Rivista Mat. Univ. Parma 5, 55-61 (1954)

LAMBERTO CESARI and JACK K. HALE (*)

Second order linear differential systems with periodic L-integrable coefficients. (**)

The well known MATHIEU equation $y'' + (\sigma^2 + \lambda \cos \omega x)y = 0$ and the HILL equation $y'' + [\sigma^2 + \lambda\varphi(x)]\, y = 0$ [φ periodic of period $T = 2\pi/\omega$], and others, can be considered as particular cases of systems of linear differential equations of the form

$$(1) \qquad y_i' + \sum_1^n {}_h\, [a_{ih} + \lambda\varphi_{ih}(x)]y_h = 0\,, \qquad (i = 1, 2, ..., n)\,,$$

where $\varphi_{ih}(x)$ are periodic functions of period $T = 2\pi/\omega$ and λ is a parameter. These systems for λ small have been con idered by L. CESARI [4] (1) in a previous paper under the hypothesis that the functions φ_{ih} have absolute convergent FOURIER series. The same systems (1) shall be further discussed under the same hypothesis by R. A. GAMBILL [**6**], and under the weaker hypothesis that the functions φ_{ih} are only L-integrable in [0, T] by J. K. HALE [**8**]. In all these papers [**4, 6, 8**], the authors use a variant of the POINCARÉ method of casting out the secular terms in the solutions of (1) by successive approximations. Another analogous variant of the same method shall be used by J. K. HALE [**8**] for autonomous non-linear systems, and by R. A. GAMBILL and J. K. HALE [**7**] for non-linear systems with periodic terms.

For $n = 2$, i.e., for systems

$$(2) \qquad \begin{cases} y_1' = (a_{11} + \lambda\varphi_{11})y_1 + (a_{12} + \lambda\varphi_{12})y_2\,, \\ y_2' = (a_{21} + \lambda\varphi_{21})y_1 + (a_{22} + \lambda\varphi_{22})y_2\,, \end{cases}$$

(*) Authors' addresses: L. CESARI, Purdue University, Lafayette, Indiana, U.S.A.; J. K. HALE, Sandia Corporation, Albuquerque, New Mexico, U.S.A..

(**) Received March 23, 1954. The present paper was written in part under a grant from the National Science Foundation [NSF Research Project 0101 at Purdue University].

(1) Numbers in brackets refer to Bibliography at the end of the paper.

results somewhat stronger than for $n > 2$ can be obtained as has been pointed out by L. CESARI [4]. Indeed, if ϱ_1, ϱ_2 are the characteristic roots of the 2×2 constant matrix $\|a_{jh}\|$, $(j, h = 1, 2)$, if either $\mathcal{R}(\varrho_j) < 0$, $(j = 1, 2)$, or $\mathcal{R}(\varrho_j) = 0$, $(j = 1, 2)$, $\varrho_j = \pm i\sigma$, $\sigma > 0$, and $m\omega \neq 2\sigma$ for all $m = 1, 2, \ldots$, if the functions φ_{jh} are periodic of period T, have mean value zero in $[0, T]$ and absolutely convergent FOURIER series, then there exists a $\delta > 0$ such that all solutions of (2) are bounded for $|\lambda| < \delta$. (For $n > 2$, the analogous statement holds provided general conditions of symmetry of the matrix $\|\varphi_{jh}\|$ are satisfied [4].)

In the present short paper, we prove that much more inclusive statements (Theorems I and II below) can be obtained for systems (2) by a direct application of general theorems concerning existence, unicity, and continuous dependence upon parameters of linear VOLTERRA integral equations, in particular, some slight generalizations of results of E. HILLE and J. D. TAMARKIN [9], or some known results of C. CARATHÉODORY [3].

