii) $l(t) < +\infty$ implies that all numbers $\lambda(\gamma, \alpha)$ are finite and that at most for a countable subfamily of sets γ we have $\lambda(\gamma, \alpha) > 0$.

(iii) $\lambda(\gamma, \alpha) < +\infty$ implies that, for each prime-end $\omega \in \{\omega\}_{\gamma,A}$, T is constant on the set E_ω.

9. – Reduction of the generalized length to ordinary length.

(i) If for some γ, α we have $\lambda(\gamma, \alpha) < +\infty$, then T is constant on each continuum E_ω, $\omega \in \{\omega\}_{\gamma,A}$ and, if $\{\omega\}_{\gamma,A}$ is thought to be ordered in one of the orderings Ω, then the equation c: $p = T(E_\omega)$, $\omega \in \{\omega\}_{\gamma,A}$, is a continuous curve (closed, or open) and $\lambda(\gamma, \alpha)$ is the JORDAN length of c [**5**, (A)].

By using mappings defined as in no. 2, (iii) we can have representations of c as a continuous mapping from a simple arc (closed, or open).

As a consequence of no. **8**, (ii), (iii), and (i) above, we have that, whenever $l(t) < +\infty$, then $T(C)$ is the countable sum of continuous closed curves c (open or closed) and of a set h (may be uncountable) of single points, and that $l(t)$ is the sum of the JORDAN lengths of the curves c. Here the set h is the image under T of a set whose components are disjoint continua of constancy for T in J.

10. – Properties of $l(t)$.

(i) If (T, J) is a continuous mapping from a closed simple JORDAN region $J \subset E_2$ into the p-space E_3, if $f(p)$, $f_n(p)$, $(n = 1, 2,...; f_n > f)$, are real single-valued continuous functions in E_3, and $f_n \rightrightarrows f$ uniformly in E_3, if $l(t)$, $l_n(t)$ are the corresponding functions defined in no. **8**, then $l(t) \leqslant \underline{\lim}\, l_n(t)$ as $n \to \infty$, $-\infty < t < +\infty$ [**5**, (A)].

(ii) If (T, J) is defined as above, if $f(p)$ is any real single-valued continuous function in E_3, then $l(t) \leqslant \underline{\lim}\, l(\tau)$ as $\tau \to t - 0$ [**5**, (A)].

(iii) If (T, J), (T_n, J), $n = 1, 2,...,$ are continuous mappings such that $T_n \rightrightarrows T$ uniformly in J, then $l(t) \leqslant \underline{\lim}_{\tau \to t-0}\, \underline{\lim}_{n\to\infty}\, l_n(\tau)$ [**5**, (A)].

(iv) Is (T, J) is defined as above, then the function $l(t)$, $[0 \leqslant l(t) \leqslant +\infty$, $-\infty < t < +\infty]$ is measurable [**5**, (A)].

(v) If (T, J): $p = T(w)$, $w \in J$, is any continuous mapping as in (i) and $L(J, T)$ denotes the LEBESGUE area of (T, J), if $f(p)$, $p \in E_3$, is any real single-valued continuous function in E_3 such that $|f(p) - f(p')| \leqslant K|p - p'|$ for all $p, p' \in E_3$, $(K > 0$ a constant), then

$$KL(J, T) \geqslant \int_{-\infty}^{+\infty} l(t)\, dt\,. \tag{3}$$

11. – Fréchet equivalence.

Let (T, J): $p = T(w)$, $w \in J$, (T', J'): $p = T'(w)$, $w \in J'$, be two FRÉCHET equivalent mappings from simple closed JORDAN regions J, J' of the w-plane into the p-space E_3, $p = (x, y, z)$. Then, for every integer n, there is a homeomorphism H_n: $w' = H_n(w)$, $w = H_n^{-1}(w')$, $w \in J$, $w' \in J'$, between J and J' such that $|T(w) - T'[H_n(w)]| < n^{-1}$ for all $w \in J$, $(n = 1, 2,...)$. It is usual to say that T and T' are representations of the same FRÉCHET surface S. A first well known implication is that $[S] = T(J) = T'(J') \subset E_3$.

Let G, G' be the collections of all maximal continua $g \subset J$, $g' \subset J'$ on which T, or T', is constant. Both G, G' are upper-semicontinuous decompositions of J, J' in disjoint continua g, g' [9, p. 38; 18, VII]; that is, if $[g_n]$ is a sequence of continua $g_n \in G$ and $\liminf g_n \neq 0$, then $\limsup g_n = K \subset g$, $g \in G$, where k, by ZORETTI's theorem (no. 1) is a continuum (analogously for G').

The FRÉCHET equivalence between T and T' implies that there exists, between the collections G and G' a one-one correspondence $\mathcal{C}$: $g' = \mathcal{C}(g)$, $g = \mathcal{C}^{-1}(g')$, $g \in G$, $g' \in G'$, with the following properties: (1) if $g' = \mathcal{C}(g)$ then $T(g) = T'(g')$; (2) $\mathcal{C}$ is semicontinuous; i.e., if $[g_n]$ is a sequence of continua $g_n \in G$ such that $\liminf g_n \neq 0$, and $\liminf g_n \subset \limsup g_n \subset g_0$, $g_0 \in G$, $g_0' = \mathcal{C}(g_0)$, $g_0' \in G'$, $g_n' = \mathcal{C}(g_n)$, $g_n' \in G'$, then $\limsup g_n' \subset g_0'$; the same statement holds by exchanging G and G'; (3) if $C = \sum g$, $C' = \sum g'$, $g' = \mathcal{C}(g)$, are two sums of corresponding continua, $g \in G$, $g' \in G'$, then C' is closed, open, open in J', connected, a continuum if and only if C is closed, open, open in J, connected, a continuum, respectively.

