Rivista Mat. Univ. Parma 5, 137-167 (1954)

JACK K. HALE (*)

On boundedness of the solutions of linear differential systems with periodic coefficients. (**)

Introduction. In the present paper, we consider linear differential systems of the form

$$(1) \qquad y_i' = \sum_{h=1}^{n} [a_{ih} + \lambda \varphi_{ih}(x)] y_h \,, \quad (i = 1, 2, ..., n; \ ' = \mathrm{d}/\mathrm{d}x),$$

where the a_{ih} are constants, λ is a «small» parameter, and the functions $\varphi_{ih}(x)$ are periodic functions of period $T = 2\pi/\omega$. Notice that system (1) contains, as important particular cases, the MATHIEU equation

$$y'' + (\sigma^2 + \lambda \cos 2x) y = 0 \,,$$

and the HILL-MEISSNER equation

$$y'' + (\sigma^2 + \lambda \psi(x)) = 0 \,,$$

where $\psi(x) = 1$, if $0 \leqslant x \leqslant \pi$, $= -1$, if $\pi < x < 2\pi$. For a discussion of the MATHIEU equation see [**15**] (¹), [**20**], and of the HILL-MEISSNER equation, see [**17**]. General systems (1) have been studied by L. CESARI [**4**]. A more general system of the type (1) with $n = 2$ is discussed by L. CESARI and J. K. HALE [**5**].

Let us suppose that the characteristic roots $\varrho_1, ..., \varrho_n$ of the matrix $\|a_{ih}\|$ are distinct and incongruent modulo ωi. According to FLOQUET [**8**], a number τ

(*) Address: Sandia Corporation, Albuquerque, New Mexico, U.S.A..

(**) Received April 2, 1954. The present research has been made possible by a grant from the National Science Foundation [NSF Research Project 0101 at Purdue University under the direction of Professor LAMBERTO CESARI].

(¹) Numbers in brackets refer to Bibliography at the end of the paper.

is said to be a characteristic exponent of (1) provided that there exists a solution $[y_i(x), (i = 1, 2, ..., n)]$ of (1) such that $y_i(x+T) = e^{\tau T} y_i(x)$, $(i = 1, 2, ..., n)$, and it is known that there are n characteristic exponents $\tau_1, ..., \tau_n$, not necessarily distinct, that coincide with $\varrho_1, ..., \varrho_n$ respectively when $\lambda = 0$. Since the characteristic exponents are continuous functions of λ, we have that $\tau_1, ..., \tau_n$ are incongruent modulo ωi for $|\lambda|$ sufficiently small. In this case, there is a fundamental system of solutions of (1) of the form $[y_{ih}(x) = e^{\tau_i x} p_{ih}(x)$, $(i, h = 1, 2, ..., n)]$, where each function $p_{ih}(x)$ is periodic of period T. Thus, in this case, if one wishes to study the boundedness of the solutions of (1), it is sufficient to determine the characteristic exponents $\tau_1, ..., \tau_n$. If a fundamental system of solutions of (1) is known, then the characteristic exponents may be determined by using FLOQUET's algebraic equations.

L. CESARI [4] has studied systems of the type (1) with the functions $\varphi_{ih}(x)$ possessing absolutely convergent FOURIER series. In the quoted paper, CESARI has studied a variant of POINCARÉ's method of casting out the secular terms in the solutions of (1) by successive approximations and has obtained a system of equations (different from the equations of FLOQUET) for the determination of the characteristic exponents $\tau_1, ..., \tau_n$. By showing that these equations have a solution $\tau_1, ..., \tau_n$ which is purely imaginary, he proved the following

Theorem. Consider the system

$$y''_\mu + \sigma_\mu^2 y_\mu + \lambda \sum_{\nu=1}^{n} \varphi_{\mu\nu}(x) y_\nu = 0\,, \qquad (\mu = 1, 2, ..., n), \tag{2}$$

where $\sigma_1, ..., \sigma_n$ are distinct, real, positive numbers, λ is a real parameter, the functions $\varphi_{\mu\nu}(x)$ are real functions, periodic of period $T = 2\pi/\omega$, possessing absolutely convergent FOURIER series and $\int_0^T \varphi_{\mu\nu}(x)\, dx = 0$. If either $\varphi_{\mu\nu}(x) = \varphi_{\mu\nu}(-x)$, $(\mu, \nu = 1, 2, ..., n)$, or, $\varphi_{\mu\nu}(x) = \varphi_{\nu\mu}(x)$, $(\mu, \nu = 1, 2, ..., n)$, and if $m\omega \neq \sigma_\mu \pm \sigma_\nu$, $(\mu, \nu = 1, 2, ..., n;\ m = 1, 2, ...)$, then, for $|\lambda|$ sufficiently small, the solutions of (2) are bounded.

It is to be noted that the MATHIEU equation is a particular case of (2), but the HILL-MEISSNER equation is not, since the FOURIER series of $\psi(x)$ is not absolutely convergent.

Using the method of FLOQUET, W. HAACKE [9] has proved the above theorem under the condition that the functions φ are even. He has also considered a slightly more general case in which equations (2) may have higher order terms in λ. The case where the functions φ are not necessarily even but satisfy the relation $\varphi_{\mu\nu}(x) = \varphi_{\nu\mu}(x)$ is not considered in the paper of W. HAACKE. Other references are given in the book of R. BELLMAN [1].

In the present paper, the same systems (1), (2) are discussed when the requirements that the functions φ possess absolutely convergent FOURIER series is replaced by the much weaker condition that the functions φ be L-integrable in $[0, T]$. In such a way, the HILL-MEISSNER equation is contained in the discussion, and even differential systems of the above type where the coefficients are unbounded functions whose integral in a period is finite. In the discussion of such a general system, we shall consider only the solutions which are absolutely continuous in every finite interval (AC solutions) and satisfy system (1) almost everywhere (a.e.).

In § **1**, the system is transformed into an integral system and the previously mentioned method of successive approximations studied by L. CESARI is described. In the same section, a new proof of the convergence of the method is given under the new conditions on the functions φ. In this proof, the results of a previous paper by the author [**10**] are used extensively. In § **2**, the following theorem is proved.

Theorem. *Consider the system*

$$y''_\mu + \sigma_\mu^2 y_\mu + \lambda \sum_{\nu=1}^{n} \varphi_{\mu\nu}(x) y_\nu = 0\,, \quad \text{a. e.}, \quad -\infty < x < +\infty\,, \qquad (\mu = 1, 2, \ldots, n)\,, \tag{3}$$

where $\sigma_1, \ldots, \sigma_n$ *are distinct, real, positive numbers,* λ *is a real parameter, the functions* $\varphi_{\mu\nu}(x)$ *are real functions, periodic of period* $T = 2\pi/\omega$, *L-integrable in* $[0, T]$ *and* $\int_0^T \varphi_{\mu\nu}(x)\,\mathrm{d}x = 0$. *If either* $\varphi_{\mu\nu}(x) = \varphi_{\mu\nu}(-x)$, $(\mu, \nu = 1, 2, \ldots, n)$, *or,* $\varphi_{\mu\nu}(x) = \varphi_{\nu\mu}(x)$, $(\mu, \nu = 1, 2, \ldots, n)$, *and if* $m\omega \neq \sigma_\mu \pm \sigma_\nu$, $(\mu, \nu = 1, 2, \ldots, n;\ m = 1, 2, \ldots)$, *then, for* $|\lambda|$ *sufficiently small, the* AC *solutions of* (3) *are bounded.*

In § **3**, the preceding theorem is extended to the case where the functions $\varphi_{\mu\nu}(x)$ and also σ_μ contain the parameter λ, i.e. $\varphi_{\mu\nu}(x) = \varphi_{\mu\nu}(x; \lambda)$, $\sigma_\mu = \sigma_\mu(\lambda)$. Finally, it is shown in § **3** that *the condition* $\int_0^T \varphi_{\mu\nu}(x)\,\mathrm{d}x = 0$ *in the theorem above can be completely omitted.*

§ 1. – Casting out method of approximation.

Solutions to a system of differential equations are usually obtained by means of successive approximations. If the solutions are functions of the independent variable x, then an important problem in applications is to determine the behavior of the solutions as x approaches ∞.

In defining a method of successive approximations, it may happen that certain terms are introduced which behave badly for large values of x, and at the same time these terms do not portray the true character of the solutions to the system of differential equations. For instance, when the solutions are expected to be periodic, it is desirable that all the successive approximations are also periodic, but it happens that terms are obtained which are not of this type. These are the terms which are generally called «secular» terms and various methods have been devised in order to eliminate or avoid these terms. These methods are called «casting out» methods of approximation (LINDSTEDT [**14**], POINCARÉ [**19**], DUFFING [**7**]; see also N. MINORSKY [**17**, p. 136]).

In the following, we shall let C_ω denote the family of all functions which are finite sums of functions of the form $f(x) = e^{\alpha x}\varphi(x)$, $-\infty < x < +\infty$, where α is any complex number and $\varphi(x)$ is any complex-valued function of the real variable x, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$. If $\varphi(x)$ has a FOURIER series,

$$\varphi(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\omega x},$$

then we shall denote the series

$$f(x) = e^{\alpha x}\varphi(x) \approx \sum_{n=-\infty}^{+\infty} c_n e^{(in\omega+\alpha)x}$$

as the series associated with $f(x)$. Moreover, we shall denote by *mean value* $m[f]$ of $f(x)$ the number $m[f] = 0$ if $in\omega + \alpha \neq 0$ for all n, $m[f] = c_n$ if $in\omega + \alpha = 0$ for some n. We shall also make use of the following

Theorem. *If $f(x) \in C_\omega$ and $m[f] = 0$, then there is one and only one primitive of $f(x)$, say $F(x)$, which belongs to C_ω and such that $m[F] = 0$. Moreover, this primitive $F(x)$ is obtained by formal integration of the series associated with $f(x)$.*

For a proof of this Theorem, essentially known, see J. K. HALE [**10**].

The following pages will be devoted to a description of a variant to the above mentioned casting out methods due to L. CESARI [**4**, (a)].

1.1. – Description of the method. Consider the differential equation

$$Y' = AY + \lambda\Phi Y, \qquad (' = d/dx), \tag{1.1.1}$$

where A is a constant $n\cdot n$ matrix, λ is a real parameter, Y is a $n\cdot n$ matrix with elements $y_{\mu\nu}(x)$, and Φ is an $n\cdot n$ matrix whose elements $\varphi_{\mu\nu}(x)$ are complex-valued functions, periodic of period $T = 2\pi/\omega$, and mean value $m[\varphi_{\mu\nu}(x)] = 0$, $(\mu, \nu = 1, 2, \dots, n)$.

By considering an auxiliary equation

$$Y' = BY + \lambda\Phi Y \tag{1.1.2}$$

and applying a convenient modification of LIOUVILLE's method of successive approximations to an integral equation, we will show that one obtains a non-singular matrix of solutions for a new system

$$Y' = (B - \lambda D)Y + \lambda \Phi Y \,, \tag{1.1.3}$$

where D is a constant matrix which depends on B, Φ, λ. Then, by determining B such that

$$B - \lambda D = A \,, \tag{1.1.4}$$

the non-singular matrix of solutions of (1.1.3) becomes a non-singular matrix of solutions of (1.1.1).

