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On the rectification of contours of a Fréchet surface. (**)

Introduction. Let S be a FRÉCHET surface and let $T: Q \to E_3$ be a representation of S, where Q is the unit square $0 \leqslant s, t \leqslant 1$ in E_2. We denote by $[S]$ the set of points of E_3 occupied by the surface S. We let $w = (s, t)$, $x = (x_1, x_2, x_3)$ denote vectors in E_2 and E_3 respectively and we let T be represented by the vector function $x = x(w)$. In the preceding paper [1], CESARI considered a certain family of subsets of $[S]$ and proved that almost all sets of the family were rectifiable continuous curves. In working with certain problems in the theory of surfaces it is convenient to have information about the inverse images of these sets, in particular, to be able to exhibit a representation of the surface such that the inverse images of these sets have certain regular properties. The purpose of this Note is to show that not only are these sets rectifiable curves, but that there exists a representation of S such that the inverse image in Q of any countable family of them is actually a union of simple arcs and closed curves.

For the most part, the notations and definitions used in this paper are the same as in [1] and [2] The sets mentioned above are defined by a real valued Lipschitzian function defined over $[S]$. Let $t' = \min_{x \in [S]} f(x)$, $t'' = \max_{x \in [S]} f(x)$. If $t' < t < t''$ the open set β_t is defined to be the set of points of Q for which $f(x(w)) < t$. We define ξ_t to be the set $\mathcal{F}(\beta_t) = \overline{\beta}_t - \beta_t$. We shall call ξ_t the contour defined by f, t, and S. It is shown in [1] that for almost all t in the interval $t' < t < t''$ the essential part of the image under T of ξ_t is a union of rectifiable continuous curves on $[S]$. For each t, in the given range a length $\lambda(t, f, S)$ can be defined for the image under T of ξ_t by using the ends of β_t ending on ξ_t in a manner similar to the standard method of defining length

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of a curve C which is the continuous image of an arc Γ as the supremum of the lengths of the polygonal lines P inscribed in C where the vertices of P are chosen as the images of an ordered finite set of points of Γ. CESARI showed that [1], this length is a FRÉCHET invariant of the surface and [2] that $K \cdot L(S) \geq \int_{t'}^{t''} \lambda(t, f, S)\, dt$ where $|f(x_1) - f(x_2)| < K|x_1 - x_2|$ and where the absolute values denote distances in the spaces involved. If there exists a map T' FRÉCHET equivalent to T under which ξ_t becomes essentially a union of simple arcs and simple closed in Q, we shall call the substitution of T' for T a *rectification of the contour* ξ_t. We shall show that if $\{\xi_{t_i}\}$, $(i = 1, 2, 3, ...)$, is any countable set of contours, each of finite length, there exists a FRÉCHET equivalent map T' which simultaneously rectifies all the $\{\xi_{t_i}\}$ and such that $|T(p) - T'(p)|$ is small for every $p \in Q$.

1. – Rectification of a single contour.

Theorem 1. *Let S be a Fréchet surface with a representation $T: x = x(w)$, $w \in Q$ and f a real valued function defined over $[S]$, with lower and upper bounds t' and t'' respectively such that for $x_1, x_2 \in S$, $|f(x_1) - f(x_2)| < K|x_1 - x_2|$. Let t be chosen so that the length $\lambda(t)$ of the image of ξ_t is finite. Then if $\varepsilon > 0$ there exists a mapping $T': x' = x'(w)$, $w \in Q$, Fréchet equivalent to T such that $|x'(w) - x(w)| < \varepsilon$, $w \in Q$ and such that each component of the contour ξ_t relative to T' is either a simple arc, a Jordan curve or a continuum of constancy for T' on Q.*