12. – Let T, T' be as in no. **11**, let $f(p)$ be any continuous function in E_3, and $t_1 = \min f[T(w)] = \min f[T'(w')]$ for all $w \in J$, $w' \in J'$, $p \in [S]$. Analogously for $t_2 = \max f[T(w)]$, $w \in J$. Obviously $l(t) = 0$ for all $t < t_1$ and $t > t_2$.

For any $t_1 < t < t_2$ we denote by $C \subset J$, $C' \subset J'$, the closed sets of the points w, w' such that $f[T(w)] = t$, $f[T'(w')] = t$. Since $f[T(w)]$, as well as $T(w)$, is constant on each set $g \in G$, $C = \sum g$ is the sum of a collection of continua $g \in G$. The same holds for C' and, by no. **11**, (1), we have $C = \sum g$, $C' = \sum g'$, $g' = \mathcal{C}(g)$. Analogous statement holds for all components α of $J - C$, α' of $J' - C'$, and, as before, $\alpha = \sum g$, $\alpha' = \sum g'$, $g' = \mathcal{C}(g)$. Thus, by no. **11**, (3), for each connected set α open in J the corresponding set α' is connected and open in J' and viceversa. Thus the mapping $\mathcal{C}$ implies a one-one correspondence between the collections $\{\alpha\}$, $\{\alpha'\}$ of the components of $J - C$ and $J' - C'$.

If $\alpha \in \{\alpha\}$, $\alpha' \in \{\alpha'\}$ are corresponding components of $J - C$ and $J' - C'$, let us consider the closed sets $F = \bar{\alpha} - \alpha$, $F' = \bar{\alpha}' - \alpha'$, $F \subset C$, $F' \subset C'$, and the collections $\{\gamma\}_\alpha$, $\{\gamma'\}_{\alpha'}$, of all components (continua) of the closed, bounded sets F, F'.

Let $\gamma \in \{\gamma\}_\alpha$ and Γ be the set $\Gamma = \sum g$ where the sum ranges over all $g \in G$ such that $g\gamma \neq 0$. We have $\Gamma \supset \gamma$ and we have to prove first that Γ is closed. If w_0 is any point of accumulation of Γ then there is a sequence of points $w_n \to w_0$, $w_n \in g_n$, $g_n \subset \Gamma$, $g_n\gamma \neq 0$. Let v_n be any point $v_n \in g_n\gamma$;

hence $v_n \in \gamma$ and there is at least a point of accumulation $v_0 \in \gamma$. Since $w_n \to w_0$, we have $\liminf g_n \neq 0$, and, therefore, $\limsup g_n \subset g$, $g \in G$, and $v_0 \in g$; hence $g\gamma \neq 0$, $g \subset \Gamma$, $w_0 \in \Gamma$. This proves that Γ is closed. Since Γ is evidently connected, Γ is a continuum.

Note that $\Gamma\alpha = 0$. Indeed, if $\Gamma\alpha \neq 0$, and w is any point $w \in \Gamma\alpha$, then $w \in g$, $g \in G$ and $g\alpha \neq 0$, $g\Gamma \neq 0$; hence $g \subset \alpha$, $g \subset \Gamma$, $g \subset \alpha\Gamma$. But $g\gamma \neq 0$, therefore $\gamma\alpha \neq 0$, what is impossible. Thus we have proved $\Gamma\alpha = 0$.

If $\gamma_1, \gamma_2 \in \{\gamma\}_\alpha$, and Γ_1, Γ_2 are the corresponding continua $\Gamma_1 \supset \gamma_1$, $\Gamma_2 \supset \gamma_2$ we can prove that $\Gamma_1\Gamma_2 = 0$. First, by no. **6**, (i), there is a simple polygonal line $l \subset \alpha$ separating γ_1 and γ_2 in J. Now $l \subset \alpha$, $\Gamma_1\alpha = 0$, $\Gamma_2\alpha = 0$, hence $l\Gamma_1 = 0$, $l\Gamma_2 = 0$ and so l separates Γ_1, Γ_2 in J. This proves $\Gamma_1\Gamma_2 = 0$.

Therefore we have a collection $\{\Gamma\}_\alpha$ of disjoint continua Γ each containing one and only one continuum $\gamma \in \{\gamma\}_\alpha$. Analogously we have a collection $\{\Gamma'\}_\alpha$ of disjoint continua Γ' each containing one and only one continuum $\gamma' \in \{\gamma'\}_{\alpha'}$.