Throughout section (1.1), we shall suppose that the functions $\varphi_{\mu\nu}(x)$ have FOURIER series which are absolutely convergent, $\varphi_{\mu\nu}(x) \sim \sum_{n=-\infty}^{+\infty} c_{\mu\nu n}\, e^{in\omega x}$, $(\mu, \nu = 1, 2, \dots, n)$. Also, let $A = \operatorname{diag}(\varrho_1, \dots, \varrho_n)$, where

$$\varrho_\mu \not\equiv \varrho_\nu \pmod{\omega i} \qquad (\mu \neq \nu;\ \mu, \nu = 1, 2, \dots, n). \tag{1.1.5}$$

Put

$$\delta_0 = \min_{\substack{\mu,\nu=1,\dots,n \\ \mu\neq\nu}} |\varrho_\mu - \varrho_\nu| \,; \qquad \delta = \min_{\substack{\mu,\nu=1,\dots,n \\ k=0,1,\dots \\ |\mu-\nu|+k\neq 0}} |ik\omega - (\varrho_\mu - \varrho_\nu)| \,; \tag{1.1.6}$$

$0 < \delta \leqslant \delta_0$, and if $0 < \varepsilon < 1/2$ is any number, let us consider, in the complex ϱ-plane, n circles $C_1, C_2, \dots, C_n$ with radius $\varepsilon\delta$ and centers $\varrho_1, \dots, \varrho_n$, respectively. Moreover, let $\tau_1, \tau_2, \dots, \tau_n$ be n points lying in the interior of the boundary of these circles, i.e., $\tau_\mu \in C_\mu$, $(\mu = 1, 2, \dots, n)$. All of these points τ_μ are thus distinct, and since

$$\min_{\substack{\mu,\nu=1,\dots,n \\ k=0,1,\dots \\ |\mu-\nu|+k\neq 0}} |ik\omega - (\tau_\mu - \tau_\nu)| \geqslant \delta - 2\varepsilon\delta = \delta(1 - 2\varepsilon) > 0 \,,$$

we have

$$\tau_\mu \not\equiv \tau_\nu \pmod{\omega i} \qquad (\mu \neq \nu;\ \mu, \nu = 1, 2, \dots, n). \tag{1.1.7}$$

Put $B = \operatorname{diag}(\tau_1, \dots, \tau_n)$ and consider the equation

$$Y' = BY + \lambda \Phi Y \,. \tag{1.1.8}$$

If we let $Z(x) = \operatorname{diag}(e^{\tau_1 x}, \dots, e^{\tau_n x})$, we see that a matrix $Y(x)$ satisfying the equation

$$Y = Z + \lambda Z \int Z^{-1} \Phi Y \, d\alpha \tag{1.1.9}$$

for any n^2 arbitrary constants (one for each element of the matrix) also satisfies (1.1.8).

Let us put $D_0 = 0$, $Y_0 = Z$, and consider

$$\|p_{rs}^{(1)}\| = Z^{-1}\Phi Y_0 = Z^{-1}\Phi Z = \begin{Vmatrix} \varphi_{11} & e^{-\tau_1 x}\varphi_{12}e^{\tau_2 x} & \dots & e^{-\tau_1 x}\varphi_{1n}e^{\tau_n x} \\ e^{-\tau_2 x}\varphi_{21}e^{\tau_1 x} & \varphi_{22} & \dots & e^{-\tau_2 x}\varphi_{2n}e^{\tau_n x} \\ \cdot & \cdot & \cdot & \cdot \\ e^{-\tau_n x}\varphi_{n1}e^{\tau_1 x} & e^{-\tau_n x}\varphi_{n2}e^{\tau_2 x} & \dots & \varphi_{nn} \end{Vmatrix}.$$

Each element of this matrix is contained in the class C_ω and since $\tau_\mu \not\equiv \tau_\nu \pmod{\omega i}$, $(\mu \neq \nu;\ \mu, \nu = 1, 2, \dots, n)$, each element of this matrix has mean value zero. Therefore, from the theorem stated at the beginning of this section, there is one and only one matrix

$$\|q_{rs}^{(1)}\| = \int Z^{-1}(\alpha)\Phi(\alpha)Y_0(\alpha)\,d\alpha = \int Z^{-1}(\alpha)\Phi(\alpha)Z(\alpha)d\alpha$$

whose elements are contained in C_ω and have mean value zero. Moreover, by this same theorem,

$$q_{rs}^{(1)} = \int e^{-\tau_r \alpha}\varphi_{rs}(\alpha)\,e^{\tau_s \alpha}\,d\alpha = \sum_{\substack{l=-\infty \\ l\neq 0}}^{+\infty} \frac{c_{rsl}}{-\tau_r + il\omega + \tau_s}\, e^{(-\tau_r + il\omega + \tau_s)x}.$$

Put $D_1 = 0$, $Y_1 = Z + \lambda Z\int Z^{-1}\Phi Y_0\,d\alpha$, where we will take the n^2 particular primitives of mean value zero. Moreover, let us consider the matrix

$$\|p_{rs}^{(2)}\| = Z^{-1}\Phi Y_1 = Z^{-1}\Phi Z + \lambda Z^{-1}\Phi Z\int Z^{-1}\Phi Z\,d\alpha,$$

where $p_{rs}^{(2)}$ has the associated series

$$p_{rs}^{(2)} \approx e^{-\tau_r x}\varphi_{rs}e^{\tau_s x} + \lambda\sum_{h=1}^{n} e^{-\tau_r x}\varphi_{rh}\,e^{\tau_h x}\int e^{-\tau_h \alpha}\varphi_{hs}e^{\tau_s \alpha}\,d\alpha =$$

$$= p_{rs}^{(1)} + \lambda\sum_{h=1}^{n}\sum_{\substack{l_1, l_2 = -\infty \\ l_1\neq 0, l_2\neq 0}}^{+\infty} \frac{c_{rhl_1}c_{hsl_2}}{-\tau_h + il\,\omega + \tau_s}\, e^{[-\tau_r + i(l_1+l_2)\omega + \tau_s]x}.$$

Then, by definition, we have $m[p_{rs}^{(2)}] = 0$, if $r \neq s$,

$$m[p_{rr}^{(2)}] = \lambda\sum_{h=1}^{n}\sum_{l_1+l_2=0} \frac{c_{rhl_1}c_{hrl_2}}{-\tau_h + il_2\omega + \tau_r},$$

and, thus, $m[p_{rs}^{(2)}]$ may be different from zero. These terms where the mean value is different from zero lead to secular terms. Put $D_2 = m[Z^{-1}\Phi Y_1]$, $D_2 = \|d_{rs}^{(2)}\|$, $d_{rs}^{(2)} = m[p_{rs}^{(2)}]$.

The matrix D_2 is diagonal and, therefore, Z^{-1}, Z, D_2 commute. As a consequence, we may write

$$\|\bar{p}_{rs}^{(2)}\| = Z^{-1}[\Phi - D_2]Y_1 = Z^{-1}\Phi Y_1 - Z^{-1}D_2(Z + \lambda Z\int Z^{-1}\Phi Z\,\mathrm{d}\alpha) = \\ = Z^{-1}\Phi Y_1 - D_2 - \lambda D_2\int Z^{-1}\Phi Z\,\mathrm{d}\alpha\,,$$

and, therefore, since $m[\sum] = \sum m[\ ]$, we have

$$m[\|\bar{p}_{rs}^{(2)}\|] = m[Z^{-1}\Phi Y_1 - D_2] - \lambda D_2 m[\int Z^{-1}\Phi Z\,\mathrm{d}\alpha] = \\ = m[Z^{-1}\Phi Y_1] - D_2 - \lambda D_2 m[\|q_{rs}^{(1)}\|] = 0\,.$$

Thus, there exists one and only one matrix

$$\|q_{rs}^{(2)}\| = \int Z^{-1}(\alpha)[\Phi(\alpha) - D_2]Y_1(\alpha)\,\mathrm{d}\alpha$$

whose elements are functions of mean value zero. With this convention, put $Y_2 = Z + \lambda Z\int Z^{-1}(\Phi - D_2)Y_1\mathrm{d}\alpha$. We have thus defined Y_2 in such a way that the secular terms are omitted, but it should be observed that in doing so, we have added some new terms. The reason for doing this is to obtain a solution of an equation of the form (1.1.3).

Suppose that we have repeated the above process $(m-1)$ times, i.e., we can calculate successively the matrices

$$D_3 = m[Z^{-1}\Phi Y_2]\,, \qquad Y_3 = Z + \lambda Z\int Z^{-1}(\Phi - D_3)Y_2\,\mathrm{d}\alpha\,,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$D_{m-1} = m[Z^{-1}\Phi Y_{m-2}]\,, \qquad Y_{m-1} = Z + \lambda Z\int Z^{-1}(\Phi - D_{m-1})Y_{m-2}\,\mathrm{d}\alpha\,,$$

where $D_2, \dots, D_{m-1}$ are diagonal matrices and we shall show that the process may be continued once more.

The integrals in Y_k $(k = 1, 2, \dots, m-1)$ are

$$\|q_{rs}^{(k)}\| \equiv \int Z^{-1}(\Phi - D_k)Y_{k-1}\,\mathrm{d}\alpha\,,$$

$$q_{rs}^{(k)} = \int\sum_{t_1=1}^{n} e^{-\tau_r\alpha}(\varphi_{rt_1} - d_{rt_1}^{(k)})y_{t_1s}^{(k-1)}\,\mathrm{d}\alpha\,.$$

Let us assume that $q_{rs}^{(k)}$ $(k = 1, 2, \dots, m-1)$ has the form

$$q_{rs}^{(k)} = e^{(-\tau_r+\tau_s)x}\sum_{\substack{l=-\infty \\ l\neq 0 \text{ if } r=s}}^{+\infty}\gamma_{rsl}^{(k)}\,e^{il\omega x}, \tag{1.1.10}$$

and observe that this is true for $q_{rs}^{(1)}$. We then have

$$\|p_{rs}^{(m)}\| \equiv Z^{-1}\Phi Y_{m-1} = Z^{-1}\Phi Z + \lambda Z^{-1}\Phi Z \int Z^{-1}(\Phi - D_{m-1}) Y_{m-2}\, d\alpha,$$

$$p_{rs}^{(m)} = p_{rs}^{(1)} + \lambda \sum_{h=1}^{n} e^{-\tau_r x} \varphi_{rh}\, e^{\tau_h x} q_{hs}^{(m-1)} = p_{rs}^{(1)} + \lambda \sum_{h=1}^{n} \sum_{\substack{l_1, l_2 = -\infty \\ l_1 \neq 0}}^{+\infty} c_{rhl_1} \gamma_{hsl_2}^{(m-1)} e^{(-\tau_r + i(l_1 + l_2)\omega + \tau_s)x},$$

and, by definition, since $\tau_\mu \not\equiv \tau_\nu \pmod{\omega i}$, $\mu \neq \nu$, we have $m[p_{rs}^{(m)}] = 0$ if $r \neq s$. The matrix $D_m = \|d_{rs}^{(m)}\| = m[\|p_{rs}^{(m)}\|] = m[Z^{-1}\Phi Y_{m-1}]$ is therefore a diagonal matrix.

Put

$$\|\bar{p}_{rs}^{(m)}\| \equiv Z^{-1}(\Phi - D_m) Y_{m-1} =$$
$$= Z^{-1}\Phi Y_{m-1} - Z^{-1} D_m Z - \lambda Z^{-1} D_m Z \int Z^{-1}(\Phi - D_{m-1}) Y_{m-2}\, d\alpha =$$
$$= Z^{-1}\Phi Y_{m-1} - D_m - \lambda D_m \int Z^{-1}(\Phi - D_{m-1}) Y_{m-2}\, d\alpha .$$

Moreover, we have

$$y_{rs}^{(m-1)} = \delta_{rs} e^{\tau_s x} + \lambda e^{\tau_r x} q_{rs}^{(m-1)} = \delta_{rs} e^{\tau_s x} + \lambda e^{\tau_s x} \sum_{\substack{l = -\infty \\ l \neq 0 \text{ if } r = s}}^{+\infty} \gamma_{rsl}^{(m-1)} e^{il\omega x},$$

or,

(1.1.11) $$y_{rs}^{(m-1)} = e^{\tau_s x} P_{rs}^{(m-1)},$$

where $P_{rs}^{(m-1)}$ is periodic of period $T = 2\pi/\omega$, $m\left[y_{rs}^{(m-1)}\right] = 0$. Consequently, using (1.1.10) and (1.1.11), we see that

(1.1.12) $$\bar{p}_{rs}^{(m)} = e^{(-\tau_r + \tau_s)x} Q_{rs}^{(m)},$$

where $Q_{rs}^{(m)}$ is periodic of period T. Therefore, $\bar{p}_{rs}^{(m)} \in C_\omega$, and since, by assumption, $m\left[\int Z^{-1}(\Phi - D_{m-1}) Y_{m-1}\, d\alpha\right] = 0$, we have

(1.1.13) $$m[\bar{p}_{rs}^{(m)}] = 0 \qquad (r, s = 1, 2, \ldots, n).$$

Thus, there exists one and only one matrix

$$\|q_{rs}^{(m)}\| \equiv \int Z^{-1}(\Phi - D_m) Y_{m-1}\, d\alpha, \qquad q_{rs}^{(m)} = \int \bar{p}_{rs}^{(m)}\, d\alpha,$$

whose elements are functions contained in C_ω and of mean value zero, and the elements are of the type

$$q_{rs}^{(m)} = \sum_{\substack{l = -\infty \\ l \neq 0 \text{ if } r = s}}^{+\infty} \gamma_{rsl}^{(m)} e^{(-\tau_r + il\omega + \tau_s)x}.$$

We can, therefore, put $Y_m = Z + \lambda Z \int Z^{-1}(\Phi - D_m) Y_{m-1}\, d\alpha$, and, in general, define the infinite algorithm as follows:

$$(1.1.14) \qquad \begin{cases} D_0 = 0\,, \qquad Y_0 = Z\,, \\ D_m = m[Z^{-1}\Phi Y_{m-1}]\,, \\ Y_m = Z + \lambda Z \int Z^{-1}(\Phi - D_m) Y_{m-1}\, d\alpha\,, \qquad (m = 1, 2, \ldots). \end{cases}$$

1.2. – Conditions on the functions φ. In the previous number, we have supposed, for the sake of simplicity, that the FOURIER series of the functions $\varphi_{\mu\nu}$ are absolutely convergent. This is also the hypotheses under which the method is discussed in the quoted paper by L. CESARI [**4**, (a)]. In the next few pages, we shall show that the method is also valid if we assume only that the $\varphi_{\mu\nu}$ ear L-integrable functions.

