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Evaluations concerning products of exponential and periodic functions. (**)

Introduction. In the present paper, we study the class C_ω of functions $f(x)$, $-\infty < x < +\infty$, namely, the class of the functions $f(x)$ which are finite sums of products of the type $e^{\alpha x}\varphi(x)$, where α is any complex number and $\varphi(x)$ is any single-valued, real or complex, periodic function of period $T = 2\pi/\omega$, integrable in the sense of LEBESGUE in $[0, T]$. For this class of functions, a concept of mean value $m[f]$ is introduced as a generalization of the ordinary concept of mean value (or average) for purely periodic functions. In extension of a well-known statement for periodic functions, here, it also occurs that a function $f \in C_\omega$ has a primitive $F \in C_\omega$ if and only if $m[f] = 0$ (§ **5**). Functions $f(x) \in C_\omega$ are considered in questions of asymptotic behavior of solutions of ordinary differential equations and the particular primitive $F(x)$ of $f(x)$, $F(x) \in C_\omega$, is commonly denoted by $\int f(x)\,dx$ (see A. LIAPOUNOFF [**4**] (1), S. LEFSCHETZ [**3**]; see also L. CESARI [**1**]). This integral $\int f(x)\,dx$ can be given as an improper integral $\int_{\pm\infty}^{x} f(x)\,dx$ when $\mathcal{R}(\alpha) \neq 0$; see [**3**].

A first question is whether the particular primitive $\int f(x)\,dx$ is the definite integral of $f(x)$, say $F(x) = \int_{\xi}^{x} f(t)\,dt$, where ξ is some point, independent of x, between 0 and T. It is known that the answer is affirmative for purely periodic, real functions $f(x)$. For general functions $f(x) = e^{\alpha x}\varphi(x)$, with $\varphi(x)$

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(1) Numbers in brackets refer to Bibliography at the end of the paper.

real, the statement is not true, as we see by examples, but there is a very particular decomposition $f = f_1 + f_2$ of f into two functions of the same type, obtained by means of Faltung integrals, such that $\int f(x)\,dx = \int_{\xi_1}^{x} f_1(t)\,dt + \int_{\xi_2}^{x} f_2(t)\,dt$, where ξ_1, ξ_2 are between 0 and T and independent of x (§ **6**). A further theorem concerning the evaluations of this primitive states that

$$\left|\int e^{\alpha x}\varphi(x)\,dx\right| < N(\alpha, T)\int_0^T |\varphi(x)|\,dx\,, \qquad 0 \leqq x \leqq T,$$

where $N(\alpha, T)$ is a constant depending only upon the complex constants α, T, provided $m[e^{\alpha x}\varphi(x)] = 0$ (§ **8**).

We will use all of these theorems in a paper concerning the asymptotic behavior of the solutions of systems of linear differential equations with periodic coefficients. This paper will appear at a later date in this same journal.

1. – The family C_ω of functions. Let $C = C_\omega$ be the family of all functions which are finite sums of functions of the form $f(x) = e^{\alpha x}\varphi(x)$, $-\infty < x < +\infty$, where α is any complex number and $\varphi(x)$ is any complex-valued function of the real variable x, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$. Thus, the functions $f(x) = e^{\alpha x}$, α complex, as well as all periodic functions of period T belong to C. Also, the functions $f(x) = c$, c a complex constant, and, in particular, the function $f(x) = 0$ for all x belong to C. The latter will be called the zero function. We will say $f(x)$ is equivalent to zero if $f(x)$ differs from zero only on a set of LEBESGUE measure zero. If a function $f(x)$ belongs to C, we will write briefly $f \in C$.

For each function $f(x)$ of the form $f(x) = e^{\alpha x}\varphi(x)$, the decomposition $e^{\alpha x}\varphi(x)$ is not unique since we have also $f(x) = e^{(\alpha + ik\omega)x}\,\psi(x)$, where $\psi(x) = e^{-ik\omega x}\varphi(x)$, $(k = 0, \pm 1, \pm 2, \ldots)$. Each function $f(x)$ of the family C is of the type

$$f(x) = \sum_{j=1}^{n} e^{\alpha_j x}\varphi_j(x)\,,$$

and in virtue of the last remark, we can suppose $\alpha_j \not\equiv \alpha_k \pmod{\omega i}$ for all $j \neq k$, $(j, k = 1, 2, \ldots, n)$. Since each function $\varphi_j(x)$ is periodic of period $T = 2\pi/\omega$ and L-integrable in $[0, T]$, we shall denote by

$$\varphi_j(x) \sim \sum_{n=-\infty}^{+\infty} C_{jn} e^{in\omega x}\,, \qquad (j = 1, 2, \ldots, n),$$

the FOURIER series of $\varphi_j(x)$ and we shall denote the series

$$f(x) \approx \sum_{j=1}^{n} \sum_{l=-\infty}^{+\infty} C_{jl}\, e^{(il\omega+\alpha_j)x} \tag{1.1}$$

as the series *associated* with $f(x) \in C_\omega$ provided that $\alpha_j \not\equiv \alpha_k \pmod{\omega i}$ for all $j \neq k$, $(j, k = 1, 2, \ldots, n)$.

2. – Linear dependence. Definition. The functions $f_1(x), f_2(x), \ldots, f_n(x)$, $a \leqq x \leqq b$, are *linearly dependent on* $[a, b]$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

almost everywhere (a. e.) in $[a, b]$. Otherwise, the functions are said to be *linearly independent on* $[a, b]$.

Lemma (2.i). If the functions $\varphi_n(x)$ $(n = 1, 2, \ldots, N)$ are periodic of period $T = 2\pi/\omega$, if no $\varphi_n(x)$ is equivalent to zero, and if $\alpha_j \not\equiv \alpha_k \pmod{\omega i}$ for all $j \neq k$ $(j, k = 1, 2, \ldots, N)$, then the functions $e^{\alpha_1 x}\varphi_1(x), \ldots, e^{\alpha_N x}\varphi_N(x)$ are linearly independent on $(-\infty, +\infty)$.

Proof. There is no constant $c_1 \neq 0$ such that $c_1 e^{\alpha_1 x}\varphi_1(x) = 0$ a. e.. Suppose that it has been shown that $e^{\alpha_1 x}\varphi_1(x), \ldots, e^{\alpha_{n-1} x}\varphi_{n-1}(x)$ are linearly independent on $(-\infty, +\infty)$. Also, suppose there exist constants $c_1, c_2, \ldots, c_n$ such that

$$c_1 e^{\alpha_1 x}\varphi_1(x) + \ldots + c_n e^{\alpha_n x}\varphi_n(x) = 0 \tag{2.1}$$

a. e. in $(-\infty, +\infty)$. Consider in the following only the set of all points x for which (2.1) holds at x as well as at $x+T$, i.e., a.e.. Then we have

$$c_1 e^{\alpha_1(x+T)}\varphi_1(x+T) + \ldots + c_n e^{\alpha_n(x+T)}\varphi_n(x+T) = 0\,, \tag{2.2}$$

and, also,

$$c_1 e^{\alpha_1(x+T)} e^{-\alpha_n T}\varphi_1(x) + \ldots + c_n e^{\alpha_n(x+T)} e^{-\alpha_n T}\varphi_n(x) = 0\,, \tag{2.3}$$

since $\varphi_i(x+T) = \varphi_i(x)$, for $i = 1, 2, \ldots$. Therefore by subtracting the above formulas (2.1) and (2.3) we get

$$c_1(1-e^{(\alpha_1-\alpha_n)T})\,e^{\alpha_1 x}\varphi_1(x)+\ldots+c_{n-1}(1-e^{(\alpha_{n-1}-\alpha_n)T})\,e^{\alpha_{n-1}x}\varphi_{n-1}(x)=0\,.$$