Theorem I. *Consider the matrix equation*

(3) $$Y' = AY + \Phi(x; \lambda)Y, \quad \text{a.e. } (^2), \quad -\infty < x < +\infty, \quad (' = d/dx),$$

where λ is a complex parameter, $A = \|a_{\mu\nu}\|$, $(\mu, \nu = 1, 2)$, is a real constant matrix, and $\Phi(x; \lambda) = \|\varphi_{\mu\nu}(x; \lambda)\|$, $(\mu, \nu = 1, 2)$, is a matrix whose elements $\varphi_{\mu\nu}(x; \lambda)$ are complex valued functions of the real variable x, periodic in x of period $T = 2\pi/\omega$, L-integrable in $[0, T]$ and each function $\varphi_{\mu\nu}(x; \lambda)$ is a continuous function of λ at $\lambda = 0$ for almost all x in $[0, T]$. Moreover, we assume that $\varphi_{\mu\nu}(x; 0) = 0$ and $|\varphi_{\mu\nu}(x; \lambda)| < \omega(x)$ a.e. in $[0, T]$, $|\lambda| \leqslant \lambda_0$ for some given $\lambda_0 > 0$, and $\omega(x)$ is L-integrable in $[0, T]$. If the characteristic roots of the matrix A are ϱ_1, ϱ_2, where either (a) *$\mathcal{R}(\varrho_j) < 0$, $(j = 1, 2)$, or,* (b) *$\varrho_1 = i\sigma$, $\varrho_2 = -i\sigma$, $\sigma > 0$, $m\omega \neq 2\sigma$, $(m = 1, 2, \ldots)$, and $\int_0^T (\varphi_{11} + \varphi_{22})\, dx \leqslant 0$ for all $|\lambda| \leqslant \lambda_0$, then the absolutely continuous solutions Y of (3) are bounded in $(0, +\infty)$ for $|\lambda|$ sufficiently small.*

Theorem II. *Consider the differential equation*

(4) $$y'' + \psi(x; \lambda)y' + \varphi(x; \lambda)y + \sigma^2 y = 0, \quad \text{a.e.}, \quad -\infty < x < +\infty,$$

where $\sigma > 0$, λ is a real parameter, the functions $\varphi(x; \lambda)$, $\psi(x; \lambda)$ are real functions, periodic in x of period $T = 2\pi/\omega$, L-integrable with respect to x in $[0, T]$ for all $|\lambda| \leqslant \lambda_0$, continuous functions of λ at $\lambda = 0$ for almost all x in $[0, T]$ and $\varphi(x; 0) = \psi(x; 0) = 0$. Moreover, assume that $\int_0^T \psi(x; \lambda)\, dx \geqslant 0$ for all $|\lambda| \leqslant \lambda_0$

(²) By « a.e. » we mean « almost everywhere ».

and some given $\lambda_0 > 0$, and that there exists a function $\eta(x)$, L-integrable in $[0, T]$, such that $|\varphi(x; \lambda)| < \eta(x)$, $|\psi(x; \lambda)| < \eta(x)$ for all x in $[0, T]$ and all $|\lambda| \leqslant \lambda_0$. If $|\lambda|$ is sufficiently small and $m\omega \neq 2\sigma$ $(m = 1, 2, \ldots)$, then the absolutely continuous solutions of (4) *are bounded in* $(0, +\infty)$.

In order to prove the preceding theorems, we shall prove the following lemmas which are extensions of a result of E. HILLE and J. D. TAMARKIN **[9]**. We shall denote by λ a real or complex parameter, $|\lambda| \leqslant \lambda_0$, by $f(x)$, $\omega(\xi) \geqslant 0$, $y(x)$ functions defined a.e. in an interval $[0, b]$ and by $K(x, \xi; \lambda)$, $\mathcal{K}(x, \xi; \lambda)$ functions defined for every $|\lambda| \leqslant \lambda_0$, a.e. in the triangle $R = [0 \leqslant \xi \leqslant x \leqslant b]$. We shall denote as usual by L the class of all L-integrable functions in $[0, b]$ or R, and by $\mathcal{L}[\omega]$ the class of all functions $F(x)$ such that $\omega(x)F(x) \in \mathrm{L}$.