In the following, we shall denote functions which are absolutely continuous in each finite interval by AC functions. We shall need the following theorems.

Theorem (1.2.i). *Consider the equation*

$$(1.2.1) \qquad Y' = AY + \lambda\Phi Y \quad \text{a. e.},$$

where A is a constant $n\cdot n$ matrix, Y is an $n\cdot n$ matrix, and Φ is an $n\cdot n$ matrix whose elements are periodic of period T, L*-integrable in* $[0, T]$. *The* AC *solutions of* (1.2.1) *coincide with the solutions of the integral equation*

$$(1.2.2) \qquad Y(x) = Z(x)K + \lambda Z(x) \int_0^x Z^{-1}(\alpha)\,\Phi(\alpha)\,Y(\alpha)\,d\alpha\,,$$

where K is a constant $n\cdot n$ matrix and $Z(x)$ is a non-singular solution of the equation $Z' = AZ$.

Proof. (The following proof is similar to a proof given in LEFSCHETZ [**12**, p. 62].) Differentiating (1.2.2), we get a. e.

$$Y'(x) = Z'(x)K + \lambda Z'(x) \int_0^x Z^{-1}(\alpha)\,\Phi(\alpha)\,Y(\alpha)\,d\alpha + \lambda Z(x)Z^{-1}(x)\,\Phi(x)\,Y(x) =$$

$$= AZK + \lambda AZ \int_0^x Z^{-1}(\alpha)\,\Phi(\alpha)\,Y(\alpha)\,d\alpha + \lambda\Phi Y = AY + \lambda\Phi Y\,.$$

Conversely, let Y be any AC solution of (1.2.1) and choose K such that $Y(0) = Z(0)K$, where $Z(x)$ is a non-singular solution of $Z' = AZ$. If we set $\bar{Z} = C_0 Z^{-1}$, where C_0 is a constant non-singular matrix, then

$$0 = d(\bar{Z}Z)/dx = \bar{Z}'Z + \bar{Z}Z' = \bar{Z}'Z + \bar{Z}AZ = (\bar{Z}' + \bar{Z}A)Z\,,$$

or, $\bar{Z}' = -\bar{Z}A$. Then,

$$d(\bar{Z}Y)/dx = \bar{Z}'Y + \bar{Z}Y' = -\bar{Z}AY + \bar{Z}(AY + \lambda\Phi Y) = \lambda\bar{Z}\Phi Y\,,$$

or,

$$\bar{Z}Y = \lambda\int_0^x \bar{Z}\Phi Y\,d\alpha + C_0K\,,$$

$$Y = \lambda\bar{Z}^{-1}\int_0^x \bar{Z}\Phi Y d\alpha + \bar{Z}^{-1}C_0K = \lambda ZC_0^{-1}\int_0^x C_0Z^{-1}\Phi Y\,d\alpha + $$
$$+ ZC_0^{-1}C_0K = ZK + \lambda Z\int_0^x Z^{-1}\Phi Y d\alpha\,,$$

and Y satisfies (1.2.2). Therefore, the theorem is proved.

Theorem (1.2.ii). *The solution to the system of integral equations*

$$y_i(x) = y_{i0} + \int_0^x a_i(y_1, \ldots, y_n; t)\,dt \qquad (i = 1, 2, \ldots, n),$$

is unique provided that

$$\sum_{i=1}^n |a_i(y_{11}, \ldots, y_{1n}; x) - a_i(y_{21}, \ldots, y_{2n}; x)| \leqslant M(x)\sum_{i=1}^n |y_{1i} - y_{2i}|\,,$$

where $M(x)$ is a summable function.

Proof. This theorem is given in C. Carathéodory [3, p. 674], but for completeness we give a proof here. Choose the closed interval $[0, x]$ so small that $\int_0^x M(t)\,dt < 1$. Then, if $\bar{y}_1(x), \ldots, \bar{y}_n(x)$ is some other solution with $\bar{y}_i(0) = y_{i0}$ $(i = 1, 2, \ldots, n)$, we have $\sum_{i=1}^n |y_i(x) - \bar{y}_i(x)|$ is a continuous function on $[0, x]$ and, therefore, attains a maximum N at some point x_0, $0 \leqslant x_0 \leqslant x$. Thus, if $N > 0$, we have

$$N = \sum_{i=1}^n |y_i(x_0) - \bar{y}_i(x_0)| =$$
$$= \sum_{i=1}^n \left|\int_0^{x_0} [a_i(y_1, \ldots, y_n; t) - a_i(\bar{y}_1, \ldots, \bar{y}_n; t)\,dt\right| \leqslant$$
$$\leqslant \int_0^{x_0} \sum_{i=1}^n |a_i(y_1, \ldots, y_n; t) - a_i(\bar{y}_1, \ldots, \bar{y}_n; t)|\,dt \leqslant$$
$$\leqslant \int_0^{x_0} M(t)\sum_{i=1}^n |y_i - \bar{y}_i|\,dt \leqslant N\int_0^{x_0} M(t)\,dt < N\,,$$

which is impossible, therefore, unless $N = 0$, i.e., $\bar{y}_i(x) = y_i(x)$ $(i = 1, 2, \ldots, n)$.

From the preceding uniqueness theorem, it follows that if Y_0 is an AC non-singular solution of (1.2.1), then, as it is well known, every other AC solution $Y(x)$ of (1.2.1) is given by $Y(x) = Y_0(x)K$, i.e., Y_0 is a fundamental matrix of AC solutions.

Now, suppose that we wish to find a non-singular matrix Y_0 of AC solutions to the equation

$$Y' = AY + \lambda \Phi Y \qquad \text{a. e.,} \tag{1.2.3}$$

where $\Phi = \|\varphi_{\mu\nu}(x)\|$, $\varphi_{\mu\nu}(x)$ periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$, $\int_0^T \varphi_{\mu\nu}(x)\,dx = 0$, and $A = \operatorname{diag}(\varrho_1, \ldots, \varrho_n)$, $\varrho_\mu \not\equiv \varrho_\nu \pmod{\omega i}$, $\mu \neq \nu$ $(\mu, \nu = 1, 2, \ldots, n)$.

We begin our problem with the auxiliary integral equation

$$Y(x) = Z(x) + \lambda Z(x) \int Z^{-1}(\alpha)\Phi(\alpha)Y(\alpha)\,d\alpha\,, \tag{1.2.4}$$

where $Z(x) = \operatorname{diag}(e^{\tau_1 x}, e^{\tau_2 x}, \ldots, e^{\tau_n x})$, and the numbers $\tau_1, \tau_2, \ldots, \tau_n$ are chosen as in section (1.1). Moreover, in the integrations, we may choose any n^2 particular primitives, since there are n^2 arbitrary constants (one for each element of the matrix). We apply the algorithm defined by (1.1.14) to equation (1.2.4) above and, if we can show the convergence of the procedure, we will have a solution to the equation

$$Y(x) = Z(x) + \lambda Z(x) \int Z^{-1}(\alpha)[\Phi(\alpha) - D]Y(\alpha)\,d\alpha\,, \tag{1.2.5}$$

where D is a constant diagonal matrix. Moreover, we shall show that the columns of the matrix $Y(x)$ satisfying (1.2.5) are linearly independent and, thus, $Y(x)$ is a non-singular AC solution of (1.2.5). Then, differentiating (1.2.5), we get

$$Y'(x) = (B - \lambda D)Y(x) + \lambda\Phi(x)Y(x) \qquad \text{a. e.}\,,$$

where $B = \operatorname{diag}(\tau_1, \ldots, \tau_n)$ and, if we can choose the numbers $\tau_1, \tau_2, \ldots, \tau_n$ in such a way that $B - \lambda D = A$, then $Y(x)$ becomes a non-singular AC solution of (1.2.3).

In the next few sections, we shall prove that under the new hypotheses the method described in (1.1) is convergent and produces a fundamental system of AC solutions of the given system (1.2.3).

1.3. - Preliminary considerations for the proof of convergence. Let us try to get more convenient expressions for the D_m and $Y_m(x)$. In order to do this, we will introduce the notation

$$(1.3.1)\qquad \begin{cases} \Phi_m = \|\varphi_{\mu\nu}^{(m)}(x)\| = \Phi - D_m\,, \\ \varphi_{\mu\nu}^{(m)}(x) = \varphi_{\mu\nu}(x) - d_{\mu\nu}^{(m)}\,, \qquad (\mu, \nu = 1, 2, \ldots, n), \end{cases}$$

and $d_{\mu\nu}^{(m)} = 0$ if $\mu \neq \nu$. Then the algorithm defined by relation (1.1.14) becomes

$$(1.3.2)\qquad \begin{cases} D_0 = 0\,, \qquad Y_0 = Z\,, \\ D_m = m[Z^{-1}\Phi Y_{m-1}]\,, \\ Y_m = Z + \lambda Z \int Z^{-1}\Phi_m Y_{m-1}\, d\alpha\,, \qquad (m = 1, 2, \ldots), \end{cases}$$

where the integrations are always performed so as to get the primitive of mean value zero. We have already seen that this primitive is unique.

We can also write (1.3.2) in the following form

$$\begin{aligned} D_m = m[Z^{-1}\Phi Y_{m-1}] &= m[Z^{-1}\Phi Z] + \lambda m[Z^{-1}\Phi Z \int Z^{-1}\Phi_{m-1} Y_{m-2}\, d\alpha] = \\ &= \lambda m[Z^{-1}\Phi Z \int Z^{-1}\Phi_{m-1} Y_{m-2}\, d\alpha] = \ldots = \\ &= \lambda m[Z^{-1}\Phi Z \int Z^{-1}\Phi_{m-1} Z\, d\alpha_2 + \lambda Z^{-1}\Phi Z \int Z^{-1}\Phi_{m-1} Z\, d\alpha_2 \int Z^{-1}\Phi_{m-2} Z\, d\alpha_3 + \\ &\qquad + \ldots + \lambda^{m-2} Z^{-1}\Phi Z \int Z^{-1}\Phi_{m-1} Z\, d\alpha_2 \int \ldots \int Z^{-1}\Phi Z\, d\alpha_m]\,, \end{aligned}$$

$$\begin{aligned} Y_m(x) &= Z + \lambda Z \int Z^{-1}\Phi_m Y_{m-1}\, d\alpha = \ldots = \\ &= Z + \lambda Z \int Z^{-1}\Phi_m Z\, d\alpha_1 + \lambda^2 Z \int Z^{-1}\Phi_m Z\, d\alpha_1 \int Z^{-1}\Phi_{m-1} Z\, d\alpha_2 + \\ &\qquad + \ldots + \lambda^m Z \int Z^{-1}\Phi_m Z\, d\alpha_1 \int \ldots \int Z^{-1}\Phi Z\, d\alpha_m]\,, \end{aligned}$$

where the integrations are always performed so as to get the primitive of mean value zero.