Proof. Let α_t be a component of β_t and let $F = \alpha_t^* - (\alpha_t \cap \alpha_t^*)$. Let γ be a component of F whose image under T is of finite generalized length, $\lambda(t) < \infty$, and consider the set $A(\alpha, \gamma)$ of points of Q not separated from α by γ as defined in [1]. By definition of $A(\alpha, \gamma)$, A is either simply connected or of genus one with Q^* as its outer boundary, while the collections $\{\eta\}_{\gamma,A}$, $\{\omega\}_{\gamma,A}$ of all ends and prime ends of γ in A either have a cyclic ordering or a linear ordering. We first consider the case in which A is simply connected and the ends and prime ends have a cyclic ordering. Let Γ be a circle plus its interior which is interior to $A(\alpha, \gamma)$. By a well known theorem of CARATHEODORY on plane sets (see, for example [3], p. 112) there exists a correspondence τ between points of Γ and A which is a homeomorphism between the interiors of A and Γ and on the boundaries sets up a one to one order preserving correspondence between points of Γ^* and ends and prime ends of A. Let $\{p_i\}$ be a countable dense subset of Γ^* such that each p_i corresponds to an end of A. This is possible since any two distinct prime ends of A are separated by an end and the ends have images dense on Γ. The points on A^*

corresponding to ends are dense on A^* since any neighborhood of a boundary point of A contains points of A accessible from A. Let arcs $\{c_i\}$ be constructed in $A - \Gamma$ connecting each point p_i to its image point $\tau(p_i)$ on A^* via an arc defining the end to which p_i corresponds and such that no two of the arcs $\{c_i\}$ intersect except possibly at end points on A^*. A new mapping T_1: $x_1 = x_1(w)$ is defined in the following manner: $x_1(w) = x(w)$, $w \in (Q - \operatorname{Int} A)$; $x_1(w) = x(\tau^{-1}(w))$, $w \in \operatorname{Int} \Gamma$; $x_1(w)$ is constant on each c_i. For other points of $\Gamma^* \cup (A - \Gamma)$ T_1 is defined as follows. If $w \notin c_i$ for any i, we consider all continua formed by closed regions bounded by two arcs c_i and portions of Γ^* and A^* which contain w. The intersection of all such regions will be a continuum μ containing only one point p of Γ^* where p is a limit point of certain of the p_i, $p = \lim_{k \to \infty} p_{i_k}$. Define $x_1(p) = \lim_{k \to \infty} x_1(p_{i_k})$ and let $x_1(w) = x_1(p)$ for $w \in \mu$. Thus T_1 is defined for all $w \in Q$ and is evidently continuous. It must next be shown that T_1 is FRÉCHET equivalent to T.

We shall show that for any $\varepsilon > 0$ there exists a homeomorphism φ_ε of Q onto itself such that for $w \in Q$, $|x_1(\varphi_\varepsilon^{-1}(w)) - x(w)| < \varepsilon$. On $Q - A$, φ_ε is defined to be the identity. On A, φ_ε is constructed as described below.

Since $x(w)$ is uniformly continuous over Q there exists a $\delta > 0$ such that $|x(w_1) - x(w_2)| < \varepsilon/6$ if $|w_1 - w_2| < \delta$. Let $\{w_i\}$ be points of A^* (not necessarily distinct) which are images of the points $\{p_i\}$ under τ and let $\{\eta_\mu\}$ be the points or continua of A^* defined by $\eta_\mu = \mu \cap A^*$ for the continua described above. The set $\{w_i\}$ is dense on A^*. Let a finite ordered subfamily $\{\xi_{i_k}\}$, $(k = 1, 2, 3, \ldots, n)$, of ends of A be chosen corresponding to distinct points $\{w_{i_k}\} \subset \{w_i\}$ and such that

$$(1) \qquad \left| \sum_{k=1}^{n} |x(w_{i_k}) - x(w_{i_{k+1}})| - \lambda(\gamma) \right| < \varepsilon/6 ,$$

where $w_{i_{n+1}} = w_1$.