If $\gamma \in \{\gamma\}_\alpha$ and $\Gamma \supset \gamma$ is the corresponding continuum $\Gamma \in \{\Gamma\}_\alpha$ then $\Gamma = \sum g$ and we have a continuum $\bar{\Gamma}' = \sum g'$ which is the sum of all continua $g' \in C_1'$, $g' = \mathcal{C}(g)$, $g \subset \Gamma$ (no. **11**, (3)). We shall prove now that each continuum $g' \subset \bar{\Gamma}'$ contains at least one point of a $\gamma' \in \{\gamma'\}_{\alpha'}$. Let $g' \subset \bar{\Gamma}'$, $g = \mathcal{C}^{-1}(g')$, $g \subset \Gamma$; therefore $g\gamma \neq 0$ and we can take a point $w \in g\gamma$. Then there is a sequence of points $w_n \in \alpha$, $w_n \to w$. If $w_n \in g_n$, $g_n \in G$, then $g_n \subset \alpha$ and, because of $w_n \to w$, $\liminf g_n \neq 0$. Hence, by [**11**, (3)], if $g_n' = \mathcal{C}(g_n)$, then $\limsup g_n \subset g$, $g\alpha = 0$, $\limsup g_n' \subset g'$, and $g_n' \subset \alpha'$, $g'\alpha' = 0$. Therefore g' contains at least a point of $\bar{\alpha}' - \alpha'$, that is at least one point of a $\gamma' \in \{\gamma'\}_{\alpha'}$, hence $\bar{\Gamma}'\gamma' \neq 0$. But two different γ', $\gamma'' \in \{\gamma'\}_{\alpha'}$ are separated by a polygonal line $\pi' \subset \alpha'$ in J' and $\Gamma'\alpha' = 0$, hence we can have $\Gamma'\gamma' \neq 0$ for only one $\gamma' \in \{\gamma'\}_{\alpha'}$. Since for each $g' \subset \bar{\Gamma}'$, we have $g'\gamma' \neq 0$ we obtain $\bar{\Gamma}' \subset \Gamma'$, where Γ' is the continuum relative to γ'. Now if $\bar{\Gamma}$ is the set corresponding to Γ' in J, we have $\Gamma \subset \bar{\Gamma}$ and, by repeating the above argument, also $\bar{\Gamma} \subset \Gamma$; hence $\Gamma = \bar{\Gamma}$ and $\bar{\Gamma}' = \Gamma'$. This proves that between the collections $\{\Gamma\}$, $\{\Gamma'\}$ there is a one-one correspondence, and therefore the same one-one correspondence is obtained between the collections $\{\gamma\}_\alpha$, $\{\gamma'\}_{\alpha'}$.

13. – Some examples.

Under the conditions of nos. **11** and **12**, let $\alpha \in \{\alpha\}$, $\alpha' \in \{\alpha'\}$ be corresponding components of $J - C$ and $J' - C'$ and $\gamma \in \{\gamma\}_\alpha$, $\gamma' \in \{\gamma'\}_{\alpha'}$ corresponding components of $\mathcal{F}(\alpha)$, $\mathcal{F}(\alpha')$. Let $A = A(\alpha, \gamma)$, $A' = A'(\alpha', \gamma')$ and $\{\eta\}_{\gamma,A}$, $\{\eta'\}_{\gamma',A'}$ be the collections of all ends η of γ in A and the ends η' of γ' in A'. Let $\{\omega\}_{\gamma,A}$, $\{\omega'\}_{\gamma',A'}$ be the collections of the prime-ends of γ in A and of γ' in A'.

Not necessarily is there a one-one correspondence between the collections $\{\eta\}$ and $\{\eta'\}$ or $\{\omega\}$ and $\{\omega'\}$. Let us consider the following examples.

I. Let (r, θ), (ϱ, ω) be polar coordinates of poles $(0, 0)$ in the xy and uv-planes; let T: $r = \varrho$, $\theta = \omega$, $z = 1 - \varrho$; T': $z = 1$, $r = 0$ if $0 \leqslant \varrho \leqslant 2^{-1}$; $z = 2 - 2\varrho$, $r = 2\varrho - 1$, $\theta = \omega$, if $2^{-1} \leqslant \varrho \leqslant 1$, be the given continuous mappings from the 2-cell $J = J' = [\varrho \leqslant 1]$ into the p-space E_3, $p = (x, y, z)$; hence $T \sim T'$. On the other hand, if $f(p) = z$ and $t = 1$, then C is the only point $(0, 0)$, C' is the disc $[\varrho \leqslant 2^{-1}]$ and $\alpha = [0 < \varrho \leqslant 1]$, $\gamma = [\varrho = 0]$, $\alpha' = [2^{-1} < \varrho \leqslant 1]$, $\gamma' = [\varrho = 2^{-1}]$, $A = \alpha$, $A' = \alpha'$. Thus $\{\eta\}_{\gamma,A}$ has a single element, while $\{\eta'\}_{\gamma',A'}$ has infinitely many elements (corresponding to the points $w \in \gamma'$).