In the following, we shall

$$(1.3.3)\qquad \begin{cases} e^{(-\tau_{t_k}+\tau_{t_{k+1}})\alpha}\varphi_{t_k,t_{k+1}}^{(m-l)} = \xi_{t_k,t_{k+1}}^{(l)}(\alpha)\,, \\ e^{(-\tau_{t_k}+\tau_{t_{k+1}})\alpha}\, d_{t_k,t_{k+1}}^{(m-l)} = \eta_{t_k,t_{k+1}}^{(l)}(\alpha)\,. \end{cases}$$

Then, from (1.3.1), we have

$$(1.3.4)\qquad \xi_{t_k,t_{k+1}}^{(l)} = \xi_{t_k,t_{k+1}}^{(m)} - \eta_{t_k,t_{k+1}}^{(l)}\,.$$

If we write the above formulas for $D_m = \|d_{rr}^{(m)}\|$, $Y_m = \|y_{rs}^{(m)}\|$, in terms of the elements of the matrices, we have

$$\text{(1.3.5)}\quad \begin{cases} d_{rr}^{(m)} = \lambda m\Big[\sum_{p=0}^{m-2}\lambda^p \sum_{t_1,\dots,t_{p+1}=1}^{n} e^{-\tau_r x}\varphi_{rt_1}e^{\tau_{t_1}x}\int e^{-\tau_{t_1}\alpha_2}\varphi_{t_1t_2}^{(m-1)}e^{\tau_{t_2}\alpha_2}\,d\alpha_2\int \dots \\ \qquad \dots\int e^{-\tau_{t_{p+1}}\alpha_{p+2}}\varphi_{t_{p+1}r}^{(m-p-1)}e^{\tau_r\alpha_{p+2}}\,d\alpha_{p+2}\Big] = \\ = \lambda m\Big[\sum_{p=0}^{m-2}\lambda^p\sum_{t_1,\dots,t_{p+1}=1}^{n}\xi_{rt_1}^{(m)}\int \xi_{t_1t_2}^{(1)}\,d\alpha_2\int\dots\int \xi_{t_{p+1}r}^{(p+1)}\,d\alpha_{p+2}\Big], \end{cases}$$

$$\text{(1.3.6)}\qquad y_{rs}^{(m)}(x) = \sum_{p=0}^{m}\lambda^p\sum_{t_1\dots,t_{p-1}=1}^{n} e^{\tau_r x}\int \xi_{rt_1}^{(0)}\,d\alpha_1\int \xi_{t_1t_2}^{(1)}\,d\alpha_2\int\dots\int \xi_{t_{p-1}s}^{(p-1)}\,d\alpha_p\,,$$

where the integrals are to be interpreted as stated below. From (1.3.2) above, we have

$$D_m = m\left[Z^{-1}\Phi Y_{m-1}\right] = \lambda m\left[Z^{-1}\Phi Z\int Z^{-1}\Phi_{m-1}Y_{m-2}\,d\alpha\right],$$

and from formula (1.1.12), we have that each of the elements of the matrix

$$\|\overline{p}_{rs}^{(m-1)}\| = Z^{-1}\Phi_{m-1}Y_{m-2} = \|e^{(-\tau_r+\tau_s)x}\,Q_{rs}^{(m-1)}\|$$

is contained in C_ω and, from (1.1.13), $m[Z^{-1}\Phi_{m-1}Y_{m-2}] = 0$. Therefore, if we make use of the results of a paper by the author [**10**], we know that the unique primitive of $Z^{-1}\Phi_{m-1}Y_{m-2}$ may be obtained as a definite integral or as a sum of definite integrals according to Lemmas (6.i) and (6.ii) of [**10**], provided that $m[Q_{rs}^{(m-1)}] = 0$. The range of integration of all these integrals is contained in the interval $[0, T]$. If $m[Q_{rs}^{(m-1)}] = c_0^{(m-1)} \neq 0$, then we may write $Q_{rs}^{(m-1)} = \overline{Q}_{rs}^{(m-1)} + c_0^{(m-1)}$, where $m[\overline{Q}_{rs}^{(m-1)}] = 0$. Then

$$\int e^{(-\tau_r+\tau_s)\alpha}\,Q_{rs}^{(m-1)}\,d\alpha = \int e^{(-\tau_r+\tau_s)\alpha}\,\overline{Q}_{rs}^{(m-1)}\,d\alpha + c_0^{(m-1)}\int e^{(-\tau_r+\tau_s)\alpha}\,d\alpha\,,$$

and the first integral may be interpreted as in Lemmas (6.i) and (6.ii) of [**10**] and the second may be interpreted as in [**10**, formula (8.1)].

Now, with these integrals being interpreted as a sum of integrals whose ranges of integration are determined by the function $Z^{-1}\Phi_{m-1}Y_{m-2}$, we may substitute for Y_{m-2} its expression in (1.3.2), and apply the same reasoning as above to $\int Z^{-1}\Phi_{m-2}Y_{m-3}\,d\alpha$ and see that each of these integrals has the same type of interpretation. The same type of reasoning holds for all the other integrals in (1.3.5) and (1.3.6). Thus, we see that each of the integrals must be interpreted as a sum of integrals whose ranges of integration are defined as in Lemmas (6.i) and (6.ii) of [**10**].

Furthermore, from (1.1.11), we see that for each $r, s=1, 2, \ldots, n$; $m=1, 2, \ldots$, we have $y_{rs}^{(m)} = e^{\tau_s x} P_{rs}^{(m)}(x)$. Also since $D_m = \lambda m [Z^{-1}\Phi Z \int Z^{-1}\Phi_{m-1} Y_{m-2}\, \mathrm{d}\alpha]$, we have

$$d_{rr}^{(m)} = \lambda m \Big[\sum_{t_1, t_2 = 1}^{n} e^{-\tau_r x} \varphi_{r t_1} e^{\tau_{t_1} x} \int e^{-\tau_{t_1}\alpha} \varphi_{t_1 t_2}^{(m-1)} y_{t_2 r}^{(m-2)}\, \mathrm{d}\alpha \Big],$$

and from (1.1.12), we have

$$\sum_{t_2=1}^{n} \int e^{-\tau_{t_1}\alpha} \varphi_{t_1 t_2}^{(m-1)} y_{t_2 r}^{(m-2)}\, \mathrm{d}\alpha = e^{(-\tau_{t_1}+\tau_r)x} Q_{t_1 r}^{(m-1)}(x),$$

where $Q_{t_1 r}^{(m-1)}(x)$ is periodic of period T. Consequently,

$$d_{rr}^{(m)} = \lambda m \Big[\sum_{t_1=1}^{n} \varphi_{r t_1} Q_{t_1 r}^{(m-1)} \Big] = (\lambda/T) \int_0^T \sum_{t_1=1}^{n} \varphi_{r t_1} Q_{t_1 r}^{(m-1)}\, \mathrm{d}x,$$

and we have the result that

$$d_{rr}^{(m)} = (\lambda/T) \int_0^T \sum_{t_1, t_2=1}^{n} e^{-\tau_r x} \varphi_{r t_1} e^{\tau_{t_1} x} \int e^{-\tau_{t_1}\alpha} \varphi_{t_1 t_2}^{(m-1)} y_{t_2 r}^{(m-2)}\, \mathrm{d}\alpha\, \mathrm{d}x. \tag{1.3.7}$$

1.4. – Proof of the convergence of the functions $d_{rr}^{(m)}$. If we remember from (1.3.4) that $\xi_{rs}^{(l)} = \xi_{rs}^{(m)}$ if $r \neq s$, $= \xi_{rr}^{(m)} - \eta_{rr}^{(l)}$ if $r = s$, then the general term in expression (1.3.5) can be transformed as follows:

$$\left\{\begin{aligned} &\xi_{r t_1}^{(m)} \int \xi_{t_1 t_2}^{(1)} \mathrm{d}\alpha \int \ldots \int \xi_{t_{p+1} r}^{(p+1)} \mathrm{d}\alpha_{p+2} = \\ &= \xi_{r t_1}^{(m)} \int \xi_{t_1 t_2}^{(m)} \mathrm{d}\alpha_2 \int \ldots \int \xi_{t_{p+1} r}^{(m)} \mathrm{d}\alpha_{p+2} - \xi_{r t_1}^{(m)} \int \eta_{t_1 t_2}^{(1)} \mathrm{d}\alpha_2 \int \ldots \int \xi_{t_{p+1} r}^{(m)} \mathrm{d}\alpha_{p+2} - \ldots \\ &\ldots - \xi_{r t_1}^{(m)} \int \xi_{t_1 t_2}^{(m)} \mathrm{d}\alpha_2 \int \ldots \int \eta_{t_{p+1} r}^{(p+1)} \mathrm{d}\alpha_{p+2} + \xi_{r t_1}^{(m)} \int \eta_{t_1 t_2}^{(1)} \mathrm{d}\alpha_2 \int \eta_{t_2 t_3}^{(2)} \mathrm{d}\alpha_3 \int \ldots \\ &\ldots \int \xi_{t_{p+1} r}^{(m)} \mathrm{d}\alpha_{p+2} + \ldots + (-1)^{p+1} \xi_{r t_1}^{(m)} \int \eta_{t_1 t_2}^{(1)} \mathrm{d}\alpha_2 \int \ldots \int \eta_{t_{p+1} r}^{(p+1)} \mathrm{d}\alpha_{p+2}. \end{aligned}\right. \tag{1.4.1}$$

Then, using (1.3.7), we have

$$\left\{\begin{aligned} m\Big[\sum_{t_1,\ldots,t_{p+1}=1}^{n} (1.4.1)\Big] &= \frac{1}{T} \int_0^T \Big\{ \sum_{t_1,\ldots,t_{p+1}=1}^{n} \xi_{r t_1}^{(m)} \int \xi_{t_1 t_2}^{(m)} \mathrm{d}\alpha_2 \int \ldots \int \xi_{t_{p+1} r}^{(m)} \mathrm{d}\alpha_{p+2} - \\ &- \sum_{\substack{t_1,\ldots,t_{p+1}=1 \\ t_1 = t_2}}^{n} \xi_{r t_1}^{(m)} \int \eta_{t_1 t_2}^{(1)} \mathrm{d}\alpha_2 \int \ldots \int \xi_{t_{p+1} r}^{(m)} \mathrm{d}\alpha_{p+2} - \ldots + \\ &+ \sum_{\substack{t_1,\ldots,t_{p+1}=1 \\ t_1 = t_2 = t_3}}^{n} \xi_{r t_1}^{(m)} \int \eta_{t_1 t_2}^{(1)} \mathrm{d}\alpha_2 \int \eta_{t_2 t_3}^{(2)} \mathrm{d}\alpha_3 \int \ldots \int \xi_{t_{p+1} r}^{(m)} \mathrm{d}\alpha_{p+2} + \ldots + \\ &+ (-1)^{p+1} \sum_{\substack{t_1,\ldots,t_{p+1}=1 \\ t_1 = \ldots = t_{p+1}}}^{n} \xi_{r t_1}^{(m)} \int \eta_{t_1 t_2}^{(1)} \mathrm{d}\alpha_2 \int \ldots \int \eta_{t_{p+1} r}^{(p+1)} \mathrm{d}\alpha_{p+2} \Big\} \mathrm{d}x, \end{aligned}\right. \tag{1.4.2}$$

where each of the integrals is to be interpreted as in section (1.3), i.e., each integral is a sum of integrals with definite ranges of integration lying in the interval $[0, T]$, the number of integrals being determined by Lemmas (6.i) and (6.ii) of [**10**]. Each of the signs $\int$ may represent as many as six definite integrals.

Each integral in (1.4.1), (1.4.2) is one of the forms

$$\int \xi_{rs}^{(m)}\,\mathrm{d}x = \int e^{(-\tau_r+\tau_s)x}\varphi_{rs}\,\mathrm{d}x\,, \qquad \int \eta_{rs}^{(k)}\,\mathrm{d}x = \int e^{(-\tau_r+\tau_s)x}\,d_{rs}^{(m-k)}\,\mathrm{d}x\,, \tag{1.4.3}$$

where $\varphi_{rs}(x)$, $d_{rs}^{(m-k)}$ are defined as above, $-\tau_r+\tau_s \not\equiv 0 \pmod{\omega i}$, $r \neq s$, and sign $\int$ is interpreted as above. Since τ_r, τ_s range over a finite set of values, and since each integral is of the same type as in Lemmas (6.i) and (6.ii) of [**10**], we see from [**10**, Lemma (8.i), Remark 2], that

$$\begin{cases} \left|\int \xi_{rs}^{(m)}\,\mathrm{d}x\right| = \left|\int e^{(-\tau_r+\tau_s)x}\varphi_{rs}\,\mathrm{d}x\right| < N\int_0^T |\varphi_{rs}(x)|\,\mathrm{d}x\,, \\ \left|\int \eta_{rs}^{(k)}\,\mathrm{d}x\right| = \left|\int e^{(-\tau_r+\tau_s)x}\,d_{rs}^{(m-k)}\,\mathrm{d}x\right| < NT\left|d_{rs}^{(m-k)}\right|, \end{cases} \tag{1.4.4}$$

where N is a constant independent of τ_r, τ_s and the specific value of the number N is given in the proof of the quoted Lemma (8.i).