Let δ_2 be chosen so that $\min_{k,l} |w_{i_k} - w_{i_l}| > 2\delta_2$, $k = l$, $\delta_2 \leqslant \delta_1/2$ and $\{\Gamma^*, A^*\} > \delta_2$. Let circles $\Gamma_i(\delta_2)$ of radius δ_2 be constructed at each point w_i on A^*. Since A^* is compact, there exists a finite subfamily $\Gamma_{i_l}(\delta_2)$, $(l = 1, 2, 3, \ldots, h)$, which covers A^*. Let this family be chosen with centers $\{w_{i_l}\}$ so as to include all circles with centers at the points $\{w_{i_k}\}$. Let $\delta_3 = \min_l \{A^*, [c_{i_l} - \operatorname{Int} \Gamma_{i_l}(\delta_2)]\}$. Let q_{i_l} be the last point of the arc (starting from $p_{i_l} \in \Gamma^*$) which c_{i_l} has in common with $\Gamma_{i_l}(\delta_2)$. Let c_{i_l}' be the sub arc of c_{i_l} from p_{i_l} to q_{i_l}. Thus the arc $c_{i_l} - c_{i_l}'$ is contained in $\Gamma_{i_l}(\delta_2)$ and has diameter less than $2\delta_2$. Let $\delta_4 > 0$ be the minimum of the distances of the arcs c_{i_l}' from A^*. Let δ_5 be the distance $\{A^*, ([\cup_l \Gamma_{i_l}(\delta_2)]^* \cap A)\}$. Let $\delta = \min(\delta_1, \delta_2, \delta_4/2, \delta_5/2)$. Let circles $\Gamma_i(\delta)$ be constructed with each w_i as center and radius δ. Choose a finite family $\Gamma_{i_m}(\delta)$, $(m = 1, 2, 3, \ldots, H)$, covering A^* and including all circles $\Gamma_{i_l}(\delta)$ with centers at the centers of the

$\Gamma_{i_l}(\delta_2)$ and radius δ. Let D be the component of $A - \cup_m \Gamma_{i_m}(\delta)$ which includes Γ and let D^* be its boundary. D^* is a JORDAN curve and is such that $D^* \subset [\cup_l \Gamma_{i_l}(\delta_2) \cap A]$. Let R_{i_k} be the open, simply connected set bounded by D^* and A^* and the minimal sub arcs c'_{i_k} and $c'_{i_{k+1}}$ of c_{i_k} and $c_{i_{k+1}}$, $(k = 1, 2, \ldots, n)$, joining A^* and D^* and let $w_1, w_2 \in R_{i_k}$. By the choice of the w_{i_k} satisfying (1) we note that the oscillation of $x(w)$ on $A^* \cap R_{i_k}$ is less than $\varepsilon/6$. Also, by the construction of R_{i_k}, if $w \in R_{i_k}$, there exists a point w_{i_l} on A^* such that the circle $\Gamma_{i_l}(\delta)$ contains w and hence $|x(w) - x(w_{i_l})| < \varepsilon/6$. If $w_{i_l} \notin R_{i_k} \cap A^*$, the line segment between w and w_{i_l} intersects $R^*_{i_k} - D^*$ in at least one point $\overline{w}$. If $\overline{w} \in A^*$ then $|x(\overline{w}) - x(w)| < \varepsilon/6$. If $\overline{w} \in c'_{i_k}$ then $|w - w_{i_k}| \leqslant |w - \overline{w}| + |\overline{w} - w_{i_k}| < 2\delta < \delta_1$ and $|x(w) - x(w_{i_k})| < \varepsilon/6$. Similarly for $c'_{i_{k+1}}$. Thus in any case if $w \in R_{i_k}$ there exists a $w' \in A^* \cap R_{i_k}$ with $|x(w) - x(w')| < \varepsilon/6$. Let $w_1, w_2 \in R_{i_k}$ and let w'_1, w'_2 be points of $A^* \cap R_{i_k}$ with $|x(w_1) - x(w'_1)| < \varepsilon/6$, $|x(w_2) - x(w'_2)| < \varepsilon/6$. Then we have

$$|x(w_1) - x(w_2)| \leqslant |x(w_1) - x(w'_1)| + |x(w'_1) - x(w'_2)| + |x(w'_2) - x(w_2)| < \varepsilon/2$$

and the oscillation of $x(w)$ on R_{i_k} does not exceed $\varepsilon/2$.