But, by assumption, $e^{\alpha_1 x}\varphi_1(x), \ldots, e^{\alpha_{n-1}x}\varphi_{n-1}(x)$ are linearly independent and, thus,

$$c_1(1-e^{(\alpha_1-\alpha_n)T})=\ldots=c_{n-1}(1-e^{(\alpha_{n-1}-\alpha_n)T})=0\,.$$

Since $\alpha_j \not\equiv \alpha_k \pmod{\omega i}$, $j\neq k$, $(j, k=1, 2, \ldots)$, we have $e^{(\alpha_j-\alpha_k)T}\neq 1$ for all $j\neq k$, and, consequently, $c_1=c_2=\ldots=c_{n-1}=0$. From (2.1), we then have $c_n=0$. Thus, $e^{\alpha_1 x}\varphi_1(x), \ldots, e^{\alpha_n x}\varphi_n(x)$ are linearly independent. Thereby, the induction is completed and the lemma has been proved.

Lemma (2.ii). If

$$\sum_{j=1}^{n} e^{\alpha_j x}\varphi_j(x)=\sum_{k=1}^{m} e^{\beta_k x}\psi_k(x) \qquad \text{a.e. in } (-\infty, +\infty),$$

where $\varphi_j(x)$, $\psi_k(x)$ are periodic of period $T=2\pi/\omega$, $(j=1, 2, \ldots, n;\ k=1, 2, \ldots, m)$, if no one of the $\varphi_j(x)$, $\psi_k(x)$ is equivalent to zero, and if $\alpha_j\not\equiv\alpha_k \pmod{\omega i}$, $j\neq k$, $\beta_j\not\equiv\beta_k \pmod{\omega i}$, $j\neq k$, $(j, k=1, 2, \ldots)$, then

a) $m=n$,

b) $e^{\alpha_j x}\varphi_j(x)=e^{\beta_j x}\psi_j(x)$ a.e. in $(-\infty, +\infty)$, $(j=1, 2, \ldots, n)$, the functions

on the right being numbered conveniently, and

c) $\alpha_j\equiv\beta_j \pmod{\omega i}$, $(j=1, 2, \ldots, n)$.

Proof. Though the proof is quite elementary, we give it here for completeness. Let us order the numbers α_j, β_k in such a way that $\alpha_1\equiv\beta_1$, $\alpha_2\equiv\beta_2, \ldots, \alpha_h\equiv\beta_h$, while the numbers α_j, $h+1\leqq j\leqq n$, and β_k, $h+1\leqq k\leqq m$ are two by two incongruent mod ωi, $0\leqq h\leqq m, n$. Then, if $\psi_j'(x)=e^{(\beta_j-\alpha_j)x}\psi_j(x)$, $(j=1, 2, \ldots, h)$, we have

$$\sum_{j=1}^{h} e^{\alpha_j x}\left[\varphi_j(x)-\psi_j'(x)\right]+\sum_{j=h+1}^{n} e^{\alpha_j x}\varphi_j(x)-\sum_{k=h+1}^{m} e^{\beta_k x}\psi_k(x)=0 \qquad \text{a.e. in } (-\infty, +\infty).$$

By Lemma (2.i), this can only be possible if $\varphi_j(x)-\psi_j'(x)$ is equivalent to zero for at least one j, say $j=1$, and then we have

$$\sum_{j=2}^{h} e^{\alpha_j x}\left[\varphi_j(x)-\psi_j'(x)\right]+\sum_{j=h+1}^{n} e^{\alpha_j x}\varphi_j(x)-\sum_{k=h+1}^{m} e^{\beta_k x}\psi_k(x)=0 \qquad \text{a.e. in } (-\infty, +\infty).$$

From here we deduce that $\varphi_j(x) - \psi_j'(x)$ is equivalent to zero for at least one j, say $j = 2$, and so on. By h repetitions of this argument, we obtain that all functions $\varphi_j(x) - \psi_j'(x)$ $(j = 1, 2, ..., h)$ are equivalent to zero and that

$$\sum_{j=h+1}^{n} e^{\alpha_j x}\varphi_j(x) - \sum_{k=h+1}^{m} e^{\beta_k x}\psi_k(x) = 0 \qquad \text{a.e. in } (-\infty, +\infty).$$

But, by Lemma (2.i), this is possible only if $h = n$ and $h = m$. Thus, $m = n$, $\alpha_j \equiv \beta_j \pmod{\omega i}$ and $\varphi_j(x) = \psi_j'(x)$ a.e. in $(-\infty, +\infty)$, $(j = 1, 2, ..., n)$, i.e., $e^{\alpha_j x}\varphi_j(x) = e^{\beta_j x}\psi_j(x)$ a.e. in $(-\infty, +\infty)$, $(j = 1, ..., n)$.

3. – Theorem. *For each $f(x) \in C_\omega$, the associated series is uniquely determined.*

Proof. Suppose $f(x) = \sum_{j=1}^{n} e^{\alpha_j x}\varphi_j(x) = \sum_{j=1}^{m} e^{\beta_j x}\psi_j(x)$ a.e. in $(-\infty, +\infty)$, where we can suppose that the numbers α_j are two by two incongruent mod ωi and the same holds for the numbers β_j. By the preceding Lemma (2.ii), we have $m = n$ and $\alpha_j \equiv \beta_j \pmod{\omega i}$, $e^{\alpha_j x}\varphi_j(x) = e^{\beta_j x}\psi_j(x)$ a. e. in $(-\infty, +\infty)$, $(j = 1, 2, ..., n)$. Therefore, it is sufficient to prove that if $f(x) = e^{\alpha x}\varphi(x) = e^{\beta x}\psi(x)$ a.e. in $(-\infty, +\infty)$, where $\varphi(x)$, $\psi(x)$ are periodic of period T, then the series associated with $f(x)$ is uniquely determined. Since $\alpha \equiv \beta \pmod{\omega i}$, we have $\beta = \alpha - ik\omega$ for some integer k. Thus, $f(x) = e^{\alpha x}\varphi(x) = e^{\beta x}\psi(x) = e^{\alpha x}e^{-ik\omega x}\psi(x)$ a.e. in $(-\infty, +\infty)$, and $\psi(x) = e^{ik\omega x}\varphi(x)$ a.e. in $(-\infty, +\infty)$. Therefore, the FOURIER series of $\psi(x)$ is

$$\psi(x) = e^{ik\omega x}\varphi(x) \sim \sum_{n=-\infty}^{+\infty} c_{n-k}\, e^{in\omega x},$$

where $\varphi(x) \sim \sum_{n=-\infty}^{+\infty} c_n\, e^{in\omega x}$, and the series associated with $f(x)$ according to the definition in section **1** is given by

$$f(x) = e^{\beta x}\psi(x) \approx \sum_{n=-\infty}^{+\infty} c_{n-k}\, e^{(in\omega + i\beta)x} = \sum_{n=-\infty}^{+\infty} c_n\, e^{(in\omega + ik\omega + \beta)x} = \sum_{n=-\infty}^{+\infty} c_n\, e^{(in\omega+\alpha)x},$$

which is the series associated with $e^{\alpha x}\varphi(x)$. Therefore, the series associated with $f(x)$ is uniquely determined.