Let $c = \max|R(\tau_r - \tau_s)|$, $(r, s = 1, 2, \ldots, n)$, and $M = \max\int_0^T |\varphi_{\mu\nu}|\,\mathrm{d}x$, $(\mu, \nu = 1, 2, \ldots, n)$. We wish to show by induction that

$$|d_{rr}^{(m)}| < \varrho K(NT)\,, \qquad K = e^{cT}\cdot MN\,, \tag{1.4.5}$$

provided that

$$|\lambda| < \varrho/[(\varrho+1)(nNM+\varrho K)]\,, \qquad 0 < \varrho < 1\,, \tag{1.4.6}$$

and N is the same constant as before.

Since each of the integrals in (1.4.2) is one of the forms in (1.4.3) whose evaluations are given by (1.4.4), we have

$$\begin{aligned} |d_{rr}^{(2)}| &\leqslant (|\lambda|/T)\int_0^T \sum_{t_1=1}^{n} |e^{(-\tau_r+\tau_{t_1})x}\varphi_{rt_1}|\cdot\left|\int e^{(-\tau_{t_1}+\tau_r)\alpha_2}\varphi_{t_1r}\,\mathrm{d}\alpha_2\right|\mathrm{d}x \leqslant \\ &\leqslant (|\lambda|/T)NMe^{cT}Mn = \{K/(NT)\}|\lambda|nMN < \varrho K/(NT) \end{aligned}$$

by (1.4.6). Thus, (1.4.5) is true for $m = 2$. Assume (1.4.5) true for 2, 3, ...,

$m-1$. Then, from (1.4.2), we have

$$\left| m\left[\sum_{t_1,\dots,t_{p+1}=1}^{n} (1.4.1)\right]\right| \leqslant (e^{cT}M/T)\left[n^{p+1}(NM)^{p+1} + \right.$$
(1.4.7)
$$\left. + \binom{p+1}{1} n^p (NM)^p(\varrho K) + \dots + \binom{p+1}{p+1}(\varrho K)^{p+1}\right] =$$
$$= (e^{cT}M/T)(nNM + \varrho K)^{p+1}.$$

Therefore, since $m[\sum] = \sum m[\ ]$, we have

$$|d_{rr}^{(m)}| \leqslant |\lambda| \sum_{p=0}^{m-2} |\lambda|^p (e^{cT}M/T)(nNM + \varrho K)^{p+1} =$$
$$= |\lambda|(e^{cT}M/T)(nNM + \varrho K)(1/[1 - |\lambda|(nNM + \varrho K)]) <$$
$$< (e^{cT}M/T)|\lambda|(nNM + \varrho K)(\varrho + 1) < \varrho(e^{cT}M/T) = \varrho K/(NT),$$

which completes the induction.

Let

(1.4.8) $$d_{rr}^{(m)} = d_{m0} + d_{m1} + \dots + d_{mp} + \dots + d_{m,m-2},$$

where d_{mp} denotes the terms in $d_{rr}^{(m)}$ with a coefficient containing λ^{p+1}. Then

(1.4.9) $$d_{rr}^{(m)} - d_{rr}^{(m-1)} = d_{m,m-2} + (d_{m0} - d_{m-1,0}) + \dots + (d_{m,m-3} - d_{m-1,m-3}).$$

From (1.4.7), we have

(1.4.10) $$|d_{m,m-2}| \leqslant |\lambda|^{m-1}(e^{cT}M/T)(nNM + \varrho K)^{m-1} =$$
$$= \{K/(NT)\}[(nNM + \varrho K)|\lambda|]^{m-1}.$$

Moreover, using (1.3.5), we may write

$$d_{mp} - d_{m-1,p} = \lambda^{p+1} m\Big[\sum_{t_1,\dots,t_{p+1}=1}^{n} \xi_{rt_1}^{(m)} \cdot \Big\{ \int \xi_{t_1t_2}^{(1)} d\alpha_2 \int \xi_{t_2t_3}^{(2)} d\alpha_3 \int \dots \int \xi_{t_{p+1}r}^{(p+1)} d\alpha_{p+2} -$$
$$- \int \xi_{t_1t_2}^{(2)} d\alpha_2 \int \xi_{t_2t_3}^{(3)} d\alpha_3 \int \dots \int \xi_{t_{p+1}r}^{(p+2)} d\alpha_{p+2} \Big\}\Big] =$$
$$= \lambda^{p+1} m\Big[\sum_{t_1,\dots,t_{p+1}=1}^{n} \xi_{rt_1}^{(m)} \cdot \Big\{ \int (\xi_{t_1t_2}^{(1)} - \xi_{t_1t_2}^{(2)}) d\alpha_2 \int \dots \int \xi_{t_{p+1}r}^{(p+1)} d\alpha_{p+2} +$$
$$+ \int \xi_{t_1t_2}^{(2)} d\alpha_2 \int (\xi_{t_2t_3}^{(2)} - \xi_{t_2t_3}^{(3)}) d\alpha_3 \int \dots \int \xi_{t_{p+1}r}^{(p+1)} d\alpha_{p+2} + \dots +$$
$$+ \int \xi_{t_1t_2}^{(2)} d\alpha_2 \int \dots \int (\xi_{t_{p+1}r}^{(p+1)} - \xi_{t_{p+1}r}^{(p+2)}) d\alpha_{p+2} \Big\}\Big].$$

Again, the integrals are to be interpreted as before, and these integrals may be evaluated by use of (1.4.4).

If we remember that $\xi_{rs}^{(l)} - \xi_{rs}^{(l+1)} = 0$, if $r \neq s$, $= \eta_{rr}^{(l)} - \eta_{rr}^{(l+1)}$, if $r = s$, we get

$$d_{mp} - d_{m-1,p} = \lambda^{p+1} m \Bigg[\sum_{\substack{t_1,\ldots,t_{p+1}=1 \\ t_1 = t_2}}^{n} \xi_{rt_1}^{(m)} \int (\eta_{t_1t_2}^{(1)} - \eta_{t_1t_2}^{(2)}) \,\mathrm{d}\alpha_2 \int \ldots \int \xi_{t_{p+1}r}^{(p+1)} \,\mathrm{d}\alpha_{p+2} +$$

$$+ \ldots + \sum_{\substack{t_1,\ldots,t_{p+1}=1 \\ t_{p+1} = r}}^{n} \xi_{rt_1}^{(m)} \int \xi_{t_1t_2}^{(2)} \mathrm{d}\alpha_2 \int \ldots \int (\eta_{t_{p+1}r}^{(p+1)} - \eta_{t_{p+1}r}^{(p+2)}) \,\mathrm{d}\alpha_{p+2} \Bigg] .$$

But, each sum in this expression is of the same type as that in (1.4.7) except here we have only p running indices. Therefore, if we write

$$\theta_m/(NT) = \max_{r=1,\ldots,n} |d_{rr}^{(m)} - d_{rr}^{(m-1)}| \qquad (m = 1, 2, \ldots),$$

and use (1.4.4), we get

$$|d_{m,p} - d_{m-1,p}| \leqslant |\lambda|^{p+1} (e^{cT} M/T)(nNM + \varrho K)^p \cdot (\theta_{m-1} + \theta_{m-2} + \ldots + \theta_{m-p-1}) =$$

$$(1.4.11) \qquad = (\theta_{m-1} + \theta_{m-2} + \ldots + \theta_{m-p-1}) \cdot (K/[NT(nNM + \varrho K)])[|\lambda|(nNM + \varrho K)]^{p+1}.$$

Combining (1.4.9), (1.4.10) and (1.4.11), we have

$$|d_{rr}^{(m)} - d_{rr}^{(m-1)}| \leqslant \{K/(NT)\}(nNM + \varrho K)^{m-1} |\lambda|^{m-1} + K/[NT(nNM + \varrho K)] \cdot$$

$$(1.4.12) \qquad \cdot \{\theta_{m-1}|\lambda|(nNM + \varrho K) + (\theta_{m-1} + \theta_{m-2})[|\lambda|(nNM + \varrho K)]^2 + \ldots +$$

$$+ (\theta_{m-1} + \theta_{m-2} + \ldots + \theta_2)[|\lambda|(nNM + \varrho K)]^{m-2}\}.$$

We shall prove by induction that

$$(1.4.13) \qquad \theta_m < K(gh)^{m-1},$$

where $h = |\lambda| \cdot (nNM + \varrho K)$, $g = 1 + K/[(nNM + \varrho K)(1 - h)]$ $(m = 2, 3, \ldots)$.

Formula (1.4.13) certainly holds for $m = 2$, since we have already observed that $|d_{rr}^{(2)}| < \{K/(NT)\}|\lambda| nNM$, and, thus, $\theta_2 < |\lambda| nNMK$. Assume that (1.4.13) holds for 2, 3, ..., $m - 1$. Then, using (1.4.12) and remembering that

$\theta_m = NT \max | d_{rr}^{(m)} - d_{rr}^{(m-1)} |$, we have

$$\theta_m < Kh^{m-1} + K/(nNM + \varrho K)\{h\theta_{m-1} + h^2(\theta_{m-1} + \theta_{m-2}) + \\ + \dots + h^{m-2}(\theta_{m-1} + \dots + \theta_2)\} <$$

$$< Kh^{m-1} + K/(nNM + \varrho K)\{hK(gh)^{m-2} + h^2K[(gh)^{m-2} + (gh)^{m-3}] + \\ + \dots + h^{m-3}K[(gh)^{m-2} + \dots + (gh)]\} =$$

$$= Kh^{m-1}\{1 + K/(nNM + \varrho K)[g^{m-2} + (hg^{m-2} + g^{m-3}) + \dots + \\ + (h^{m-3}g^{m-2} + \dots + hg^2 + g)]\} =$$

$$= Kh^{m-1}\{1 + K/(nNM + \varrho K)[g + g^2(1 + h) + g^3(1 + h + h^2) + \dots + \\ + g^{m-2}(1 + h + \dots + h^{m-3})]\} <$$

$$< Kh^{m-1}\{1 + K/(nNM + \varrho K)\cdot[g/(1 - h)](1 + g + g^2 + \dots + g^{m-3})\} =$$

$$= Kh^{m-1}\left\{1 + [K/(nNM + \varrho K)]\cdot\frac{g(g^{m-2} - 1)}{(1 - h)(g - 1)}\right\} =$$

$$= Kh^{m-1}\{1 + g^{m-1} - g\} < K(gh)^{m-1}$$

since $1 - g$ is negative. Therefore, relation (1.4.13) holds for all m. But, from (1.4.6), we have

$$gh = \left[1 + \frac{K}{nNM + \varrho K}\cdot\frac{1}{1 - |\lambda|(nNM + \varrho K)}\right]\cdot|\lambda|(nNM + \varrho K) \leqslant$$

$$\leqslant \left[1 + \frac{K(\varrho + 1)}{nNM + \varrho K}\right]\cdot|\lambda|(nNM + \varrho K) = |\lambda|(nNM + 2\varrho K + K).$$

Consequently, if $|\lambda| < 1/(nNM + 2\varrho K + K)$, the series $\sum_{j=2}^{\infty} \theta_j$ will converge and this implies the following limit exists and

$$\lim_{m\to\infty} D_m = D .$$

1.5. – Completion of the proof of convergence. In a manner similar to the above, we prove $y_{rs}^{(m)}$ approaches a limit as $m \to \infty$. From expression (1.3.6), we have

$$y_{rs}^{(m)}(x) = \sum_{p=0}^{m} \lambda^p \sum_{t_1, \dots, t_{p+1}=1}^{n} e^{\tau_r x} \int \xi_{rt_1}^{(0)} \mathrm{d}\alpha_1 \int \xi_{t_1 t_2}^{(1)} \mathrm{d}\alpha_2 \int \dots \int \xi_{t_{p-1}s}^{(p-1)} \mathrm{d}\alpha_p .$$