II. Let $J = [\varrho \leqslant 1]$, $\Gamma = [\varrho \leqslant 2^{-1}]$, $w_n = [\varrho = 2^{-1} + 4^{-1}n^{-1},\ \omega = n\pi]$, $n = 1, 2, \ldots$, $\Gamma_0 = \Gamma + \sum w_n$. Then Γ_0 is a closed set, sum of Γ and of the points w_n. Let $\delta(w) = \{(w), \Gamma_0\}$, and T: $p = T(w)$, $w \in J$, be the continuous mapping defined by $x = 0$, $y = 0$, $z = 1 - \delta(w)$. If $f(p) = z$ and $t = 1$ we have $C = \Gamma_0$; hence there is only one $\alpha = J - \Gamma_0$, and we have $\mathcal{F}(\alpha) = \bar{\alpha} - \alpha = \Gamma^* + \sum w_n$. Thus $\gamma = \Gamma^*$ is a component of $\mathcal{F}(\alpha)$ and all points $w \in \gamma$ are accessible from α. Let $J' = J$, $\Gamma' = \Gamma$, $W_n = [\varrho = \varrho_n = 2^{-1} + 4^{-1}n^{-1}$, $(n-1)\pi + n^{-1} \leqslant \omega \leqslant (n+1)\pi - n^{-1}]$, $n = 1, 2, \ldots$, and $\Gamma_0' = \Gamma' + \sum W_n$. We can define, in an elementary way, a continuous mapping τ: $w = \tau(w')$ from $J' - \Gamma'$ onto $J - \Gamma$, which maps W_n onto w_n. We can also suppose that τ is one-one between $\alpha = J - \Gamma_0$ and $\alpha' = J' - \Gamma_0'$, and maps each δ-neighborhood of Γ onto the δ-neighborhood of Γ' and viceversa. For instance, let τ: $\varrho = \varrho'$, $\omega = l_n(\varrho')\varphi_n(\omega') + [1 - l_n(\varrho')]\psi_n(\omega')$ for all $\varrho_n' \leqslant \varrho' \leqslant \varrho_{n-1}'$, where $\varrho_n' = \varrho_n - 2^{-1}(\varrho_n - \varrho_{n+1})$, $(n = 1, 2, \ldots)$, $\varrho_0' = 1$, where $l_n(\varrho')$ is the function which is linear in both intervals (ϱ_n', ϱ_n), $(\varrho_n, \varrho_{n-1}')$ with $l_n(\varrho_n') = 1$, $l_n(\varrho_n) = 0$, $l_n(\varrho_{n-1}') = 1$, and where $\varphi_n(\omega') = \omega'$, $(n-1)\pi \leqslant \omega' \leqslant (n+1)\pi$; $\psi_n(\omega') = (n-1)\pi + n\pi[\omega' - (n-1)\pi]$ if $(n-1)\pi \leqslant \omega' \leqslant (n-1)\pi + n^{-1}$; $\psi_n(\omega') = n\pi$ if $(n-1)\pi + n^{-1} \leqslant \omega' \leqslant (n+1)\pi - n^{-1}$; $\psi_n(\omega') = (n+1)\pi - n\pi[(n+1)\pi - \omega']$ if $(n+1)\pi - n^{-1} \leqslant \omega' \leqslant (n+1)\pi$.

In such a way τ is completely defined. Let T': $p = T'(w)$, $w \in J'$, be the mapping from J' into E_3 defined by $T' = T$ in Γ', and by $T' = T\tau$ in $J' - \Gamma'$. Since T is continuous on J and constant on Γ^*, T' is continuous in J' and $T' \sim T$, as it can be proved without difficulties. On the other hand $\alpha' = J' - \Gamma_0'$, $\mathcal{F}(\alpha') = \bar{\alpha}' - \alpha' = \Gamma'^* + \sum W_n$, and $\gamma' = \Gamma'^*$ is a component of $\mathcal{F}(\alpha)$. While all points of γ are accessible from α, no point of γ', is accessible from α'. Nevertheless $A(\alpha, \gamma) = J - \Gamma$, $A'(\alpha', \gamma') = J' - \Gamma'$ and the collections $\{\eta\}_{\gamma,A}$, $\{\eta'\}_{\gamma',A'}$ coincide.

III. Let $J = [\varrho \leqslant 1]$, $s = [-1 \leqslant u \leqslant 1,\ v = 0]$, $\Gamma = J^* + s$; hence $J - \Gamma$ is an open set whose two components α_1, α_2 are two open semicircles. Let T:

$p = T(w)$, $w \in J$, be a continuous mapping from J into E_3 which maps Γ into the point $p_0 = (0, 0, 1)$ and each of the open sets α_1, α_2 into $S - p_0$, where S is the sphere $x^2+y^2+z^2=1$. If $f(p) = z$, $t = 1$, we have $C = \Gamma$, $\{\alpha\} = \{\alpha_1, \alpha_2\}$, and $\mathcal{F}(\alpha_i) = \bar{\alpha}_i - \alpha_i = \alpha_i^*$ is a continuum $\gamma_i = \alpha_i^*$, $i = 1, 2$. Thus $A_i = A_i(\alpha_i, \gamma_i) = \alpha_i$ and all points of γ_i are accessible from $A_i = \alpha_i$. Let $J' = [\varrho \leqslant 1]$, s' be the double spiral $s' = [0 \leqslant \varrho < 1,\ \omega = \pm \operatorname{tg}(2^{-1}\varrho\pi)]$, $\Gamma' = J'^* + s'$; hence $J' - \Gamma'$ is an open set whose two components α_1', α_2' are such that $\alpha_1'^* = \alpha_2'^* = \Gamma'$ [9, p. 117]. Let τ: $w = \tau(w')$ be any homeomorphism between α_1' and α_1, and between α_2' and α_2 [9, p. 110] preserving the direction of the rotations. Let T': $p = T'(w)$, $w \in J'$, be the mapping defined by $T'(w) = (0, 0, 1)$ for all $w \in \Gamma'$; $T' = T\tau$ for all $w \in \alpha_1' + \alpha_2'$. Then T' is continuous in J' and $T' \sim T$. We have $C' = \Gamma'$, $\{\alpha'\} = \{\alpha_1', \alpha_2'\}$, $\mathcal{F}(\alpha_i') = \bar{\alpha}_i' - \alpha_i' = \alpha_i^*$, i.e. $\mathcal{F}(\alpha_i')$ is a continuum $\gamma_i' = \alpha_i'^*$, $i = 1, 2$. While all points of γ_i are accessible from α_i, γ_i' presents a prime end ω (not corresponding to any end) with $E_\omega = J'^*$ and no point of $E_\omega = J'^*$ is accessible from α_i nor from $A_i' = \alpha_i'$, $i = 1, 2$.

14. – Generalized prime ends.

We shall use the notations of numbers **7** and **8**. Let $\alpha \in \{\alpha\}$ be a component of $J - C$, $\gamma \in \{\gamma\}_\alpha$ be a component of $\mathcal{F}(\alpha)$ and let $A = A(\alpha, \gamma)$. As usually we denote by $\{\eta\}_{\gamma,A}$, $\{\omega\}_{\gamma,A}$ the families of all ends and prime ends of γ in A. To each ω we have associated a set E_ω and hence we shall denote by $\{E_\omega\}_{\gamma,A}$ the corresponding ordered collection of all sets E_ω, $w \in \{\omega\}_{\gamma,A}$. If an interval (ω', ω'') of prime ends has the property that the mapping $T(w)$ is constant on the set $\sum E_\omega$, where $\sum$ ranges over all $\omega \in (\omega', \omega'')$, then there is a maximal interval having the same property. By *generalized prime end u*, or briefly a *gp*-end, of γ in A we mean, either (i) a maximal (open) interval (ω', ω'') having the above property (and then T is constant on the set $I(\omega_1, \omega_2)$ (no. 2)), or (ii) any prime end ω such that T is constant on $E_\omega^{(0)}$ and is not contained, or is an end element, of any interval (ω', ω'') as in (i). Hence each end η of γ in A, which does not belong to an interval as above, is itself a *gp*-end because $E_\omega^{(0)} = w_\eta$, $\omega = \eta$, and T is certainly constant in the set $E_\omega^{(0)}$ reduced to a single point. In either case (i), or (ii), we set $U = U_u = I(\omega', \omega'')$, or $U = E_\omega^{(0)}$, respectively.

It is obvious that the collection $\{u\}_{\gamma,A}$ of all *gp*-ends of γ in A can be ordered as $\{\eta\}_{\gamma,A}$ and $\{\omega\}_{\gamma,A}$, with the exception of the trivial case where T is constant on γ and then $\{u\}_{\gamma,A}$ has only one element.

For each *gp*-end u, the set U is a continuum of constancy for T; hence $U \subset g$, $g \in G$, and also $U \subset g\gamma$, $gA = 0$.

Let $u \in \{u\}_{\gamma,A}$, $U = U_u$ be the corresponding continuum, $U \subset g_0\gamma$, $g_0 \in G$, and let $p \in E_3$ be the point $p_0 = T(U)$. Let ω be any prime end $\omega \in u$, let $[l_n, b_n]$ be any f-system relative to ω (no. 3). Then $E^{(0)}_\omega \subset \limsup b_n \subset E_\omega$ (no. 3, (i)). In addition we can always choose $[l_n, b_n]$ in such a way that $E^{(0)}_\omega = \limsup b_n$. Since $E^{(0)}_\omega \subset U$, we have $\limsup b_n \subset U$, $T(U) = p_0$. As a consequence, $\{p_0, T(b_n)\} \to 0$, $\operatorname{diam} T(b_n) \to 0$ as $n \to \infty$. If $\delta_n = \{p_0, T(b_n)\} + \operatorname{diam} T(b_n)$, we have $|p(w) - p_0| \leqslant \delta_n$ for all $w \in b_n$, $n = 1, 2, \ldots$, and $\delta_n \to 0$ as $n \to \infty$. Let Γ_n be the set $\sum g$ where $\sum$ ranges over all $g \in G$, $gb_n \neq 0$. Also Γ_n is a continuum, and $\Gamma_n \subset A$, $w_n \in \Gamma_n\Gamma_{n+1}$ and hence $\Gamma_n\Gamma_{n+1} \neq 0$. Finally, since T is constant on each g, we have also $|T(w) - p_0| \leqslant \delta_n$ for all $w \in \Gamma_n$, $n = 1, 2, \ldots$. Let $K = \limsup \Gamma_n$; hence K is closed and since A is bounded, also K is bounded. Also, $K \subset \bar{A}$ since $\Gamma_n \subset A$. If $w_0 \in K$, then there is a sequence $[w_n]$, $w_n \to w_0$, $w_n \in g_n$, $g_n \in G$, $g_n b_m \neq 0$, $m = m(n) \to \infty$ as $n \to \infty$. If w'_n is any point $w'_n \in g_n b_m$, then there is at least a subsequence of $[w'_n]$ with $w'_n \to w'$, $w' \in E^{(0)}_\omega$. Let g_0 be the continuum, $g_0 \in G$, with $w' \in g_0$. Then, because of the upper semicontinuity of G, we have $\limsup g_n \subset g_0$, where $g_0\gamma \neq 0$ and $w_0 \in g_0$. Here $g_0\gamma \neq 0$ implies $g_0\alpha = 0$. Since, by no. 6, any two components, γ, γ' of $\mathcal{F}(\alpha)$ can be separated in J by a polygonal line $l \subset \alpha$, we conclude that $g_0\alpha^* \subset \gamma$. On the other hand w_0 is a limit point of points of α and hence $w_0 \in g_0\alpha^*$, and finally $w_0 \subset \gamma$. Here w_0 is any point of K and this implies $K \subset \gamma$.