4. – **Mean value.** Definition. For each function $f(x) = e^{\alpha x}\varphi(x) \approx \sum_{n=-\infty}^{+\infty} c_n\, e^{(in\omega+\alpha)x}$, we shall denote by *mean value* $m[f]$ of $f(x)$ the number

$$m[f] = \begin{cases} 0 & \text{if } \; in\omega + \alpha \neq 0 \;\text{ for all } n, \\ c_n & \text{if } \; in\omega + \alpha = 0 \;\text{ for some } n. \end{cases}$$

For each function $f(x) \in C_\omega$, i.e.,

$$f(x) = \sum_{j=1}^{m} e^{\alpha_j x}\varphi_j(x) \approx \sum_{j=1}^{m} \sum_{n=-\infty}^{+\infty} c_{jn}\, e^{(in\omega+\alpha_j)x},$$

with $\alpha_j \not\equiv \alpha_k \pmod{\omega i}$ for all $j \neq k$ $(j, k = 1, 2, \ldots, m)$, let

$$m[f] = \sum_{j=1}^{m} m\left[e^{\alpha_j x}\varphi_j(x)\right].$$

This concept of mean value was studied by L. Cesari [**1**, p. 649] for functions $f(x)$ of the form $f(x) = e^{\alpha x}\varphi(x)$ with α complex and $\varphi(x)$ periodic of period T, L-integrable in $[0, T]$, and having absolutely convergent Fourier series.

Remark. If $f(x) = e^{\alpha x}\varphi(x)$, and if $in\omega + \alpha = 0$ for some n, then

$$m[f] = c_n = (1/T)\int_0^T \varphi(x)\, e^{-in\omega x}\,\mathrm{d}x.$$

In particular, if $\alpha = 0$, then

$$m[f] = m[\varphi] = (1/T)\int_0^T \varphi(x)\,\mathrm{d}x.$$

Lemma (4.i). If $f(x) = \sum_{j=1}^{n} e^{\alpha_j x}\,\varphi_j(x)$, $\alpha_j \equiv \alpha_k \pmod{\omega i}$ for every $j, k = 1, 2, \ldots, n$, $\varphi_j(x)$ periodic of period $T = 2\pi/\omega$, $(j = 1, 2, \ldots, n)$, then

$$m[f] = \sum_{j=1}^{n} m\left[e^{\alpha_j x}\varphi_j(x)\right].$$

Proof. Suppose $\alpha_j - \alpha_1 = k_j\omega i$, k_j an integer, $j = 1, 2, \ldots, n$, $k_1 = 0$. Then

$$f(x) = e^{\alpha_1 x}\sum_{j=1}^{n} e^{(\alpha_j-\alpha_1)x}\varphi_j(x)$$

and if $\varphi_j(x) \sim \sum_{h=-\infty}^{+\infty} c_{jh}\, e^{ih\omega x}$, also

$$e^{(\alpha_j-\alpha_1)x}\varphi_j(x) = e^{k_j\omega i x}\varphi_j(x) \sim \sum_{h=-\infty}^{+\infty} c_{j,h-k_j}\, e^{ih\omega x}.$$

Therefore, by definition,

$$f(x) \approx \sum_{h=-\infty}^{+\infty}\left(\sum_{j=1}^{n} c_{j,h-k_j}\right) e^{(ih\omega+\alpha_1)x}.$$

If $ih\omega + \alpha_1 \neq 0$ for all h, then $ih\omega + \alpha_j \neq 0$ for all h and $j = 1, 2, \ldots, n$ and $m[f] = 0$, $m\left[e^{\alpha_j x}\varphi_j(x)\right] = 0$ $(j = 1, 2, \ldots, n)$.

If $ih\omega + \alpha_1 = 0$ for some h, then for the same h we have $(h - k_j)\omega i + \alpha_j = 0$, and

$$m[f] = \sum_{j=1}^{m} c_{j,h-k_j}, \qquad m\left[e^{\alpha_j x}\varphi_j(x)\right] = c_{j,h-k_j}.$$

In any case,

$$m[f] = \sum_{j=1}^{n} m\left[e^{\alpha_j x}\varphi_j(x)\right].$$

Lemma (4.ii). If $f(x) = e^{\alpha x}\varphi(x)$, $\varphi(x)$ periodic of period T and if c is any complex number, then $m[cf] = c\, m[f]$.

Proof. It follows directly from the definition.

Theorem (4.i). *If* $f_1(x), f_2(x), \ldots, f_N(x) \in C_\omega$ *and* $c_1, c_2, \ldots, c_N$ *are complex constants, then* $F(x) = \sum_{h=1}^{N} c_h f_h(x) \in C_\omega$ *and* $m[F] = \sum_{h=1}^{N} c_h\, m[f_h]$.

Proof. Suppose $f_h(x) = \sum_{j=1}^{m_h} e^{\alpha_{jh}x}\varphi_{jh}(x)$, $\alpha_{jh} \not\equiv \alpha_{kh}$ (mod ωi) for all $j \neq k$ ($j,\ k = 1, 2, \ldots, m_h$; $h = 1, 2, \ldots, N$), and $\varphi_{jh}(x)$ periodic of period T. Then

$$F(x) = \sum_{h=1}^{N} c_h \sum_{j=1}^{m_h} e^{\alpha_{jh}x}\varphi_{jh}(x) = \sum_{h=1}^{N}\sum_{j=1}^{m_h} e^{\alpha_{jh}x} c_h \varphi_{jh}(x) = \sum_{h=1}^{N}\sum_{j=1}^{m_h} e^{\alpha_{jh}x}\varphi^*_{jh}(x),$$

where $\varphi^*_{jh}(x) = c_h\varphi_{jh}(x)$ is periodic of period $T = 2\pi/\omega$. Thus, by definition, $F(x) \in C_w$. Let $\beta_1, \beta_2, \ldots, \beta_M$ be all the numbers of the set α_{jh} ($j = 1, 2, \ldots, m_h$; $h = 1, 2, \ldots, N$), which are two by two incongruent, and let $\alpha_{j_{\nu,k}h_{\nu,k}}$ be the numbers of the set α_{jh} ($j = 1, 2, \ldots, m_h$; $h = 1, 2, \ldots, N$), such that $\alpha_{j_{\nu,k}h_{\nu,k}} \equiv \beta_\nu$ ($k = 1, \ldots, \lambda_\nu$). Then $F(x)$ can be written as

$$F(x) = \sum_{\nu=1}^{M} e^{\beta_\nu x}\psi_\nu(x),$$

where

$$\psi_\nu(x) = \sum_{k=1}^{\lambda_\nu} e^{(\alpha_{j_{\nu,k}h_{\nu,k}} - \beta_\nu)x} c_{h_{\nu,k}}\varphi_{j_{\nu,k}h_{\nu,k}}(x).$$

By definition, $m[F] = \sum_{\nu=1}^{M} m\left[e^{\beta_\nu x}\psi_\nu(x)\right]$, and by Lemmas (4.i) and (4.ii), we have

$$m\left[e^{\beta_\nu x}\psi_\nu(x)\right] = \sum_{k=1}^{\lambda_\nu} c_{h_{\nu,k}}\, m\left[e^{\alpha_{j_{\nu,k}h_{\nu,k}}x}\varphi_{j_{\nu,k}h_{\nu,k}}(x)\right].$$

As a consequence, upon rearranging terms, we get

$$m[F]=\sum_{h=1}^{N} c_h \sum_{j=1}^{m_h} m\left[e^{\alpha_{jh}x}\varphi_{jh}(x)\right]=\sum_{h=1}^{N} c_h\, m[f_h]\,.$$

5. – Primitives of functions of the class C_ω. We shall need the following theorems from the theory of FOURIER series.