Moreover, we may write

$$\int \xi_{rt_1}^{(0)} d\alpha_1 \int \dots \int \xi_{t_{p-1}s}^{(p-1)} d\alpha_p =$$

$$(1.5.1)\quad \begin{aligned} &= \int \xi_{rt_1}^{(m)} d\alpha_1 \int \dots \int \xi_{t_{p-1}s}^{(m)} d\alpha_p - \int \eta_{rt_1}^{(0)} d\alpha_1 \int \dots \int \xi_{t_{p-1}s}^{(m)} d\alpha_p - \dots \\ &\dots - \int \xi_{rt_1}^{(m)} d\alpha_1 \int \dots \int \eta_{t_{p-1}s}^{(p-1)} d\alpha_p + \int \eta_{rt_1}^{(0)} d\alpha_1 \int \eta_{t_1t_2}^{(1)} d\alpha_2 \int \dots \int \xi_{t_{p-1}s}^{(m)} d\alpha_p + \dots \\ &\dots + (-1)^p \int \eta_{rt_1}^{(0)} d\alpha_1 \int \dots \int \eta_{t_{p-1}s}^{(p-1)} d\alpha_p . \end{aligned}$$

Therefore,

$$\sum_{t_1,\dots,t_{p-1}=1}^{n} \left| (1.5.1) \right| \leqslant \left[n^p (NM)^p + \tbinom{p}{1} n^{p-1} (NM)^{p-1} (\varrho K) + \dots + \tbinom{p}{p} (\varrho K)^p \right] = = (nNM + \varrho K)^p ,$$

and

$$(1.5.2)\quad \begin{aligned} &\left| y_{rs}^{(m)}(x) - \delta_{rs} e^{\tau_r x} \right| \leqslant \sum_{p=1}^{m} |\lambda|^p e^{\mathcal{R}(\tau_r)x} (nNM + \varrho K)^p \leqslant \\ &\leqslant e^{\mathcal{R}(\tau_r)x} / [1 - |\lambda| (nNM + \varrho K)] < \varrho e \mathcal{R}^{R(\tau_r)x} < e\mathcal{R}^{R(\tau_r)x} . \end{aligned}$$

In a procedure similar to the above, we find that $|y_{rs}^{(m)} - y_{rs}^{(m-1)}| < e^{\mathcal{R}(\tau_r)x} (gh)^m$ and if $|\lambda| < 1/(nNM + 2\varrho K + K)$, then the following limit exists and

$$\lim_{m \to \infty} Y_m(x) = Y(x) .$$

Moreover, the convergence is uniform in each finite interval and, consequently, the functions $y_{rs}(x)$ are continuous in every finite interval.

From relation (1.1.14) we have, for $\lambda \neq 0$,

$$\int Z^{-1}(\Phi - D_m) Y_{m-1} d\alpha = (1/\lambda)[Z^{-1} Y_m - I]$$

and, therefore,

$$\int Z^{-1}(\Phi - D_m) Y_{m-1} d\alpha = \int_0^x Z^{-1}(\Phi - D_m) Y_{m-1} d\alpha + (1/\lambda)[Z^{-1}(0) Y_m(0) - I] .$$

The second member converges as $m \to \infty$ to

$$\int_0^x Z^{-1}(\Phi - D) Y d\alpha + (1/\lambda)[Z^{-1}(0) Y(0) - I] ,$$

and, thus, the limit of the integral in the first member also exists and we have a particular integral which we shall indicate by $\int Z^{-1}(\Phi - D)Y\,d\alpha$. Therefore if we take the limit as $m \to \infty$ in expression (1.1.14), we have

$$Y = Z + \lambda Z \int Z^{-1}(\Phi - D)Y\,d\alpha.$$

Thus, the elements $y_{rs}(x)$ of the matrix $Y(x)$ are AC functions in every finite interval and

$$Y' = (B - \lambda D)Y + \lambda \Phi Y \quad \text{a. e.}, \quad -\infty < x < +\infty.$$

It remains to show that the columns of the matrix Y are linearly independent. Observe that if $m \to \infty$ in formula (1.4.15), we have

$$|y_{rs}(x) - \delta_{rs} e^{\tau_r x}| \leqslant \varrho\, e^{\mathcal{R}(\tau_r)x} < e^{\mathcal{R}(\tau_r)x},$$

and, at $x = 0$, $|y_{rs}(0) - \delta_{rs}| < 1$, or $y_{rr}(0) \neq 0$. Consequently, at least one function in each column is not identically zero. Moreover, from (1.1.11), taking the limit as $m \to \infty$, we have

$$y_{rs}(x) = e^{\tau_s x}\, p_{rs}(x)$$

and $p_{rs}(x)$ is periodic of period T. Therefore, since $\tau_r \not\equiv \tau_s$ (mod ωi), and $p_{rr}(x)$ is not identically zero, we have, by [**10**, Lemma (2.i)], that the columns of Y are linearly independent.

1.6. – Determination of the characteristic exponents. It remains only to show that we can choose the numbers $\tau_1, \tau_2, \ldots, \tau_n$ such that $B - \lambda D = A$, or, since $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, we must show that the equations

$$f_k(\tau_1, \ldots, \tau_n; \lambda) \equiv \tau_k - \lambda d_k(\tau_1, \ldots, \tau_n; \lambda) - \varrho_k = 0 \qquad (k = 1, 2, \ldots, n), \tag{1.6.1}$$

have a unique solution $\tau_1, \ldots, \tau_n$ as functions of the numbers $\varrho_1, \varrho_2, \ldots, \varrho_n, \lambda$ and, also, for $|\lambda|$ small enough, each $\tau_i \in C_i$ with center ϱ_i, where $C_1, C_2, \ldots, C_n$ are the circles described in section (1.1).

We know that $f_k(\varrho_1, \varrho_2, \ldots, \varrho_n; 0) = 0$, $(k = 1, 2, \ldots, n)$, and, morevoer, the Jacobian $\partial(f_1, f_2, \ldots, f_n)/\partial(\tau_1, \tau_2, \ldots, \tau_n)$, taken in the complex field, is equal to 1 for $\lambda = 0$, $\tau_k = \varrho_k$, $(k = 1, 2, \ldots, n)$. Thus, by the theorem for implicit functions in the complex field [**18**, p. 267], for $|\lambda|$ sufficiently small, there exists a solution of the system of equations (1.6.1) of the form

$$\tau_i = \varrho_i + \sum_{h=1}^{\infty} a_{hi} \lambda^h \qquad (i = 1, 2, \ldots, n), \tag{1.6.2}$$

where a_{hi} are functions of $\varrho_1, \varrho_2, \ldots, \varrho_n$, and the series are convergent in a sufficiently small neighborhood of $\lambda = 0$. Moreover, we see from (1.6.2) that, if $|\lambda|$ is sufficiently small, $|\tau_i - \varrho_i| < \varepsilon\delta$, where ε, δ are given as in section (1.1), i.e., $\tau_1, \tau_2, \ldots, \tau_n$ lie in the circles $C_1, C_2, \ldots, C_n$, respectively. We summarize in the next few lines the conclusions of the present § **1**.

1.7. – Theorem. *Consider the system of equations*

$$y'_\mu = \varrho_\mu y_\mu + \sum_{\nu=1}^{n} \varphi_{\mu\nu}(x)\, y_\nu(x) \quad \text{a.e.}, \quad -\infty < x < +\infty$$

$$(\mu = 1, 2, \ldots, n),$$

where $\varrho_1, \ldots, \varrho_n$ are distinct complex numbers, λ is a complex parameter, the functions $\varphi_{\mu\nu}(x)$ are complex-valued functions of the real variable x, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$ and $\int_0^T \varphi_{\mu\nu}(x)\,\mathrm{d}x = 0$. If $\varrho_\mu \not\equiv \varrho_\nu$ (mod ωi), $\mu \neq \nu$, then, for $|\lambda|$ sufficiently small, the functions $d_{\mu\mu}^{(m)}$ given by (1.1.14) approach a limit d_μ as $m \to \infty$. Moreover, the solutions $\tau_1, \ldots, \tau_n$ of the system of equations

$$\tau_\mu - \lambda d_\mu(\tau_1, \ldots, \tau_n; \lambda) = \varrho_\mu, \qquad (\mu = 1, 2, \ldots, n),$$

are the characteristic exponents of the above system of differential equations.

§ 2. – A theorem on boundedness.

2.1. – Theorem. *Consider the system of differential equations*

$$y''_\mu + \sigma_\mu^2 y_\mu + \lambda \sum_{\nu=1}^{n} \psi_{\mu\nu}(x)\, y_\nu = 0, \quad \text{a.e.}, \quad -\infty < x < +\infty \tag{2.1.1}$$

$$(\mu = 1, 2, \ldots, n),$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are real, distinct, positive numbers, λ is a real parameter, the functions $\psi_{\mu\nu}(x)$ are real functions, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$ and $\int_0^T \psi_{\mu\nu}(x)\,\mathrm{d}x = 0$. If either

a) $$\psi_{\mu\nu}(x) = \psi_{\mu\nu}(-x) \qquad (\mu, \nu = 1, 2, \ldots, n),$$

or

b) $$\psi_{\mu\nu}(x) = \psi_{\nu\mu}(x) \qquad (\mu, \nu = 1, 2, \ldots, n),$$

and if $m\omega \neq \sigma_\mu \pm \sigma_\nu$ $(\mu, \nu = 1, 2, \ldots, n;\ \ m = 1, 2, \ldots)$, then, for $|\lambda|$ sufficiently small, the AC solutions of system (2.1.1) are bounded.

2.2. – Preliminary formulas for the proof of the Theorem. The following considerations follow closely those of CESARI. If we put

$$y_\mu = -(1/2)(Z_{2\mu-1} + Z_{2\mu}), \qquad y'_\mu = (1/i)(\sigma_\mu Z_{2\mu-1} - \sigma_\mu Z_{2\mu}),$$
$$Z_{2\mu-1} = -y_\mu + (i/\sigma_\mu)y'_\mu, \qquad Z_{2\mu} = -y_\mu - (i/\sigma_\mu)y'_\mu,$$

then (2.1.1) becomes

$$(2.2.1)\qquad \begin{cases} Z'_{2\mu-1} = i\sigma_\mu Z_{2\mu-1} + \{\lambda i/(2\sigma_\mu)\}\sum_{\nu=1}^{n}[\psi_{\mu\nu}Z_{2\nu-1} + \psi_{\mu\nu}Z_{2\nu}] \\ Z'_{2\mu} = -i\sigma_\mu Z_{2\mu} - \{\lambda i/(2\sigma_\mu)\}\sum_{\nu=1}^{n}[\psi_{\mu\nu}Z_{2\nu-1} + \psi_{\mu\nu}Z_{2\nu}] \end{cases}$$
$$(\mu = 1, 2, \ldots, n).$$

Let us write system (2.2.1) in the form

$$Z'_\mu = \varrho_\mu Z_\mu + \lambda \sum_{\nu=1}^{2n} \varphi_{\mu\nu}(x) Z_\nu \qquad (\mu = 1, 2, \ldots, 2n),$$

where $\varrho_1, \varrho_2, \ldots, \varrho_{2n}$ are written in place of $i\sigma_1, -i\sigma_1, i\sigma_2, -i\sigma_2, \ldots, i\sigma_n, -i\sigma_n$, respectively.