Thus we have proved that the sequence $[\Gamma_n]$ defined above is in the conditions of no. 4, (i), and hence K is a continuum and there is an interval $[\omega'_0, \omega''_0]$ with $I(\omega'_0, \omega''_0) \subset K \subset I^*(\omega'_0, \omega''_0)$, where necessarily $\omega \in [\omega'_0, \omega''_0]$ and where it is not excluded that $[\omega'_0, \omega''_0]$ reduces to ω itself and thus the relation $I \subset K \subset I^*$ becomes $E^{(0)}_\omega \subset K \subset E_\omega$.

The relation $|T(w) - p_0| \leqslant \delta_n$ for all $w \in \Gamma_n$ implies $T(K) = p_0$, and consequently, $E^{(0)}_\omega \subset K \subset I^*(\omega', \omega'') \subset g_0$, where $g_0\alpha = 0$, $g_0 \in G$.

15. – We shall use now the notations of nos. **11**, **12**. Let (T, J), (T', J'), $T \sim T'$, be given mappings and let $\alpha \in \{\alpha\}$, $\gamma \in \{\gamma\}_\alpha$, $A = A(\alpha, \gamma)$ be elements relative to T and $\alpha' \in \{\alpha'\}$, $\gamma' \in \{\gamma'\}_{\alpha'}$, $A' = A(\alpha', \gamma')$ the corresponding elements for T'. Let $\{\eta\}_{\gamma,A}$, $\{\omega\}_{\gamma,A}$, $\{u\}_{\gamma,A}$ be the collections of ends, prime ends, gp-ends of γ in A, and $\{\eta'\}_{\gamma',A'}$, $\{\omega'\}_{\gamma',A'}$, $\{u'\}_{\gamma',A'}$ the collections of ends, prime ends, gp-ends of γ' in A'.

(i) *There exists a one-one ordered correspondence between the collections $\{u\}_{\gamma,A}$ and $\{u'\}_{\gamma',A'}$ of the gp-ends of γ in A and of γ' in A', such that if u, u' are corresponding gp-ends, then T is constant on the set $U_u \subset \gamma$, T' is constant on the set $U'_{u'} \subset \gamma'$ and $T(U_u) = T'(U'_{u'})$.*

P r o o f. Let $u \in \{u\}_{\gamma,A}$, and ω be any prime end $\omega \in u$. Let $[l_n, b_n]$ be any f-system relative to ω, such that $E^{(0)}_\omega = \limsup b_n$ (no. 3, (i)), and let $\Gamma_n = \sum g$

be the set $\Gamma_n \subset A$ which is the sum of all $g \in G$ with $gb_n \neq 0$. Thus, if $\Gamma'_n = \sum g'$, $g' = \mathfrak{T}(g)$, also Γ'_n is a continuum, $\Gamma'_n \subset \alpha'$, $\Gamma'_n \Gamma'_{n+1} \neq 0$. Therefore, also the collection $[\Gamma'_n]$ satisfies the conditions of no. 4, (i), and hence if $K' = \lim\sup \Gamma'_n$, there is a closed interval $[\omega_0, \omega'_0]$ in $\{\omega'\}_{\gamma',A'}$ such that $I(\omega_0, \omega'_0) \subset K' \subset I^*(\omega_0, \omega'_0)$, where it is not excluded that $[\omega_0, \omega'_0]$ is reduced to a single prime end ω_0 and then $E^{(0)}_{\omega_0} \subset K \subset E_{\omega_0}$. The relation $|T(w) - p_0| \leqslant \delta_n$ for all $w \in \Gamma_n$ implies $|T'(w) - p_0| \leqslant \delta_n$ for all $w \in \Gamma'_n$, and hence $T(K') = p_0$. Consequently (ω_0, ω'_0) belongs to a gp-end $u' \in \{u'\}_{\gamma',A'}$ and we have $I(\omega_0, \omega'_0) \subset U'$, and also $U' \subset g'_0$, $g'_0 \in G'$. Because of the properties of the mapping $\mathfrak{T}$ we have $g'_0 = \mathfrak{T}(g_0)$.