Theorem (5.i). *If the function $f(x)$ is periodic of period 2π and* L-*integrable in $[0, 2\pi]$, and the function $g(x)$ is of bounded variation in the finite interval (α, β), then $\int_\alpha^\beta f(x)\,g(x)\,\mathrm{d}x$ may be evaluated by substituting for $f(x)$ its Fourier series, and applying term by term integration, and the series obtained is convergent.*

Theorem (5.ii). *Let $f(x)$, $g(x)$ be periodic functions of period 2π,* L-*integrable in $[0, 2\pi]$, and let*

$$f(x)\sim\sum_{m=-\infty}^{+\infty} c_m e^{imx},\qquad g(x)\sim\sum_{m=-\infty}^{+\infty} d_m e^{imx},$$

be their Fourier series. Then

$$\text{(a)}\qquad h(x)=\frac{1}{2\pi}\int_0^{2\pi} f(x+t)\,g(t)\,\mathrm{d}t$$

exists for almost all x and is L-*integrable, and*

$$\text{(b)}\qquad h(x)\sim\sum_{m=-\infty}^{+\infty} c_m\, d_{-m}\, e^{imx}\,.$$

For a proof of Theorem (5.i), see E. W. HOBSON [**2**, Vol. II, p. 582], or L. TONELLI [**5**, p. 343]. For a proof of Theorem (5.ii), see A. ZYGMUND [**6**, p. 14]. These theorems are proved for real-valued functions of a real variable, but it is only a formal procedure to show that they also hold true for complex-valued functions of a real variable.

Lemma (5.i). If $f(x)=e^{\alpha x}\varphi(x)$, α complex, $\varphi(x)\sim\sum_{n=-\infty}^{+\infty} c_n e^{in\omega x}$, and if $m[f]=0$, then there is a primitive of $e^{\alpha x}\varphi(x)$, say $\int e^{\alpha t}\varphi(t)\,\mathrm{d}t$, which belongs to C_ω, and there is one and only one primitive of $e^{\alpha x}\varphi(x)$ such that $m\left[\int e^{\alpha t}\varphi(t)\,\mathrm{d}t\right]=0$. Moreover, this unique primitive of mean value zero is given by

$$\int e^{\alpha t}\varphi(t)\,\mathrm{d}t=e^{\alpha x}\psi(x)=e^{\alpha x}\sum_{n=-\infty}^{+\infty} c_n(in\omega+\alpha)^{-1}\, e^{in\omega x}\,,$$

where $\psi(x)$ is periodic of period $T=2\pi/\omega$.

Proof. Every primitive of $e^{\alpha t}\varphi(t)$ is of the form

$$\Phi(x) = \int_0^x e^{\alpha t}\varphi(t)\,dt + C,$$

where C is an arbitrary constant. But, by Theorem (5.i), we may evaluate this integral by susbtituting for $\varphi(t)$ its FOURIER series and integrating term by term to get

$$\Phi(x) = \sum_{n=-\infty}^{+\infty} \left[c_n(in\omega + \alpha)^{-1} e^{(in\omega+\alpha)t}\right]_0^x + C.$$

We know $\sum_{n=-\infty}^{+\infty} c_n\,(in\omega)^{-1}\cdot e^{in\omega x}$ converges uniformly [**6**, p. 27]. Also, since the sequence $in\omega(in\omega + \alpha)^{-1}$ $(n = 1, 2, \ldots;\ in\omega + \alpha \neq 0)$, is bounded and of bounded variation, we know that the series $\sum_{n=-\infty}^{+\infty} c_n\,(in\omega + \alpha)^{-1}\cdot e^{in\omega x}$ converges uniformly [**6**, p. 3].

Therefore, we may write

$$\Phi(x) = \left[e^{\alpha t} \sum_{n=-\infty}^{+\infty} c_n(in\omega + \alpha)^{-1} e^{in\omega t}\right]_0^x + C = \left[e^{\alpha t}\psi(t)\right]_0^x + C = e^{\alpha x}\psi(x) - \psi(0) + C,$$

where $\psi(x)$ is periodic of period $T = 2\pi/\omega$. Moreover, $m\left[e^{\alpha x}\psi(x)\right] = 0$, and if we choose $C = \psi(0)$, we have $m[\Phi] = 0$, as was to be shown. That there is only one such primitive of mean value zero is easily shown. For, suppose there is another, say $\Phi^*(x)$; then $\Phi = \Phi^* + C$ and, thus, $m[C] = 0$, or $C = 0$.

Theorem (5.iii). *Any function $f(x) \in C_\omega$ has a primitive $F(x) \in C_\omega$ if and only if $m[f] = 0$.*

Proof. Let $f(x) = \sum_{j=1}^{n} e^{\alpha_j x}\varphi_j(x)$, where each α_j is a complex constant and each function $\varphi_j(x) \sim \sum_{n=-\infty}^{+\infty} c_{j,n}e^{in\omega x}$. Moreover, suppose $in_j\omega + \alpha_j = 0$ for some n_j and every $j = 1, 2, \ldots, n$, that is $m\left[e^{\alpha_j x}\varphi_j(x)\right] = c_{j,n_j}$ $(j = 1, 2, \ldots, n)$. Then, the functions

$$g_j(x) = e^{\alpha_j x}\left[\varphi_j(x) - c_{j,n_j} e^{in_j\omega x}\right] \qquad (j = 1, 2, \ldots, n),$$

are such that $m[g_j] = 0$ $(j = 1, 2, \ldots, n)$. Moreover, we have

$$\int f(x)\,dx = \sum_{j=1}^{n} \int g_j(x)\,dx + \int \left(\sum_{j=1}^{n} c_{j,n_j}\right) dx\,.$$

Therefore, from the preceding Lemma (5.i), we know that there is a primitive of $g_j(x)$ contained in C_ω and there will be a primitive of $\sum_{j=1}^{n} c_{j,n_j}$ contained in C_ω if and only if $\sum_{j=1}^{n} c_{j,n_j} = 0$, i.e., $m[f] = 0$, by theorem (4.i). If for some $j = 1, 2, \ldots, n$, we have $in\omega + \alpha_j \neq 0$ for all n, then Lemma (5.i) can be applied directly to the corresponding term $e^{\alpha_j x}\varphi_j(x)$.

6. – Integral form of the primitives in the class C_ω.

The following lemmas will show how these primitives may be obtained as definite integrals.

Lemma (6.i). If $\varphi(x)$ is a real-valued function, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$, and if $m[\varphi] = 0$, then there exists a number ξ, $0 < \xi < T$, such that the function

$$\Phi(x) = \int_{\xi}^{x} \varphi(t)\,dt$$

is periodic of period T, continuous in $(-\infty, +\infty)$ and $m[\Phi] = 0$.