Let R'_{1_k} be the region bounded by the sets c_{i_k}, $c_{i_{k+1}}$, $\tau(c_{i_k})$, $\tau(c_{i_{k+1}})$, A^*, and $\tau(D^*)$. The homeomorphism φ_ε is defined as follows. On $\tau(\overline{D})$ we let $\varphi_\varepsilon = \tau^{-1}$, φ_ε is the identity on A^* and φ_ε takes $c_{i_k} \cup \tau(c_{i_k})$, $c_{i_{k+1}} \cup \tau(c_{i_{k+1}})$ onto the portions of c_{i_k}, $c_{i_{k+1}}$ included between A^* and D^*. On the interior of R'_{i_k}, φ_ε is defined onto the interior of R_{i_k} in such a manner as to agree with the homeomorphism already constructed on its boundary. On successive regions, φ_ε is constructed so as to agree on the common portions of the boundaries of the regions. Outside A, φ_ε is the identity. Then if $w \in \overline{D}$, $x_1(\varphi_k^{-1}(w)) = x[\tau^{-1}(\tau(w))] = x(w)$. Also $x_1(\varphi_\varepsilon^{-1}(w)) = x(w)$ outside A. If $w \in R_{i_k}$ for some k, $x_1(\varphi_\varepsilon^{-1}(w)) = x[\tau^{-1}(\varphi_\varepsilon^{-1}(w))]$ and $\tau^{-1}[\varphi_\varepsilon^{-1}(w)] \in R_{i_k}$. Hence $|x_1(\varphi_\varepsilon^{-1}(w)) - x(w)| < \varepsilon$ for all points in Q and T and T_1 are FRÉCHET equivalent. Since by [1] the lengths involved are FRÉCHET invariant, the length of $x(\gamma)$ equals the length of $x_1(D^*)$ and T_1 rectifies γ.

It can be seen that the circle Γ can be replaced by any JORDAN region interior to A. In particular, a region of the same type as D can be used and can be chosen in such a way that $|x(w) - x_1(w)| < \varepsilon$ for any $\varepsilon > 0$.

In case the region $A = A(\alpha, \gamma)$ is simply connected and the collection of ends and prime ends is linearly ordered (i.e. when A has a portion of Q^* as part of its boundary) then D^* will be an arc joining two points of Q^* instead of a JORDAN curve. The above arguments also hold in this case.

In case $A(\alpha, \gamma)$ is not simply connected, it is of genus one with Q^* as its outer boundary. By the same technique as was used in the first part of the theorem, A can be mapped onto an annular region with outer boundary Q^*

and inner boundary a JORDAN curve interior to A. This rectification can be carried out so as to give $|x(w) - x_1(w)| < \varepsilon$ for any $w \in Q$.

Consider now all components of F for a given α. At most a countable family of these components have images under T of non zero length. Let $\{\gamma_i\}$ be these components. Since the $\{\gamma_i\}$ are disjoint closed sets there is a positive distance between any two of them. Let γ_1 be rectified in such a way that the new representation function $x_1(w)$ of S satisfies $|x_1(w) - x(w)| < 1/2$, $w \in Q$. Denote the rectified γ_1 by γ_1', while the other components $\gamma_2, \gamma_3, \ldots$ are replaced by new continua which for simplicity we still call $\gamma_2, \gamma_3, \ldots$. Separate γ_2 from γ_1' by a polygonal line P_2. By the rectification process of the first part of the proof, a new map T_2: $x_2 = x_2(w)$ can be constructed in such a way that $x_1(w) = x_2(w)$ in $Q - A$ as well as in the portion of Q separated from γ_2 by P_2 and T_2 rectifies γ_2 into an arc or JORDAN curve γ_2' in such a manner that $|x_2(w) - x_1(w)| < 1/4$, $w \in Q$, and does not modify γ_1'. The same process can be carried outfor successive γ_i so that at the k-th step a mapping T_k is obtained which does not alter the $\{\gamma_i'\}$, $i < k$, but which rectifies γ_k in such a way that $|x_k(w) - x_{k-1}(w)| < 1/2^k$, $w \in Q$. Since for any $\eta > 0$ there exists k_η such that $|x_k(w) - x_l(w)| < \eta$, $k, l > k_\eta$, T_k is a uniformly convergent sequence of FRÉCHET equivalent maps and its limit T_0 is a map FRÉCHET equivalent to each T_k and hence to T. Also T_0 rectifies all the contours $\{\gamma_i\}$ and as in the previous part of the proof, if $\varepsilon > 0$, the sequence $\{T_k\}$ can be constructed in such a way that at each step $|x_k(w) - x_{k-1}(w)| < \varepsilon/2^k$ and hence $|x_0(w) - x(w)| < \varepsilon$.