By the procedure above we have associated to each gp-end $u \in \{u\}_{\gamma,A}$ another gp-end $u' = \sigma(u)$, $u' \in \{u'\}_{\gamma',A'}$. We have to prove that this correspondence does not depend upon the choice of the prime end $\omega \in u$ we have used. Let $\bar{\omega}$ be any other prime end, $\bar{\omega} \in u$, and let $\bar{l}_n$, $\bar{b}_n$, $\bar{\Gamma}_n$, $\bar{\Gamma}'_n \subset A'$, $K' \subset \gamma'$ be the corresponding sets. Then we will have $I(\bar{\omega}_0, \bar{\omega}'_0) \subset K' \subset I^*(\bar{\omega}_0, \bar{\omega}'_0)$, and it is quite possible that the intervals $[\omega_0, \omega'_0]$, $[\bar{\omega}_0, \bar{\omega}'_0]$ are disjoint. It is possible to join the points $l_n b_n$ and $\bar{l}_n \bar{b}_n$ by a polygonal line $\lambda_n \subset A$, such that each point $w \in \lambda_n$ is at a distance $\leqslant \mu_n$ from $I(\omega, \bar{\omega})$, with $\mu_n \to 0$ as $n \to \infty$. Let $\beta_n = b_n + \lambda_n + \bar{b}_n$ and observe that, since T is constant on U (and hence on $I(\omega, \bar{\omega}) \subset U$), and $T(U) = p_0$, the set $T(\beta_n)$ is contained in a sphere of center p and radius a number $\delta_n > 0$ with $\delta_n \to 0$ as $n \to \infty$. Set $B_n = \sum g$ where $\sum$ ranges over all $g \in G$ with $g\beta_n \neq 0$, and $B'_n = \sum g'$, for all $g' = \mathfrak{T}(g)$, $g \subset B_n$. Then $B' \subset A'$, $B'_n B'_{n+1} \neq 0$, and also $\lim\sup B'_n = K^* \subset \gamma'$ by the same reasoning used for the continua Γ_n. Therefore, by no. 4, (i), there is a closed interval $[\omega^*, \omega'^*]$ in $\{\omega'\}_{\gamma',A'}$, such that $I(\omega^*, \omega'^*) \subset K^* \subset I^*(\omega^*, \omega'^*)$, and T' is constant on K^*, $T'(K^*) = p_0$. This proves that both $[\omega_0, \omega'_0]$, $[\bar{\omega}_0, \bar{\omega}'_0]$ are contained in $[\omega^*, \omega'^*]$ and hence in the interval $[\omega', \omega'']$ relative to the same gp-end $u' \in \{u'\}_{\gamma',A'}$. Thus we have proved that the correspondence $u' = \sigma(u)$ we have established does not depend upon the choice of the prime end $\omega \in u$.

Now let $\bar{\omega}'$ be any prime end $\bar{\omega}' \in u' = \sigma(u)$ and let us apply to u' the same procedure above in order to obtain a gp-end $u \in \{u\}_{\gamma,A}$. Let $[\bar{l}'_n, \bar{b}'_n]$ be any f-system relative to $\bar{\omega}'$, $\bar{\Gamma}'_n \subset A'$, $\bar{\Gamma}_n \subset A$ the corresponding continua. Thus there is a closed interval $[\bar{\omega}_0, \bar{\omega}'_0]$ in $\{\omega\}_{\gamma,A}$ such that $I(\bar{\omega}_0, \bar{\omega}'_0) \subset \bar{K} \subset I^*(\bar{\omega}_0, \bar{\omega}'_0)$, $\bar{K} = \lim\sup \bar{\Gamma}_n$. Let us prove that $(\bar{\omega}, \bar{\omega}')$ belongs to u. Indeed let $\omega' \in u'$ be a prime end which is reached by a sequence $[\Gamma'_n]$ of the direct procedure. Let $[l'_n, b'_n]$ be any f-system relative to ω' and observe that $l'_n \Gamma'_{m(n)} \neq 0$ for some $m = m(n)$, with $m(n) \to \infty$ as $n \to \infty$. Let λ'_n be an arc joining the point $l'_n \Gamma'_{m(n)}$ to the point $\bar{l}'_n \bar{b}'_n$, where $\lambda'_n \subset A'$. We can suppose that all points of λ'_n are at a distance $\leqslant \mu_n$ from the set $I(\omega', \bar{\omega}')$ with $\mu_n \to 0$ as $n \to \infty$. Since T' is constant on U' (and hence on $I(\omega', \bar{\omega}')$), and $T'(U') = p_0$, the set $T(\Gamma_{m(n)} + \lambda'_n + \bar{b}'_n)$ is contained in a sphere of center p_0 and radius a number $\delta'_n > 0$, $\delta'_n \to 0$ as $n \to \infty$. For each $n = 1, 2, \ldots$, let $\beta'_n = \lambda'_n + \bar{b}_n$, $B_n = \sum g'$, where $\sum$ ranges over all $g \in G$, with $g\beta'_n \neq 0$, and set $\Gamma'^*_m = \Gamma'_m$ if $m \neq m(n)$, $\Gamma'^*_m = \Gamma'_m + B_n$ if $m = m(n)$ [or $= \Gamma'_m + \sum B_n$, where $\sum$ ranges over all n such that $m(n) = m$]. Finally, let Γ^*_m be corresponding continua, $\Gamma^*_m \subset A$. By no. **4**, (i), there is a closed interval $[\omega^*, \omega'^*]$ in $\{\omega\}_{\gamma,A}$, with $I(\omega^*, \omega'^*) \subset \subset K^* \subset I^*(\omega^*, \omega'^*)$, $K^* = \lim\sup \Gamma^*_n$, and $[\omega^*, \omega'^*]$ contains both ω and $(\bar{\omega}, \bar{\omega}')$. Thus it is proved that $(\bar{\omega}, \bar{\omega}')$ belongs to u.