Proof. Let $\psi(x) = \int_0^x \varphi(t)\,dt$, $C = -(1/T)\int_0^T dx \int_0^x \varphi(t)\,dt$, and $\Phi(x) = \psi(x) + C$. Then

$$\Phi(x + T) = \psi(x + T) + C = \int_0^T \varphi(t)\,dt + \int_T^{T+x} \varphi(t)\,dt + C\,.$$

By the remark after the definition of mean value in section **4**, we have $m[\varphi] = (1/T)\int_0^T \varphi(t)\,dt$, and, thus, $m[\varphi] = 0$ implies $\int_0^T \varphi(t)\,dt = 0$. Moreover, if we let $t = T + \tau$ in the second integral, then

$$\Phi(x + T) = \int_0^x \varphi(T + \tau)\,d\tau + C = \int_0^x \varphi(\tau)\,d\tau + C = \Phi(x)\,,$$

or, $\Phi(x)$ is periodic of period T. Moreover, $\Phi(x)$ is continuous in $(-\infty, +\infty)$. On the other hand,

$$\int_0^T \Phi(x)\,dx = \int_0^T dx \int_0^x \varphi(t)\,dt + CT = \int_0^T dx \int_0^x \varphi(t)\,dt - \int_0^T dx \int_0^x \varphi(t)\,dt = 0\,,$$

and, finally, $m[\Phi] = 0$.

Since $\Phi(x)$ is continuous in $[0, T]$ and $\int_0^T \Phi(x)\,dx = 0$, $\Phi(x)$ has at least one zero ξ such that $0 < \xi < T$. We, therefore, have

$$\Phi(x) = \int_{\xi}^{x} \varphi(t)\,dt\,,$$

since both the first and the second members are primitive functions of $\varphi(t)$ and both are zero at $x=\xi$.

Remark 1. If $\varphi(x)=\varphi_1(x)+i\varphi_2(x)$, where $\varphi_1(x)$, $\varphi_2(x)$ are real-valued functions, periodic of period T, L-integrable in $[0, T]$, and if $m[\varphi]=0$, then there exist numbers ξ_1, ξ_2, $0<\xi_1, \xi_2<T$, such that the function

$$\Phi(x)=\int \varphi(t)\,dt=\int_{\xi_1}^{x}\varphi_1(t)dt+i\int_{\xi_2}^{x}\varphi(t)\,dt$$

is periodic of period T, continuous in $(-\infty, +\infty)$, and $m[\Phi]=0$.

Remark 2. That it is sometimes necessary to take $\xi_1\neq\xi_2$ is shown by the following simple example. Let

$$\Phi(x)=\int(\cos t+i\sin t)\,dt=\int_{\xi_1}^{x}\cos t\,dt+i\int_{\xi_2}^{x}\sin dt=\sin t\Big]_{\xi_1}^{x}-\mathrm{i}\cos t\Big]_{\xi_2}^{x}.$$

If $m[\Phi]=0$, we must choose ξ_1, ξ_2 such that $\sin\xi_1=0$, $\cos\xi_2=0$ and, thus, $\xi_1\neq\xi_2$.

Suppose we are given a function $f(x)=e^{\alpha x}\varphi(x)$, where α is a complex number and $\varphi(x)$ is complex-valued and periodic of period T, L-integrable in $[0, T]$ and $m[f]=0$. From the last remark, one might suspect that there exist numbers ξ_1, ξ_2, $0<\xi_1, \xi_2<T$, such that the function

$$\Phi(x)=\int e^{\alpha t}\varphi(t)\,dt=\int_{\xi_1}^{x}\mathcal{R}(e^{\alpha t}\varphi(t))\,dt+i\int_{\xi_2}^{x}\mathcal{J}(e^{\alpha t}\varphi(t))\,dt$$

is continuous in $(-\infty, +\infty)$, and $m[\Phi]=0$. The following example shows that the numbers ξ_1, ξ_2 cannot always be chosen such that $0<\xi_1$, $\xi_2<T$ if we decompose $e^{\alpha t}\varphi(t)$ into its real and imaginary parts as above.

Example. Let $f(x)=e^{i\beta x}\sin x$, β real, $\beta=1+\varepsilon$, ε arbitrary >0. Try to determine ξ_1, ξ_2 such that

$$\Phi(x)=\int e^{i\beta t}\sin t\,dt=\int_{\xi_1}^{x}\cos\beta t\cdot\sin t\,dt+i\int_{\xi_2}^{x}\sin\beta t\cdot\sin t\,dt$$

is continuous in $(-\infty, +\infty)$ and $m[\Phi]=0$. If we integrate this equation, we get

$$\Phi(x)=\left[\psi(x)-\psi(\xi_1)\right]+i\left[\eta(x)-\eta(\xi_2)\right],$$

where

$$\psi(t) = -\frac{\cos(1-\beta)t}{2(1-\beta)} - \frac{\cos(1+\beta)t}{2(1+\beta)},$$

$$\eta(t) = \frac{\sin(1-\beta)t}{2(1-\beta)} - \frac{\sin(1+\beta)t}{2(1+\beta)}.$$

If we want $m[\Phi] = 0$, we must choose ξ_1, ξ_2 such that $\psi(\xi_1) = 0$, $\eta(\xi_2) = 0$. Since $\beta = 1 + \varepsilon$, $\psi(\xi_1) = 0$ implies $\cos(2+\varepsilon)\xi_1 = [1 + (2/\varepsilon)]\cos \varepsilon\xi_1$. But, for ε very small, a value of ξ_1 which satisfies this equation would have to be much greater than 2π. In fact, for $\varepsilon = .05$, $\xi_1 > 9\pi$.

Therefore, if we want to be sure that the points are to be always contained in $(0, T)$, we must decompose $e^{\alpha x}\varphi(x)$ in some other way.

Lemma (6.ii). Let $f(t) = e^{(\alpha+i\beta)t}\varphi(t)$, where α, β are real numbers, $\varphi(t)$ is a real function, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$, $m[\varphi] = 0$, and $\alpha + i\beta \not\equiv 0 \pmod{\omega i}$. Then, the function $\varphi(t)$ can be decomposed into

$$\varphi(t) = \varphi_1(t) + i\varphi_2(t), \tag{6.1}$$

where $\varphi_1(t)$, $\varphi_2(t)$ are L-integrable functions (not necessarily real), periodic of period T, and such that

$$\int e^{(\alpha+i\beta)t}\varphi(t)\,dt = \int_{\xi_1}^{x} e^{(\alpha+i\beta)t}\varphi_1(t)\,dt + i\int_{\xi_2}^{x} e^{(\alpha+i\beta)t}\varphi_2(t)\,dt = e^{(\alpha+i\beta)x}[\psi(x) + i\eta(x)], \tag{6.2}$$

$0 < \xi_1, \xi_2 < T$, and $\psi(x)$, $\eta(x)$ are continuous, real-valued functions in $(-\infty, +\infty)$, periodic of period T, and $m[\psi] = 0$, $m[\eta] = 0$.