Consider now the countable set of all components $\{\alpha\}$ corresponding to a given β_t where $\lambda(\xi_t) < \infty$ and let $\lambda(t, \alpha, T)$ denote the length of the image under T of the set $\alpha^* - (\alpha \cap \alpha^*)$. Thus $0 \leqslant \lambda(\alpha, t, T) < \infty$, $\alpha \in \{\alpha\}$. Let $\{\alpha_i\}$ be the sub family of $\{\alpha\}$ for which $\lambda(t, \alpha, T) > 0$. Let $F_i = \alpha_i^* - (\alpha_i \cap \alpha^*)$, $(i = 1, 2, 3, \ldots)$. For $\varepsilon > 0$ construct a mapping T^1: $x^1 = x^1(w)$ which rectifies F_1 in such a way that $|x^1(w) - x(w)| < \varepsilon/2$. $\alpha_1 \cap \alpha_2 = 0$ and, although $F_1 \cap F_2$ may be non void and for some $\gamma \subset \alpha_2^* - (\alpha_2 \cap \alpha_2^*)$ it may happem that $\bar{\alpha} \subset A(\alpha_2, \gamma)$, it is still possible to proceed as in the first part of the proof for α_2, choosing a family of arcs and simple closed curves interior to α_2' and constructing a map T^2 with $x^2(w) = x^1(w)$, $w \in \bar{\alpha}$, which rectifies F_2 with $|x^2(w) - x^1(w)| < \varepsilon/2^k$, $w \in Q$. Mappings $T^3, T^4, \ldots$ may be similarly constructed in such a way that $x^k(w) = x^{k-1}(w)$, $w \in \overline{\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \ldots \cup \alpha_{k-1}}$ which rectify the F_k and such that for each k, $|x^k(w) - x^{k-1}(w)| < \varepsilon/2^k$, $w \in Q$. Again, a limit mapping T^0 exists, FRÉCHET equivalent to T which rectifies the contour ξ_t and such that $|x^0(w) - x(w)| < \varepsilon$.

2. – Rectification of a dense set of contours.

Theorem 2. *Let T be as in Theorem 1 and let $\{t_i\}$, $(i = 1, 2, \ldots)$, be any* *countable sequence of real numbers, $t' < t_i < t''$, for which $\lambda(t_i) < \infty$. Then for any $\varepsilon > 0$ there exists a representation T_ε of S which simultaneously rectifies all the contours $\{\xi_{t_i}\}$ and for which $|x_\varepsilon(w) - x(w)| < \varepsilon$, $w \in Q$.*

Proof. Construct T_1: $x_1 = x_1(w)$ as in Theorem 1 rectifying ξ_{t_1} and with $|x_1(w) - x(w)| < \varepsilon/2$. $f(x(w)) = t_1$ for all $w \in \xi_{t_1}$ and $f(x(w)) = t_2$ for $w \in \xi_{t_2}$ with $t_1 \neq t_2$, and the same is true for the rectified contours. Thus $\{\xi_{t_1}, \xi_{t_2}\} > 0$ and by a procedure similar to that of Theorem 1, ξ_{t_2} can be rectified by a mapping T_2 in such a way that the rectification ξ'_{t_1} of ξ_{t_1} is left unchanged and $|x_2(w) - x_1(w)| < \varepsilon/4$. By the same method as in Theorem 1 we construct a sequence $\{T_i\}$ of mappings, each FRÉCHET equivalent to T such that for each k, $|x_k(w) - x_{k-1}(w)| < \varepsilon/2^k$, $x_k(w) = x_{k-1}(w)$, $w \in \xi'_{t_1} \cup \xi'_{t_2} \cup \cup \dots \cup \xi'_{t_{k-1}}$ and T_k rectifies ξ_k. Thus the limit T_ε: $x = x_\varepsilon(w)$, $w \in Q$, is a continuous mapping FRÉCHET equivalent to T for which $|x_\varepsilon(w) - x(w)| < \varepsilon$, $w \in Q$ and such that T_ε rectifies all the $\{\xi_{t_i}\}$, $(i = 1, 2, 3, \dots)$.

Corollary. It is possible to rectify the contours of Theorem 2 in such a manner that under the new mapping T_ε none of the contours of finite length defined by the function f contain triodic continua.

Proof. By a theorem of R. L. MOORE [5], at most a countable family of disjoint continua in the plane contain triodic continua. Those contours of finite length can be rectified as in Theorem 2 and the rectification process introduces no new contours containing such continua.

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