Lemma 1. Consider the VOLTERRA integral equation

$$y(x) = f(x) + \int_0^x K(x, \xi; \lambda)\, y(\xi)\, d\xi\,, \qquad 0 \leqslant x \leqslant b\,, \tag{5}$$

where $K(x, \xi; \lambda) \in \mathrm{L}$ in R for all $|\lambda| \leqslant \lambda_0$ for some given $\lambda_0 > 0$, and we assume there exists a function $\omega(\xi) \in \mathrm{L}$ such that

$$|K(x, \xi; \lambda)| < \omega(\xi) \tag{6}$$

a.e. in R and $|\lambda| \leqslant \lambda_0$. For every $f(x) \in \mathcal{L}[\omega]$, there is a unique solution $y(x; \lambda)$ of (5) defined for every $|\lambda| \leqslant \lambda_0$ a.e. in $[0, b]$, $y(x; \lambda) \in \mathcal{L}[\omega]$ and the solution $y(x; \lambda)$ is given by

$$y(x; \lambda) = f(x) + \int_0^x \mathcal{K}(x, \xi; \lambda)\, f(\xi)\, d\xi$$

where $\mathcal{K}(x, \xi; \lambda)$ is defined for every $|\lambda| \leqslant \lambda_0$, a.e. in R, and $\mathcal{K}(x, \xi; \lambda) \in \mathrm{L}$ in R for every $|\lambda| \leqslant \lambda_0$. If we further assume that $K(x, \xi; \lambda)$ is a continuous function of λ for almost all $(x, \xi) \in R$ and some $\lambda = \lambda_1$, then $y(x; \lambda)$ is also a continuous function of λ for almost all x in $[0, b]$ and $\lambda = \lambda_1$.

Proof. In order to show the existence of the resolvent $\mathcal{K}(x, \xi; \lambda)$, we proceed exactly as in E. HILLE and J. D. TAMARKIN to obtain evaluations for the iterated kernels $K_n(x, \xi; \lambda)$. If we let $C = \int_0^b \omega(t)\, dt$, we have $|K_n(x, \xi; \lambda)| < C^{n-1}\omega(\xi)/(n-1)!$ and, hence, the series $|\sum K_n(x, \xi; \lambda)| = |\mathcal{K}(x, \xi; \lambda)| < \omega(\xi)\, e^C$ is absolutely convergent at all points where $\omega(\xi)$ is finite, and at each of these points the same series is uniformly convergent in λ for $|\lambda| \leqslant \lambda_0$. The proof that the function $\mathcal{K}(x, \xi; \lambda)$ is really a resolvent and the proof of the uniqueness is exactly the same as in E. HILLE and J. D. TAMARKIN. Thus, the solution $y(x; \lambda)$ is given by

$$y(x; \lambda) = f(x) + \int_0^x \mathcal{K}(x, \xi; \lambda)\, f(\xi)\, d\xi\,. \tag{7}$$

If $K(x, \xi; \lambda)$ is a continuous function of λ at $\lambda = \lambda_1$ for almost all $(x, \xi) \in R$, then, by (6), we see from a theorem in E. W. HOBSON [**10**, p. 323] that each of the iterated kernels $K_n(x, \xi; \lambda)$ is continuous at $\lambda = \lambda_1$ for almost all $(x, \xi) \in R$. Moreover, since $\sum K_n(x, \xi; \lambda)$ is uniformly convergent with respect to λ, for $|\lambda| \leqslant \lambda_0$ and almost all $(x, \xi) \in R$, we conclude that $\mathcal{K}(x, \xi; \lambda)$ is continuous at $\lambda = \lambda_1$ for almost all $(x, \xi) \in R$. Finally, from (7) and the same theorem in HOBSON, we have the solution $y(x; \lambda)$ is a continuous function of λ at $\lambda = \lambda_1$ for almost all x in $[0, b]$, and the Lemma is proved.

Lemma 2. If all the conditions of Lemma 1 are satisfied, if, in addition, $K(x, \xi; \lambda)$ is finite for all x in $[0, b]$, $|\lambda| \leqslant \lambda_0$ and almost all ξ and $|K(x, \xi; \lambda)| < \omega(\xi)$ for all $(x, \xi) \in R$ and $|\lambda| \leqslant \lambda_0$, and if $f(x) \in \mathcal{L}[\omega]$ is finite for all x in $[0, b]$, then $y(x; \lambda)$ is a continuous function of λ at $\lambda = \lambda_1$ for all x in $[0, b]$.