Proof. Let

$$\varphi_2(t) = i\beta\int_0^T \varphi(t+\tau)g(\tau)\,d\tau, \qquad \varphi_1(t) = \varphi(t) - i\varphi_2(t), \tag{6.3}$$

where $g(t)$, $-\infty < t < +\infty$, is the periodic function of period T defined for all $0 \leqq t < T$ by

$$\begin{cases} g(t) = H(\alpha,\beta)e^{(\alpha-i\beta)t}, \\ H(\alpha,\beta) = i(e^{T(\alpha+i\beta)} - 1)/|e^{T(\alpha+i\beta)} - 1|^2, \end{cases} \tag{6.4}$$

and defined in $(-\infty, +\infty)$ by the periodicity of period T. Since $\alpha + i\beta \not\equiv 0 \pmod{\omega i}$, we have $|e^{T(\alpha+i\beta)} - 1| \neq 0$, and, thus, the function $g(t)$ is of bounded variation in $[0, T]$. As a first consequence, we have also $\alpha - i\beta \not\equiv 0 \pmod{\omega i}$. From Theorem (5.ii) and formula (6.3) above, we have

$\varphi_1(t)$, $\varphi_2(t)$ are summable in $[0, T]$. In the next few lines the symbol $\sum_{n=-\infty}^{+\infty}{}'$ shall mean that the term with $n = 0$ is zero. Let

$$\varphi(t) \sim \sum_{n=-\infty}^{+\infty}{}' c_n e^{in\omega t}, \qquad g(t) \sim \sum_{n=-\infty}^{+\infty} d_n e^{in\omega t}.$$

Then,

$$d_n = (1/T)\int_0^T g(t)e^{-in\omega t}\,dt = [H(\alpha, \beta)/T]\int_0^T e^{(\alpha-i\beta)t}e^{-in\omega t}\,dt =$$
$$= [H(\alpha, \beta)/T]\cdot[(e^{T(\alpha-i\beta)} - 1)/(\alpha - i\beta - in\omega)],$$

and, thus, from (6.3) and Theorem (5.ii), we have

$$(6.5)\quad \begin{cases} \varphi_2(t) \sim i\beta T \sum_{n=-\infty}^{+\infty}{}' c_n d_{-n} e^{in\omega t} = i\beta \sum_{n=-\infty}^{+\infty}{}' \dfrac{H(\alpha, \beta)[e^{T(\alpha-i\beta)} - 1]}{\alpha - i\beta + in\omega} c_n e^{in\omega t} = \\ \qquad = \sum_{n=-\infty}^{+\infty}{}' \dfrac{-\beta c_n}{\alpha - i\beta + in\omega} e^{in\omega t}. \end{cases}$$

Therefore,

$$(6.6)\quad \begin{aligned} \varphi_1(t) = \varphi(t) - i\varphi_2(t) &\sim \sum_{n=-\infty}^{+\infty}{}' c_n\left(1 + \frac{i\beta}{\alpha - i\beta + in\omega}\right) e^{in\omega t} = \\ &= \sum_{n=-\infty}^{+\infty}{}' \frac{\alpha + in\omega}{\alpha - i\beta + in\omega} c_n e^{in\omega t}. \end{aligned}$$

Moreover, since $\varphi_1(t)$ is summable and $e^{(\alpha+i\beta)t}$ is of bounded variation in $[0, T]$, we have, using Theorem (5.i) and formula (6.6), that

$$(6.7)\quad \int_{\xi_1}^{x} e^{(\alpha+i\beta)t}\varphi_1(t)\,dt = \sum_{n=-\infty}^{+\infty}{}' \left[\frac{\alpha + in\omega}{(\alpha + i\beta + in\omega)(\alpha - i\beta + in\omega)} c_n e^{(\alpha+i\beta+in\omega)t}\right]_{\xi_1}^{x},$$

where ξ_1 is a constant to be determined later. Consider the series

$$(6.8)\quad \begin{aligned} &\sum_{n=-\infty}^{+\infty}{}' \frac{(\alpha + in\omega)c_n}{(\alpha - i\beta + in\omega)(\alpha + i\beta + in\omega)} e^{in\omega t} = \\ &= \sum_{n=-\infty}^{+\infty}{}' \frac{(\alpha + in\omega)(\alpha - in\omega + i\beta)(\alpha - i\beta - in\omega)}{[\alpha^2 + (n\omega - \beta)^2]\cdot[\alpha^2 + (n\omega + \beta)^2]} c_n e^{in\omega t} = \\ &= \sum_{n=-\infty}^{+\infty}{}' [Q_1(n) + iQ_2(n)] \frac{c_n}{in\omega} e^{in\omega t}, \end{aligned}$$

where $Q_1(n)$, $Q_2(n)$ are real bounded functions of n, positive and monotone for all $n > n_0$, sufficiently large. We know that the series $\sum_{n=-\infty}^{+\infty}{}' c_n(in\omega)^{-1}e^{in\omega t}$ converges uniformly [,6 p. 27]. Therefore, from the character of $Q_1(n)$, $Q_2(n)$, the series (6.8) converges uniformly [6, p. 3]. Consequently, the series (6.7) can be written in a simpler form

$$\int_{\xi_1}^{x} e^{(\alpha+i\beta)t}\varphi_1(t)\,dt = \left[e^{(\alpha+i\beta)t}\psi(t)\right]_{\xi_1}^{x}, \tag{6.9}$$

where

$$\psi(t) = \sum_{n=-\infty}^{+\infty}{}' \frac{\alpha + in\omega}{(\alpha + i\beta + in\omega)(\alpha - i\beta + in\omega)}\, c_n e^{in\omega t}. \tag{6.10}$$

The series (6.10) is the FOURIER series of $\psi(t)$ since the sum of the squares of the coefficients is convergent [6, p. 74]. The function $\psi(t)$ is periodic of period $T = 2\pi/\omega$, and since $c_0 = 0$, also, $m[\psi] = 0$. Moreover, since $e^{(\alpha+i\beta)t}\psi(t)$ is a primitive function of $e^{(\alpha+i\beta)t}\varphi_1(t)$, we have $\psi(t)$ is continuous in $(-\infty, +\infty)$. Also, $\psi(t)$ is a real function; for,

$$\bar{\psi}(t) = \sum_{n=-\infty}^{+\infty}{}' \frac{\alpha - in\omega}{(\alpha - i\beta - in\omega)(\alpha + i\beta - in\omega)}\, c_{-n} e^{-in\omega t} =$$

$$= \sum_{n=-\infty}^{+\infty}{}' \frac{\alpha + in\omega}{(\alpha - i\beta + in\omega)(\alpha + i\beta + in\omega)}\, c_n e^{in\omega t} = \psi(t).$$

Therefore, since $\psi(t)$ is a continuous, real-valued function in $(-\infty, +\infty)$ and $m[\psi] = 0$, there exists a number ξ_1, $0 < \xi_1 < T$, such that $\psi(\xi_1) = 0$.