Thereby it is proved that not only for each $u \in \{u\}_{\gamma,A}$ the procedure above defines an $u' \in \{u'\}_{\gamma',A'}$, $u' = \sigma(u)$, but also that the same procedure applied to u' reproduces u. The same result holds by exchanging $\{u\}_{\gamma,A}$ and $\{u'\}_{\gamma',A'}$. This implies that the correspondence $u' = \sigma(u)$ is one-one between $\{u\}_{\gamma,A}$ and $\{u'\}_{\gamma',A'}$. Indeed if we consider any u' and we apply the procedure above we have an $u \in \{u\}_{\gamma,A}$ and the same procedure applied to u gives u'. We have now to prove that σ preserves the order of $\{u\}_{\gamma,A}$ and $\{u'\}_{\gamma',A'}$. Indeed, if $\omega_1 \in u_1$, $\omega_2 \in u_2$, $u_1 \neq u_2$, then there are two other $\omega_3 \in u_3$, $\omega_4 \in u_4$, such that u_1, u_2 separates u_3, u_4 in $\{u\}_{\gamma,A}$, and we can suppose $p_1 \neq p_3, p_4$; $p_2 \neq p_3, p_4$, where $p_i = T(U_i)$, $(i = 1, 2, 3, 4)$. If $u'_i = \sigma(u_i)$, $(i = 1, 2, 3, 4)$, we can prove, by a reasoning similar to the one in no. **4**, (i), that u'_1, u'_2 separate u'_3, u'_4 in $\{u'\}_{\gamma',A'}$. Thereby (i) is proved.

16. – Invariance of $l(t)$ with respect to Fréchet equivalence.

(i) *If (T, J), (T', J') are continuous mappings from the simple closed Jordan regions J, J' into E_3 and $T \sim T'$, then for each $t_1 < t < t_2$* (no. **12**), *for each pair of corresponding components $\alpha \in \{\alpha\}_t$, $\alpha' \in \{\alpha'\}_t$ of $J - C$ and $J' - C'$, and for each pair of corresponding components $\gamma \in \{\gamma\}_\alpha$, $\gamma' \in \{\gamma'\}_{\alpha'}$ of $\mathcal{F}(\alpha)$ and $\mathcal{F}(\alpha')$ we have $\lambda(\gamma, \alpha) = \lambda(\gamma', \alpha')$. Hence $l(t) = l'(t)$ for all $t_1 < t < t_2$.*

Proof. We have $\lambda = \operatorname{Sup} S$, $S = \sum |T(w_i) - T(w_{i+1})|$, $w_i = w_{\eta_i}$, $\eta_i \in \in \{\eta\}_{\gamma,A}$; hence, given $\varepsilon > 0$, there is a finite ordered system $[\eta]$ of N ends η such that $S > \lambda - \varepsilon$ if $\lambda < +\infty$, $> \varepsilon^{-1}$ if $\lambda = +\infty$. We can suppose that the ends $\eta \in [\eta]$ belong to different gp-ends u of γ in A, because of the fact that T is constant on each U_u. If $[u]$ is the ordered collection of the gp-ends to which the ends $\eta \in [\eta]$ belong, if $[u']$ is the collection of the different gp-ends u' of γ' in A', if $U' = U'_{u'}$, we have $T'(U') = T(U)$. If u' contains an end η', then $T'(w'_{\eta'}) = T(w_\eta)$. If u' is just a prime end ω', then $U' = E^{(0)}_{\omega'}$ and we can approach ω' with ends η' such that $w_{\eta'}$ is as close to the set $E^{(0)}_{\omega}$ as we want. We can choose η' in such a way that $|T'(w_{\eta'}) - T(w_\eta)| < 2N^{-1}\varepsilon^{-1}$ and that the new set $[\eta']$ of ends η' of γ' in A' is ordered as $[u']$. We have $S' > S - \varepsilon$, and finally $\lambda' \geqslant S' > S - \varepsilon > \lambda - 2\varepsilon$, or $> \varepsilon^{-1} - \varepsilon$. Therefore, $\lambda' \geqslant \lambda$. By exchanging T with T' we have also $\lambda \geqslant \lambda'$; hence $\lambda = \lambda'$. Since $l(t) = \sum^{\alpha} \sum_{\gamma} \lambda(\gamma, \alpha)$, $l'(t) = \sum_{\alpha'} \sum_{\gamma'} \lambda(\gamma', \alpha')$ we have $l(t) = l'(t)$. Thereby (i) is proved.

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