Proof. We then have $|\mathcal{K}(x, \xi; \lambda)| < \omega(\xi) e^c$ for $|\lambda| \leqslant \lambda_0$ and all x in $[0, b]$ and the Lemma follows immediately.

Proof of Theorem I. We may assume without loss of generality that $A = \left\| \begin{matrix} \varrho_1 & 0 \\ \varepsilon & \varrho_2 \end{matrix} \right\|$, where ε is either 0 or 1, and ε is certainly 0 if $\varrho_1 \neq \varrho_2$. The absolutely continuous (AC) solutions of (3) coincide with the solutions of the matrix integral equation

$$Y(x; \lambda) = Z(x)K + \int_0^x Z(x-s)\, \Phi(s; \lambda) Y(s; \lambda)\, ds, \tag{8}$$

where K is a constant matrix and $Z(x)$ is a non-singular solution of the equation $Z' = AZ$. This can be proved following the proof in LEFSCHETZ [**11**, p. 62] for the case where Φ is continuous. Furthermore, the matrix $Z(x)$ can be chosen to be

$$Z(x) = \left\| \begin{matrix} e^{\varrho_1 x} & 0 \\ \varepsilon x\, e^{\varrho_2 x} & e^{\varrho_2 x} \end{matrix} \right\|,$$

where ε is the same as above. As a consequence, the elements of the matrix kernel $Z(x-s)\,\Phi(s; \lambda)$ are of the form

$$K(x, s; \lambda) = e^{\varrho(x-s)} \varphi_1(s; \lambda) + \varepsilon(x-s)\, e^{\varrho(x-s)} \varphi_2(s; \lambda), \tag{9}$$

where ϱ is one of the numbers ϱ_1, ϱ_2 and φ_1, φ_2 correspond to some one of the functions $\varphi_{\mu\nu}$. Then $|K(x, s; \lambda)| \leqslant (1 + \varepsilon T) e^{R(\varrho)T} \omega(s)$ for all x and almost all s such that $0 \leqslant x \leqslant s \leqslant T$ and all $|\lambda| \leqslant \lambda_0$. Moreover, since $Z(x)$ is continuous and bounded for all x, we see that the conditions of Lemma 2 are satisfied and, thus, any solution $Y(x; \lambda)$ of (8) is unique and is a continuous function of λ at $\lambda = 0$ for all x in $[0, T]$. Finally, this implies that any AC solution of (3) is a continuous function of λ at $\lambda = 0$ for all x in $[0, T]$.

Since the AC solutions of (3) are unique, we may proceed exactly as in S. LEFSCHETZ [**11**, p. 68] to show that there is a fundamental matrix of AC solutions of (3) of the form $\|y_{ih}(x; \lambda)\| = \|e^{\tau_i x} p_{ih}(x; \lambda)\|$, where the functions $p_{ih}(x; \lambda)$ are periodic in x of period T if $\tau_1 \not\equiv \tau_2 \pmod{\omega i}$, and polynomials in x with coefficients periodic in x of period T if $\tau_1 \equiv \tau_2 \pmod{\omega i}$. The numbers τ_1, τ_2 are called the characteristic exponents of (3) and are determined up to a multiple of ωi.

If we let $\beta_j = e^{\tau_j T}$, $(j = 1, 2)$, then since $y(x; \lambda)$ is continuous in λ at $\lambda = 0$ for all x in $[0, T]$, it is known [5] that β_1, β_2 satisfy an equation of the form

$$\beta^2 - 2A\beta + B = 0, \tag{10}$$

where $A = A(\lambda)$, $B = B(\lambda)$ are continuous functions of λ at $\lambda = 0$. Since the numbers $\beta_j(\lambda)$, $\tau_j(\lambda)$ can be regarded as continuous functions of A, B, they are continuous functions of λ at $\lambda = 0$ and, from (3), we see that we may take $\tau_1(0) = \varrho_1$, $\tau_2(0) = \varrho_2$.