In (6.9), if we choose ξ_1 such that $\psi(\xi_1) = 0$, then

$$\int_{\xi_1}^{x} e^{(\alpha+i\beta)t}\varphi_1(t)\,dt = e^{(\alpha+i\beta)x}\psi(x). \tag{6.11}$$

In a similar manner, from Theorem (5.ii) and formula (6.5), we get

$$\int_{\xi_2}^{x} e^{(\alpha+i\beta)t}\varphi_2(t)\,dt = \left[e^{(\alpha+i\beta)t}\eta(t)\right]_{\xi_2}^{x}, \tag{6.12}$$

where

$$\eta(t) = \sum_{n=-\infty}^{+\infty}{}' \frac{-\beta c_n}{(\alpha + i\beta + in\omega)(\alpha - i\beta + in\omega)}\, e^{in\omega t}. \tag{6.13}$$

Moreover, $\eta(t)$ is a continuous, real-valued function in $(-\infty, +\infty)$, periodic of period T and $m[\eta] = 0$. Thus, there exists a number ξ_2, $0 < \xi_2 < T$ such that $\eta(\xi_2) = 0$, and if, in (6.12), ξ_2 is chosen such that $\eta(\xi_2) = 0$, we have

$$\int_{\xi_2}^{x} e^{(\alpha+i\beta)t}\,\varphi_2(t)\,\mathrm{d}t = e^{(\alpha+i\beta)x}\,\eta(x)\,. \tag{6.14}$$

Combining (6.11) and (6.14), we have

$$\int e^{(\alpha+i\beta)t}\varphi(t)\,\mathrm{d}t = e^{(\alpha+i\beta)x}\left[\psi(x) + i\,\eta(x)\right],$$

as was to be shown. Moreover, combining (6.10) and (6.13), we have

$$\int e^{(\alpha+i\beta)t}\,\varphi(t)\,\mathrm{d}t = e^{(\alpha+i\beta)x}\sum_{n=-\infty}^{+\infty}{}' \frac{c_n\,e^{in\omega x}}{\alpha + in\omega + i\beta}\,.$$

This last result would also have followed from Theorem (5.iii).

Theorem (6.i). *If* $f(x) = \sum_{j=1}^{n} e^{\alpha_j x}\varphi_j(x) \in C_\omega$, $m[\varphi_j] = 0$ $(j = 1, 2, \dots, n)$, *and if* $m[f] = 0$, *then there is a finite decomposition* $f = \sum_{i=1}^{m} f_i$ *in* C_ω *such that*

$$\int f(x)\,\mathrm{d}x = \sum_{i=1}^{m}\int_{\xi_i}^{x} f_i(x)\,\mathrm{d}x\,,$$

with $0 < \xi_i < T$ $(i = 1, 2, \dots, m)$.

Proof. If, as in Theorem (5.iii), we suppose $in_j\omega + \alpha_j = 0$ for some n_j $(j = 1, 2, \dots, n)$, and we let $g_j(x) = e^{\alpha_j x}[\varphi_j(x) - c_{j,n_j}e^{in_j\omega x}]$, where $c_{j,n_j} = m[e^{\alpha_j x}\varphi_j(x)]$, then $m[g_j] = 0$, $(j = 1, 2, \dots, n)$. Therefore, we have

$$\int f(x)\,\mathrm{d}x = \sum_{j=1}^{n}\int g_j(x)\,\mathrm{d}x + \int\Big(\sum_{j=1}^{n} c_{j,n_j}\Big)\,\mathrm{d}x = \sum_{j=1}^{n}\int g_j(x)\,\mathrm{d}x$$

since $m[f] = 0$. Moreover, by assumption, $m[\varphi_j] = 0$, and, thus, $m[\varphi_j(x) - c_{j,n_j}e^{in_j\omega x}] = 0$ $(j = 1, 2, \dots, n)$. Finally, since $m[g_j] = 0$, we may apply the preceding Lemma (6.ii) to obtain each of the primitives $\int g_j(x)\,\mathrm{d}x$. If for some $j = 1, 2, \dots, n$, we have $in\omega + \alpha_j \neq 0$ for all n, then Lemma (6.ii) can be applied directly to the corresponding term $e^{\alpha_j x}\varphi_j(x)$. Thus, the theorem is proved.

7. – Further remarks. If, in Lemma (6.ii), $\varphi(t)$ were a complex-valued function, the above procedure could be carried out for both real and the imaginary parts of $\varphi(t)$.

It is also interesting to observe that in Lemma (6.ii), since $f(t) = e^{(\alpha+i\beta)t}\varphi(t)$, we have

$$\int f(t)\,dt = \int_{\xi_1}^{x} e^{(\alpha+i\beta)t}\varphi_1(t)\,dt + i\int_{\xi_2}^{x} e^{(\alpha+i\beta)t}\varphi_2(t)\,dt =$$

$$= \int_{\xi_1}^{x} e^{(\alpha+i\beta)t}\varphi(t)\,dt + \beta H(\alpha,\beta)\int_{\xi_1}^{\xi_2} e^{(\alpha+i\beta)t}\int_0^T e^{(\alpha-i\beta)\tau}\varphi(t+\tau)\,dt\,d\tau,$$

or,

$$\int f(t)\,dt = \int_{\xi_1}^{x} f(t)\,dt + \beta H(\alpha,\beta)\int_0^T e^{-2i\beta\tau}\Big\{\int_{\xi_1}^{\xi_2} f(t+\tau)dt\Big\}d\tau\,. \tag{7.1}$$

Example 1. Let us return to the preceding example where $f(x) = e^{i\beta x}\sin x$, $\beta = 1+\varepsilon$, $\varepsilon > 0$. Using the notation of Lemma (6.ii), we have $\alpha = 0$, $\beta = \beta$, $\varphi(t) = \sin t$, $\omega = 1$. Calculating $\varphi_1(t)$, $\varphi_2(t)$ from (6.3) and (6.4), we get

$$\varphi_2(t) = \frac{\beta}{2}\left(\frac{e^{it}}{1-\beta} + \frac{e^{-it}}{1+\beta}\right), \qquad \varphi_1(t) = \sin t - i\varphi_2(t)\,,$$

and $\psi(t) = -(\cos t)(1-\beta^2)^{-1}$, $\eta(t) = \beta(1-\beta^2)^{-1}\sin t$. Therefore, we may take $\xi_1 = \pi/2$, $\xi_2 = \pi$, and we will have

$$\int e^{i\beta t}\sin t\,dt = \int_{\xi_1}^{x} e^{i\beta t}\varphi_1(t)\,dt + i\int_{\xi_2}^{x} e^{i\beta t}\varphi_2(t)\,dt = e^{i\beta x}\left[-\frac{\cos x}{1-\beta^2} + i\,\frac{\beta}{1-\beta^2}\sin x\right].$$

Example 2. Suppose that $f(x) = e^{-x}\cos x + e^{i\beta x}\sin x$, where β is a real number. Then, for the function $e^{-x}\cos x$, we have using (6.3), (6.4), that $\varphi_1(t) = \cos t$, $\varphi_2(t) = 0$, and $\psi(t) = (1/2)(-\cos t + \sin t)$, $\eta(t) = 0$. Moreover, $\psi(\pi/4) = 0$, and from the preceding example, we see that in this case $\int f(x)\,dx$ consists of a sum of three definite integrals.