Designate by $\bar\tau_1, \bar\tau_2, \ldots, \bar\tau_{2n}$ complex numbers such that $\bar\tau_1 = i\tau_1$, $\bar\tau_2 = -i\tau_1$, ..., $\bar\tau_{2n-1} = i\tau_n$, $\bar\tau_{2n} = -i\tau_n$, and $\tau_1, \tau_2, \ldots, \tau_n$ are real numbers satisfying only the condition that $i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n$ lie in the circles $C_1, C_2, \ldots, C_{2n-1}, C_{2n}$, of radius $\varepsilon\delta$ [see (1.1)] and centers at the points $i\sigma_1, -i\sigma_1, \ldots, i\sigma_n, -i\sigma_n$. Let us apply the preceding algorithm to the auxiliary equations

$$(2.2.2)\qquad Z_\mu = e^{\bar\tau_\mu x} + \lambda \sum_{\nu=1}^{2n} e^{\bar\tau_\mu x}\int e^{-\bar\tau_\mu \alpha}\varphi_{\mu\nu}(\alpha)\, Z_\nu(\alpha)\, d\alpha \qquad (\mu = 1, 2, \ldots, 2n).$$

From expression (1.3.5), we see that

$$d_{rr}^{(m)} = \lambda m \left[\sum_{p=0}^{m-2} \lambda^p \sum_{t_1,\ldots,t_{p+1}=1}^{2n} e^{-\tau_s x}\varphi_{r t_1} e^{\tau_{t_1} x}\int e^{-\tau_{t_1}\alpha_2}\varphi_{t_1 t_2}^{(m-1)} e^{\tau_{t_2}\alpha_2}\, d\alpha_3 \int \ldots \right.$$
$$\left. \ldots \int e^{-\tau_{t_{p+1}}\alpha_{p+2}}\varphi_{t_{p+1} r}^{(m-p-1)} e^{\tau_r \alpha_{p+2}}\, d\alpha_{p+2}\right].$$

But, from a known theorem on FOURIER series [**11**, p. 582], we may calculate each of the above integrals by substituting the FOURIER series for $\varphi_{\mu\nu}(x)$ and integrating term by term. If $\varphi_{\mu\nu}(x) \sim \sum_{k=-\infty}^{+\infty} \gamma_{\mu\nu k}\, e^{ik\omega x}$, $\gamma_{\mu\nu 0} = 0$, $(\mu, \nu = 1, 2, \ldots, 2n)$, we obtain

$$d_{rr}^{(m)} = \sum_{p=0}^{m-2} \lambda^{p+1} \sum_{t_1,\dots,t_{p+1}=1}^{2n} \sum_{k_1+\dots+k_{p+2}=0} \prod_{l=0}^{p+1} \gamma_{t_l t_{l+1} k_{l+1}} \cdot \\ \cdot[-\bar{\tau}_{t_l} + i(k_{2+1} + \dots + k_{p+2})\omega + \bar{\tau}_r]^{-1}, \tag{2.2.3}$$

where we put $t_0 = t_{p+2} = r$ and we understand in this formula that the first term in the product is $\gamma_{rt_1k_1}$, and also, $\gamma_{rsk}^{(m)} = -d_{rr}^{(m)}$ if $r = s$, $k = 0$, $= \gamma_{rsk}$, otherwise. Moreover, we must exclude from the sum all combinations of the indices $t_1, \dots, t_{p+1}$, $k_1, \dots, k_{p+2}$ which make any of the denominators zero. These expressions are contained in the paper by CESARI [**4**, (a)].

Put $\psi_{\mu\nu}(x) \sim \sum_{k=-\infty}^{\infty+} c_{\mu\nu k} e^{ik\omega x}$, $c_{\mu\nu 0} = 0$, and put $r = 2r' - 2 + u$, $s = 2s' - - 2 + v$, $(r, s = 1, 2, \dots, 2n;\ r', s' = 1, 2, \dots, n;\ u, v = 1, 2)$, and identify the γ's with the c's, we get $\gamma_{rsk} = (-1)^{u-1} \cdot i c_{r's'k} (2\sigma_{r'})^{-1}$, $\bar{\tau}_r = (-1)^{u-1} \cdot i\tau_{r'}$. Let us put

$$d_{1rr}^{(m)} = (2\sigma_r/i) d_{2r-1,2r-1}^{(m)}, \qquad d_{2rr}^{(m)} = -(2\sigma_r/i) d_{2r,2r}^{(m)}, \tag{2.2.4}$$

and

$$c_{rsk}^{(u,v,m)} = \begin{cases} -d_{1rr}^{(m)}, & \text{if} \quad u = v = 1, \ r = s, \ k = 0, \\ -d_{2rr}^{(m)}, & \text{if} \quad u = v = 2, \ r = s, \ k = 0, \\ c_{rsk}, & \text{otherwise.} \end{cases}$$

Then $\gamma_{rsk}^{(m)} = (-)^{u-1} \cdot i (2\sigma_r')^{-1} c_{r's'k}^{(u,v,m)}$, and, from (2.2.3), we get

$$d_{1rr}^{(m)} = \sum_{p=0}^{m-2} (\lambda/2)^{p+1} \sum_{t_1,\dots,t_{p+1}=1}^{n} \sum_{k_1+\dots+k_{p+2}=0} \sum_{u_1,\dots,u_{p+1}=1}^{2} \prod_{h=1}^{p+1} (-1)^{u_h-1} \cdot \\ \cdot \prod_{l=0}^{p+1} c_{t_l t_{l+1} k_{l+1}}^{(u_l, u_{l+1}, m-l)} \sigma_{t_l}^{-1} [-(-1)^{u_l-1} \tau_{t_l} + (k_{l+1} + \dots + k_{p+2})\omega + \tau_r]^{-1}, \tag{2.2.5}$$

$$d_{2rr}^{(m)} = \sum_{p=0}^{m-2} (\lambda/2)^{p+1} \sum_{t_1,\dots,t_{p+1}=1}^{n} \sum_{k_1+\dots+k_{p+2}=0} \sum_{u_1,\dots,u_{p+1}=1}^{2} \prod_{h=1}^{p+1} (-1)^{u_h-1} \cdot \\ \cdot \prod_{l=0}^{p+1} c_{t_l t_{l+1} k_{l+1}}^{(u_l, u_{l+1}, m-l)} \sigma_{t_l}^{-1} [-(-1)^{u_l-1} \tau_{t_l} + (k_{l+1} + \dots + k_{p+2})\omega - \tau_r]^{-1}, \tag{2.2.6}$$

where the first term in the second product of both (2.2.5) and (2.2.6) is $c_{rt_1k_1}$.

We shall prove that, in all cases,

$$\bar{d}_{1rr}^{(m)} = d_{2rr}^{(m)}. \tag{2.2.7}$$

Relation (2.2.7) is true for $m = 0$, $m = 1$. Assume that it has been shown true for $0, 1, \dots, m-1$. Then

$$c_{rs,-k}^{(u,v,p)} = \bar{c}_{rsk}^{(3-u,3-v,p)} \qquad (p = 1, 2, \dots, m-1); \tag{2.2.8}$$

for, if $k \neq 0$, $r = s$, or $r \neq s$, this is just a property of the FOURIER coefficients of $\psi_{rs}(x)$. Finally, if $r = s$, $k = 0$, $u = v$, then $c_{rs,-k}^{(u,v,p)} = -d_{urr}^{(p)}$, $c_{rsk}^{(3-u,3-v,p)} = -d_{3-u,rr}^{(p)}$, and these quantities are assumed complex conjugate. Therefore, if in (2.2.5) we replace $k_1, \dots, k_{p+2}$ by $-k_1, -k_2, \dots, -k_{p+2}$, put $u_1 = 3 - u_1', \dots, u_{p+1} = 3 - u_{p+1}'$, and make use of (2.2.8), we get $\bar{d}_{2rr}^{(m)}$. Consequently, (2.2.7) holds for all m.

In order to prove theorem (2.1), it is only necessary to show that the equations

$$\bar{\tau}_\mu - \lambda\, d_\mu(\bar{\tau}_1, \dots, \bar{\tau}_{2n}; \lambda) = \varrho_\mu \qquad (\mu = 1, 2, \dots, 2n)$$

have a purely imaginary solution $\bar{\tau}_1, \dots, \bar{\tau}_{2n}$. By (2.2.7) and the definitions of the ϱ_μ and $\bar{\tau}_\mu$, this means that the equations

$$i\tau_\mu - \lambda d_{2\mu-1}(i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n; \lambda) = i\sigma_\mu \qquad (\mu = 1, 2, \dots, n)$$

must have a real solution τ_μ; or, $d_{2\mu-1}$ must be purely imaginary. Finally, from (2.2.4), this implies that the limit of $d_{1rr}^{(m)}$ as $m \to \infty$ must be real.

2.3. – Proof of a). If $\psi_{\mu\nu}(x) = \psi_{\mu\nu}(-x)$, then

$$\psi_{\mu\nu}(x) \sim \sum_{k=1}^{\infty} a_{\mu\nu k} \cos k\omega x$$

and $c_{\mu\nu k} = 2^{-1} a_{\mu\nu|k|}$ is real for $k = \pm 1, \pm 2, \dots$. Therefore, $d_{1rr}^{(m)}$ is real for $r = 1, 2, \dots, n$; $m = 1, 2, \dots$, as was to be shown.

2.4. – Proof of b). We assume that $\psi_{\mu\nu}(x) = \psi_{\nu\mu}(x)$, $(\mu, \nu = 1, 2, \dots, n)$. By replacing $c_{rsk}^{(u,v,p)}$ by their corresponding expressions, we may write (2.2.5) in the following form

$$\begin{aligned} d_{1rr}^{(m)} = \sum_{p=0}^{m-2} (\lambda/2)^{p+1} \Bigg\{ & \sum_{t_1,\dots,t_{p+1}=1}^{n} \; \sum_{k_1+\dots+k_{p+2}=0} \; \sum_{u_1,\dots,u_{p+1}=1}^{2} \; \sum_{h=1}^{p+1} (-1)^{u_h-1} \cdot \\ & \cdot \prod_{l=0}^{p+1} c_{t_l t_{l+1} k_{l+1}}\, \sigma_{t_l}^{-1} \left[-(-1)^{u_l-1}\tau_{t_l} + (k_{l+1} + \dots + k_{p+2})\omega + \tau_r \right]^{-1} - \\ & - \sum_{j=1}^{p+1} \sum_{s=1}^{n} \sum_{v=1}^{2} d_{vss}^{(m-j)} \sum_{\substack{t_1,\dots,t_{p+1}=1 \\ t_j = t_{j+1} = s}}^{n} \; \sum_{\substack{k_1+\dots+k_{p+2}=0 \\ k_{j+1}=0}} \; \sum_{\substack{u_1,\dots,u_{p+1}=1 \\ u_j = u_{j+1} = v}}^{2} \; \prod_{h=1}^{p+1} (-1)^{u_h-1} \cdot \\ & \cdot \sigma_s^{-1} \left[-(-1)^{v-1}\tau_s + (k_{j+1} + \dots + k_{p+2})\omega + \tau_r \right]^{-1} \cdot \\ & \cdot \prod_{\substack{l=0 \\ l \neq j}}^{p+1} c_{t_l t_{l+1} k_{l+1}}\, \sigma_{t_l}^{-1} \left[-(-1)^{u_l-1}\tau_{t_l} + (k_{l+1} + \dots + k_{p+2})\omega + \tau_r \right]^{-1} + \\ & + \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{p+1} \; \sum_{s_1,s_2=1}^{n} \; \sum_{v_1,v_2=1}^{2} d_{v_1 s_1 s_1}^{(m-j_1)}\, d_{v_2 s_2 s_2}^{(m-j_2)} \sum_{\substack{t_1,\dots,t_{p+1}=1 \\ t_{j_1}=t_{j_1+1}=s_1 \\ t_{j_2}=t_{j_2+1}=s_2}}^{n} \; \sum_{\substack{k_1+\dots+k_{p+2}=0 \\ k_{j_1+1}=0 \\ k_{j_2+1}=0}} \; \sum_{\substack{u_1,\dots,u_{p+1}=1 \\ u_{j_1}=u_{j_1+1}=v_1 \\ u_{j_2}=u_{j_2+1}=v_2}}^{2} \dots \Bigg\} . \end{aligned}$$

Thus, we have

$$d_{1rr}^{(m)} = \sum_{p=0}^{m-2} (\lambda/2)^{p+1} \Big\{ \gamma_{1pr} - \sum_{j=1}^{p+1} \sum_{s=1}^{n} \sum_{v=1}^{2} d_{vss}^{(m-j)} \gamma_{pvsj} + \tag{2.4.1}$$
$$+ \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{p+1} \sum_{s_1,s_2=1}^{n} \sum_{v_1,v_2=1}^{2} d_{v_1s_1s_1}^{(m-j_1)} d_{v_2s_2s_2}^{(m-j_2)} \gamma_{pv_1v_2s_1s_2j_1j_2} - \dots \Big\},$$

where the $\gamma's$ are constants independent of λ.