If we assume that $\mathcal{R}(\varrho_j) < 0$, then, by continuity, $\mathcal{R}(\tau_j(\lambda)) < 0$, $(j = 1, 2)$, for $|\lambda|$ sufficiently small, and the AC solutions of (3) are bounded in $(0, +\infty)$.

Let us now assume that $\varrho_1 = i\sigma$, $\varrho_2 = -i\sigma$, $\sigma > 0$, $m\omega \neq 2\sigma$, $(m = 1, 2, \ldots)$, and $\int_0^T (\varphi_{11} + \varphi_{22})\, dx = \gamma(\lambda) T \leqslant 0$, where this relation defines the function $\gamma(\lambda)$. From our assumption on φ_{11}, φ_{22}, we have $\gamma(\lambda)$ is continuous at $\lambda = 0$ and $\gamma(0) = 0$. From the FLOQUET theory [**5**], we know that B in (10) is given by

$$B(\lambda) = e^{\int_0^T [(a_{11}+\lambda\varphi_{11})+(a_{22}+\lambda\varphi_{22})]dt} = e^{(a_{11}+a_{22})T+\gamma(\lambda)T} = e^{(i\sigma - i\sigma)T+\gamma(\lambda)T} = e^{\gamma(\lambda)T} .$$

Moreover, for $\lambda = 0$, we have $\beta_1 = e^{i\sigma T}$, $\beta_2 = e^{-i\sigma T}$, $B(0) = 1$, and, if we put $A_0 = A(0)$, $B_0 = B(0)$, and make use of (10), we obtain $e^{2i\sigma T} - 2A_0 e^{i\sigma T} + 1 = 0$, or, $A_0 = \cos \sigma T$. Since $m\omega \neq 2\sigma$, $(m = 1, 2, \ldots)$, we have $A_0^2 - B_0 = A_0^2 - 1 < -\delta < 0$, where δ is fixed > 0. Since $A(\lambda)$, $B(\lambda)$ are continuous functions of λ, we have $A^2(\lambda) - B(\lambda) < 0$ for $|\lambda|$ sufficiently small. Finally, since β_1, β_2 are given by $\beta_{1,2} = A \pm \sqrt{A^2 - B}$, we obtain $|\beta_1| = |\beta_2| = B(\lambda) \leqslant 1$, thus the AC solutions of (3) are bounded in $(0, +\infty)$ for $|\lambda|$ sufficiently small, and Theorem I is proved.

Note 1. Instead of using Lemma 2 in the proof, we could have used a result of C. CARATHÉODORY [**3**, p. 678, Satz 5].

Note 2. The case (a) of Theorem I could be obtained from a previous paper of D. CALIGO [**2**, (a)] as follows. We shall first evaluate each of the expressions in (9). If we let $\eta(\lambda) = \max \int_0^T |\varphi_{\mu\nu}(x; \lambda)| dx$, $(\mu, \nu = 1, 2)$, then, since $\mathcal{R}(\varrho) < 0$, we have

$$\int_0^x |e^{\varrho(x-s)}\varphi_1(s; \lambda)| ds = \Big(\int_{x-T}^{x} + \int_{x-2T}^{x-T} + \dots\Big) |e^{\varrho(x-s)}\varphi_1(s; \lambda)| ds \leqslant$$

$$\leqslant \eta(\lambda)(1 + e^{\mathcal{R}(\varrho)T} + e^{2\mathcal{R}(\varrho)T} + \dots) \leqslant \eta(\lambda)(1 - e^{\mathcal{R}(\varrho)T})^{-1}.$$

Likewise, the other term in (9) has the majorant $T\eta(\lambda)(1 - e^{\mathcal{R}(\varrho)T})^{-1}$. Furthermore, from our assumption on the $\varphi_{\mu\nu}(x; \lambda)$, we have $\eta(\lambda)$ is a continuous function of λ at $\lambda = 0$ and $\eta(0) = 0$. Thus, for $|\lambda|$ sufficiently small, we can make $\int_0^x |K(x, s; \lambda)| ds < \sigma < 1$ for all x. Finally, the conditions of [**2**, (a), p. 178] are satisfied and the AC solutions of (3) are bounded.