8. – Evaluations for the primitives in the class C_ω. Let $f(x) = e^{(\alpha+i\beta)x}$, α, β real. In order to obtain a primitive $\int e^{(\alpha+i\beta)t}\,dt$ of mean value zero, we will perform the integrations as follows:

$$\begin{cases} \alpha > 0, & \int e^{(\alpha+i\beta)t}\,dt = \int_{-\infty}^{x} e^{(\alpha+i\beta)t}\,dt = \dfrac{e^{(\alpha+i\beta)x}}{\alpha+i\beta}\,, \\ \alpha < 0, & \int e^{(\alpha+i\beta)t}\,dt = \int_{+\infty}^{x} e^{(\alpha+i\beta)t}\,dt = \dfrac{e^{(\alpha+i\beta)x}}{\alpha+i\beta}\,, \\ \alpha = 0, & \int e^{i\beta t}\,dt = \int_{\pi/\beta}^{x}\cos\beta t\,dt + i\int_{\pi/(2\beta)}^{x}\sin\beta t\,dt = \dfrac{e^{i\beta x}}{i\beta}\,. \end{cases} \tag{8.1}$$

Lemma (8.i). If $f(x) = e^{(\alpha+i\beta)x}\varphi(x)$, α, β real, and if $\varphi(t)$ is a real-valued L-integrable function in $[0, T]$, periodic of period $T=2\pi/\omega$, and if $m[f]=0$, then the primitive $\int e^{(\alpha+i\beta)t}\varphi(t)\,dt$ may be chosen so that

$$\left|\int e^{(\alpha+i\beta)t}\varphi(t)\,dt\right| \leqq 2^{-1}MN(\alpha, \beta, T)$$

for all $0 \leqq x \leqq T$, where $M = \int_0^T |\varphi(t)|\,dt$ and $N(\alpha, \beta, T)$ is a constant depending only on α, β, T.

Proof. *Case* 1. In case $m[\varphi] = 0$, we may apply Lemma (6.ii) and formula (7.1), to get

$$(8.2)\qquad \begin{cases} \int e^{(\alpha+i\beta)t}\varphi(t)\,dt = \int_{\xi_1}^{x} e^{(\alpha+i\beta)t}\varphi(t)\,dt + \\ + \beta H(\alpha,\beta)\int_0^T e^{-2i\beta\tau}\left[\int_{\xi_1}^{\xi_2} e^{(\alpha+i\beta)(t+\tau)}\varphi(t+\tau)\,dt\right]d\tau, \end{cases}$$

where $0 < \xi_1, \xi_2 < T$ and $H(\alpha, \beta) = i(e^{T(\alpha+i\beta)} - 1)\,|e^{T(\alpha+i\beta)} - 1|^{-2}$.

If we let $t = u - v$, $\tau = v$, then the last integral in (8.2) becomes

$$(8.3)\qquad I = \beta H(\alpha, \beta)\int_0^T e^{-2i\beta v}\left[\int_{\xi_1+v}^{\xi_2+v} e^{(\alpha+i\beta)u}\varphi(u)\,du\right]dv\,,$$

and the interval $(\xi_1 + v, \xi_2 + v)$ is contained in the interval $[0, 2T]$. Therefore, $|I| \leqq 2MT|\beta|e^{2|\alpha|T}/|e^{T(\alpha+i\beta)} - 1|$, and from (8.2), we see that

$$\left|\int e^{(\alpha+i\beta)t}\varphi(t)\,dt\right| \leqq Me^{|\alpha|T} + \frac{2MT|\beta|e^{2|\alpha|T}}{|e^{T(\alpha+i\beta)} - 1} = MR(\alpha, \beta, T)/2,$$

where $R(\alpha, \beta, T) = 2\{e^{|\alpha|T} + 2T|\beta|e^{2|\alpha|T} - |e^{T(\alpha+i\beta)} - 1|\}$.

Case 2. $m[\varphi] = c_0 \neq 0$. We may then write $\varphi(x) = \overline{\varphi}(x) + c_0$ where $m[\overline{\varphi}] = 0$, and, then,

$$\int e^{(\alpha+i\beta)t}\varphi(t)\,dt = \int e^{(\alpha+i\beta)t}\,\overline{\varphi}(t)\,dt + c_0\int e^{(\alpha+i\beta)t}\,dt\,.$$

We have $|c_0| = (1/T)\left|\int_0^T \varphi(t)\,dt\right| \leqq (M/T)$, and, thus,

$$\int_0^T |\overline{\varphi}(t)|\,dt = \int_0^T |\varphi - c_0|\,dt \leqq M + |c_0|\,T \leqq 2M\,.$$

Also, from (8.1), we see that

$$\left|\int e^{(\alpha+i\beta)t}\,dt\right| \leqq e^{\alpha x}/\,|\alpha + i\beta| \leqq e^{|\alpha|T}/\,|\alpha + i\beta|$$

for all α, β, and, thus,

$$\left|\int e^{(\alpha+i\beta)t}\varphi(t)\,\mathrm{d}t\right| \leqq \left|\int e^{(\alpha+i\beta)t}\bar{\varphi}(t)\,\mathrm{d}t\right| + |c_0|\cdot\left|\int e^{(\alpha+i\beta)t}\,\mathrm{d}t\right| \leqq$$

$$\leq \frac{2MR(\alpha,\beta,T)}{2} + \frac{Me^{|\alpha|T}}{T|\alpha+i\beta|} = \frac{MN(\alpha,\beta,T)}{2},$$

where $N(\alpha,\beta,T) = 2R(\alpha,\beta,T) + 2e^{|\alpha|T}/(T|\alpha+i\beta|)$. As a consequence, we have in all cases that

$$\left|\int e^{(\alpha+i\beta)t}\varphi(t)\,\mathrm{d}t\right| \leqq 2^{-1}MN(\alpha,\beta,T).$$

Remark 1. Suppose $\varphi(t) = \varphi_1(t) + i\varphi_2(t)$ where $\varphi_1(t)$, $\varphi_2(t)$ are real. Then, since

$$|\varphi_1(t)| \leqq |\varphi(t)|, \qquad |\varphi_2(t)| \leqq |\varphi(t)|,$$

we have

$$\left|\int e^{(\alpha+i\beta)t}\varphi(t)\,\mathrm{d}t\right| \leqq \left|\int e^{(\alpha+i\beta)t}\varphi_1(t)\,\mathrm{d}t\right| + \left|\int e^{(\alpha+i\beta)t}\varphi_2(t)\,\mathrm{d}t\right| \leq MN(\alpha,\beta,T).$$

Remark 2. If α, β are considered as parameters and allowed to assume only a finite set of values and $\alpha + i\beta \not\equiv 0 \pmod{\omega i}$, then there exist constants K, L, independent of α, β, such that $|H(\alpha,\beta)| = 1/|e^{T(\alpha+i\beta)} - 1| \leq K$, $1/|\alpha+i\beta| \leq L$. Therefore, we have $\left|\int e^{(\alpha+i\beta)t}\varphi(t)\,\mathrm{d}t\right| \leq MN$, independent of α, β.

Theorem (8.i). *For every function* $f(x) = \sum_{j=1}^{n} e^{\alpha_j x}\varphi_j(x) \in C_\omega$ *with* $m[f] = 0$, *there are constants* $N(\alpha_j, T)$ *depending only on* α_j, T *such that*

$$\left|\int f(x)\,\mathrm{d}x\right| \leqq \sum_{j=1}^{n} N(\alpha_j, T)\int_0^T |\varphi_j(x)|\,\mathrm{d}x$$

for all $0 \leqq x \leqq T$.

Proof. Again, as in Theorem (5.iii) and Theorem (6.i), we let $g_j(x) = e^{\alpha_j x}\left[\varphi_j(x) - c_{j,n_j}e^{in_j\omega x}\right]$, $(j = 1, \ldots, n)$, and apply the preceding Lemma (8.i) to each of the functions $g_j(x)$, $(j = 1, 2, \ldots, n)$, or to each of the functions $e^{\alpha_j x}\varphi_j(x)$ if $in\omega + \alpha_j \neq 0$ for all n.

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