Proof of Theorem II. If we let $y = (2i\sigma)^{-1}(Z_1 + Z_2)$, $y' = 2^{-1}(Z_1 - Z_2)$, then equation (4) is transformed into the canonical system

$$Z_1' = i\sigma Z_1 + [-(2i\sigma)^{-1}\varphi(x; \lambda) - 2^{-1}\psi(x; \lambda)] Z_1 + [-(2i\sigma)^{-1}\varphi(x; \lambda) + 2^{-1}\psi(x; \lambda)] Z_2,$$

$$Z_2' = -i\sigma Z_2 + [(2i\sigma)^{-1}\varphi(x; \lambda) + 2^{-1}\psi(x; \lambda)] Z_1 + [(2i\sigma)^{-1}\varphi(x; \lambda) - 2^{-1}\psi(x; \lambda)] Z_2,$$

which is a special case of system (3). Moreover, we see that the conditions of Theorem I are satisfied and, thus, Theorem II is proved.

Note 3. It is not possible to give adequate references to the subject. Nevertheless, we may mention here the papers of G. Calamai [**1**] and of D. Caligo [**2**] for completely independent quantitative conditions for boundedness or unboundedness of the solutions of the equation $y'' + p(x)y' + q(x)y = 0$, p, q continuous and periodic, and to the recent paper of C. Taam [**13**] on the self-adjoint second order differential equation with coefficients L-integrable in each finite interval.

Bibliography.

1. G. Calamai: (a) *Sul sistema canonico di una classe di equazioni differenziali del secondo ordine a coefficienti periodici*, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 560-566 (1934); (b) *Sulla stabilità delle soluzioni per l'equazione differenziale del secondo ordine a coefficienti periodici*, Atti Accad. Italia, Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 183-193 (1941-42); (c) *Sulla soluzione della equazione lineare omogenea del secondo ordine a coefficienti periodici*, Boll. Un. Mat. Ital. (2) **3**, 370-372 (1941).

2. D. Caligo: (a) *Un criterio sufficiente di stabilità per le soluzioni dei sistemi di equazioni integrali lineari e sue applicazioni ai sistemi di equazioni differenziali lineari*, Atti 2° Congr. Un. Mat. Ital. 177-185 (1940); (b) *Sulle equazioni differenziali lineari del secondo ordine a coefficienti periodici*, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 1025-1033 (1943).

3. C. Carathéodory, *Vorlesungen über reelle Funktionen*, Teubner, Berlin 1918.

4. L. Cesari, *Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici*. Atti Accad. Italia, Mem. Cl. Sci. Fis. Mat. Nat. (6) **11**, 633-692 (1940).

5. G. Floquet, *Sur les équations différentielles linéaires à coefficients périodiques*, Ann. Sci. École Norm. Sup. (2) **12**, 47-89 (1883).

6. R. A. Gambill: (a) *Stability criteria for linear differential systems with periodic coefficients*, Rivista Mat. Univ. Parma **5**, 169-181 (1954); (b) *Criteria for parametric instability for linear differential systems with periodic coefficients*; (c) *A fundamental system of real solutions for linear differential systems with periodic coefficients*; to appear in this same journal.

7. R. A. Gambill and J. K. Hale, *Subharmonic and ultraharmonic solutions for weakly non-linear systems*, to appear.

8. J. K. Hale: (a) *On boundedness of the solutions of linear differential systems with periodic coefficients*, Rivista Mat. Univ. Parma **5**, 137-167 (1954); (b) *Periodic solutions of non-linear systems of differential equations*, to appear in this same journal.

9. E. Hille and J. D. Tamarkin, *On the theory of linear integral equations*, I, Ann. of. Math. **31**, 479-528 (1930).

10. E. W. Hobson, *Theory of functions of a real variable*, Vol. 2, Harren Press, Washington 1950.

11. S. Lefschetz, *Lectures on differential equations*, Ann. Math. Studies, 14, Princeton Univ. Press, 1948.

12. G. Sansone, *Equazioni differenziali*, Zanichelli, Bologna 1948.

13. C. Taam, *Linear differential equations with small perturbations*, Duke Math. J. **20**, 13-25 (1953).