We shall show by induction that $d_{1rr}^{(m)}$ has coefficients in $\lambda, \lambda^2, \dots, \lambda^{m-1}$ real for every m. We shall first prove that

$$\bar{d}_{1rr}^{(2)} = d_{1rr}^{(2)}, \qquad \bar{d}_{2rr}^{(2)} = d_{2rr}^{(2)}, \qquad (r = 1, 2, \dots, n). \tag{2.4.2}$$

We have

$$\bar{d}_{1rr}^{(2)} = (\lambda/2) \sum_{h=1}^{n} \sum_{k_1+k_2=0} \bar{c}_{rhk_1} \bar{c}_{hrk_2} \sigma_h^{-1} \left[\frac{1}{-\tau_h + k_2\omega + \tau_r} - \frac{1}{\tau_h + k_2\omega + \tau_r} \right].$$

If we use the fact that $\bar{c}_{rsk} = c_{rs,-k}$, then use the fact that $k_1 + k_2 = 0$, $c_{rsk} = c_{srk}$, we get $\bar{d}_{1rr}^{(2)} = d_{1rr}^{(2)}$, or, $d_{1rr}^{(2)}$ is real, and, from (2.2.7), we have $d_{2rr}^{(2)}$ is real.

Next, we shall show that, in (2.4.1), we have

$$\bar{\gamma}_{1pr} = \gamma_{1pr}. \tag{2.4.3}$$

From (2.4.1), we know that

$$\bar{\gamma}_{1pr} = \sum_{t_1,\dots,t_{p+1}=1}^{n} \sum_{k_1+\dots+k_{p+2}=0} \sum_{u_1,\dots,u_{p+1}=1}^{2} \prod_{h=1}^{p+1} (-1)^{u_h-1} \cdot$$
$$\cdot \prod_{l=0}^{p+1} c_{t_l t_{l+1} k_{l+1}} \sigma_{t_l}^{-1} [-(-1)^{u_l-1} \tau_{t_l} + (k_{l+1} + \dots + k_{p+2})\omega + \tau_r]^{-1}.$$

If in this expression for $\bar{\gamma}_{1pr}$, we make use of the fact that $\bar{c}_{rsk} = c_{rs,-k}$ and replace $k_1, \dots, k_{p+2}$ by $-k_1, \dots, -k_{p+2}$, we have

$$\bar{\gamma}_{1pr} = \sum_{t_1,\dots,t_{p+1}=1}^{n} \sum_{k_1+\dots+k_{p+2}=0} \sum_{u_1,\dots,u_{p+1}=1}^{2} \prod_{h=1}^{p+1} (-1)^{u_h-1} \cdot$$
$$\cdot \prod_{l=0}^{p+1} c_{t_l t_{l+1} k_{l+1}} \sigma_{t_l}^{-1} [-(-1)^{u_l-1} \tau_{t_l} - (k_{l+1} + \dots + k_{p+2})\omega + \tau_r]^{-1}.$$

But, since $k_1 + \dots + k_{p+2} = 0$, we have $-(k_{l+1} + \dots + k_{p+2}) = k_1 + \dots + k_l$. If we make this substitution, and replace t_l by t_{p-l+2}, u_l by u_{p-l+2}, k_{l+1} by

k_{p-l+2}, $(l=0,1,\ldots,p+1)$, and let $\delta=p-l+1$, and use the fact that $c_{rsk}=$ $=c_{srk}$, then we have the result that

$$\bar{\gamma}_{1pr}=\sum_{t_1,\ldots,t_{p+1}=1}^{n}\ \sum_{k_1+\ldots+k_{p+2}=0}\ \sum_{u_1,\ldots,u_{p+1}=1}^{2}\ \prod_{h=1}^{p+1}(-1)^{u_h-1}\cdot$$
$$\cdot\prod_{\delta=0}^{p+1}c_{t_\delta t_{\delta+1}k_{\delta+1}}\sigma_{t_\delta}^{-1}\left[-(-1)^{u_\delta-1}\tau_{t_\delta}+(k_{\delta+1}+\ldots+k_{p+2})\omega+\tau_r\right]^{-1}=\gamma_{1pr},$$

as was to be shown.

Next, we shall show that

$$(2.4.4)\qquad\begin{cases}\gamma_{pvsj}=\gamma_{pvs,p-j+1},\\ \gamma_{pv_1v_2s_1s_2j_1j_2}=\gamma_{pv_1v_2s_1s_2,p-j_1+1,p-j_2+1},\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\end{cases}$$

Exactly as in the preceding proof, we obtain

$$\bar{\gamma}_{pvsj}=\sum_{\substack{t_1,\ldots,t_{p+1}=1\\ t_j=t_{j+1}=s}}^{n}\ \sum_{\substack{k_1+\ldots+k_{p+2}=0\\ k_{j+1}=0}}\ \sum_{\substack{u_1,\ldots,u_{p+1}=1\\ u_j=u_{j+1}=v}}^{2}\ \prod_{h=1}^{p+1}(-1)^{u_h-1}\sigma_s^{-1}\cdot\left[-(-1)^{v-1}\tau_s+\right.$$
$$\left.+(k_1+\ldots+k_j)\omega+\tau_r\right]^{-1}\prod_{\substack{l=0\\ l\neq j}}^{p+1}c_{t_lt_{l+1}k_{l+1}}\sigma_{t_l}^{-1}\left[-(-1)^{u_l-1}\tau_{t_l}+(k_1+\ldots+k_l)\omega+\tau_r\right]^{-1}.$$

Again, if we replace t_l by t_{p-l+2}, u_l by u_{p-l+2}, k_{l+1} by k_{p-l+2}, $(l\neq j;\ l=0,1,\ldots,p+1)$, and let $\delta=p-l+1$, and use the fact that $c_{rsk}=c_{srk}$, we have

$$\bar{\gamma}_{pvsj}=\sum_{\substack{t_1,\ldots,t_{p+1}=1\\ t_{p-j+1}=t_{p-j+2}=s}}^{n}\ \sum_{\substack{k_1+\ldots+k_{p+2}=0\\ k_{p-j+2}=0}}\ \sum_{\substack{u_1,\ldots,u_{p+1}=1\\ u_{p-j+1}=u_{p-j+2}=v}}^{2}\ \prod_{h=1}^{p+1}(-1)^{u_h-1}\cdot$$
$$\cdot\,\sigma_s^{-1}\left[-(-1)^{v-1}\tau_s+(k_{p-j+2}+\ldots+k_{p+2})\omega+\tau_r\right]^{-1}\cdot$$
$$\cdot\prod_{\substack{\delta=0\\ \delta\neq p-j+1}}^{p+1}c_{t_\delta t_{\delta+1}k_{\delta+1}}\sigma_{t_\delta}^{-1}\left[-(-1)^{u_\delta-1}\tau_{t_\delta}+(k_{\delta+1}+\ldots+k_{p+2})\omega+\tau_r\right]^{-1}=\gamma_{pvs,p-j+1},$$

as was to be shown. The other identities in (2.4.4) follow by a similar reasoning. The proof of the above properties is in CESARI's paper [**4**, (a)].

In the following, the notation $f(\lambda)=0(\lambda^p)$ shall mean that $f(\lambda)$ is a power series in λ which begins with a term of the power at least λ^p. Now, from relation (1.4.13), we have

$$\theta_m/(NT)=\max_{r=1,2,\ldots,n}\left|d_{rr}^{(m)}-d_{rr}^{(m-1)}\right|=0(\lambda^{m-1})$$

and, thus, $d_{rr}^{(m)} \equiv d_{rr}^{(m-1)} \pmod{\lambda^{m-1}}$. Therefore, we have

$$(2.4.5) \qquad \begin{cases} d_{vss}^{(m-j)} \equiv d_{vss}^{(m-p+j-1)} \pmod{\lambda^{m-p+j-1}} & \text{if } j \leqslant \mathrm{E}\left[\dfrac{p+1}{2}\right], \\ d_{vss}^{(m-p+j-1)} \equiv d_{vss}^{(m-j)} \pmod{\lambda^{m-j}} & \text{if } j \geqslant \mathrm{E}\left[\dfrac{p+1}{2}\right], \end{cases}$$

where $\mathrm{E}[x]$ is the integer part of x.

Furthermore, using (2.4.3) and (2.4.4), we may write

$$(2.4.6) \qquad \begin{aligned} \bar{d}_{1rr}^{(m)} - d_{1rr}^{(m)} = \sum_{p=0}^{m-2} (\lambda/2)^{p+1} \Big\{ & -\sum_{j=1}^{p+1} \sum_{s=1}^{n} \sum_{v=1}^{2} \left(\bar{d}_{vst}^{(m-j)} - d_{vss}^{(m-p+j-1)}\right) \gamma_{pvsj} + \\ & + \sum_{j_1,j_2=1}^{p+1} \sum_{s_1,s_2=1}^{n} \sum_{v_1,v_2=1}^{2} \left(\bar{d}_{v_1s_1s_1}^{(m-j_1)} \bar{d}_{v_2s_2s_2}^{(m-j_2)} - \right. \\ & \left. - d_{v_1s_1s_1}^{(m-p+j_1-1)} d_{v_2s_2s_2}^{(m-p+j_2-1)}\right) \gamma_{pv_1v_2s_1s_2j_1j_2} + \ldots \Big\} . \end{aligned}$$

We know that $d_{uss}^{(2)}$ is real for every u, s. Assume $d_{uss}^{(k)}$ has coefficient in λ, $\lambda^2, \ldots, \lambda^{k-1}$ real for every u and s and for $k = 1, 2, \ldots, m-1$. We wish to show that this is also true for $k = m$. Under this assumption, we then have from (2.4.5) that $\bar{d}_{vss}^{(m-j)}$ and $d_{vss}^{(m-p+j-1)}$ coincide up to terms in either λ^{m-j} or $\lambda^{m-p+j-1}$ depending on whether $j \geqslant \mathrm{E}\left[\dfrac{p+1}{2}\right]$ or $j \leqslant \mathrm{E}\left[\dfrac{p+1}{2}\right]$. Consequently, we may write (2.4.6) in the form

$$\begin{aligned} \bar{d}_{1rr}^{(m)} - d_{1rr}^{(m)} = \sum_{p=0}^{m-2} (\lambda/2)^{p+1} \Big\{ & -\sum_{j=1}^{\mathrm{E}\left[\frac{p+1}{2}\right]} \sum_{s=1}^{n} \sum_{v=1}^{2} \gamma_{pvsj}\, O(\lambda^{m-p+j-1}) - \\ & - \sum_{j=\mathrm{E}\left[\frac{p+1}{2}\right]+1}^{p+1} \sum_{s=1}^{n} \sum_{v=1}^{2} \gamma_{pvsk} O(\lambda^{m-j}) + \\ & + \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{\mathrm{E}\left[\frac{p+1}{2}\right]} \sum_{s_1,s_2=1}^{n} \sum_{v_1,v_2=1}^{2} \gamma_{pv_1v_2s_1s_2j_1j_2} O(\lambda^{2(m-p)+j_1+j_2-2}) + \\ & + \sum_{\substack{j_1,j_2=\mathrm{E}\left[\frac{p+1}{2}\right]+1 \\ j_1<j_2}}^{p+1} \sum_{s_1,s_2=1}^{n} \sum_{v_1,v_2=1}^{2} \gamma_{pv_1v_2s_1s_2j_1j_2} O(\lambda^{2m-j_1-j_2}) + \ldots \Big\}. \end{aligned}$$

Therefore, $\bar{d}_{1rr}^{(m)} - d_{1rr}^{(m)}$ is at least $O(\lambda^m)$. Thus, $d_{1rr}^{(m)}$ has coefficient in $\lambda, \lambda^2, \ldots, \lambda^{m-1}$ real, and the induction is completed. Moreover, we know that the limit of $d_{1rr}^{(m)}$ as $m \to \infty$ exists and, as a consequence, the term in λ^m approaches zero as

$m \to \infty$. Finally, we have the limit of $d_{1rr}^{(m)}$ as $m \to \infty$ is real, and the Theorem is proved.

Note. For other research in which previous statements concerning differential systems have been extended in such a way to require only L-integrability of the coefficients instead of continuity, see D. CALIGO [2], L. CESARI [4, (b); 5] and C. TAAM [21].

§ 3. - Remarks.

I. - The discussion in § 1 and § 2 may be extended to systems of a much more general type. In fact, consider a system of the form

$$y_j' = \varrho_j(\lambda)\, y_j + \lambda \sum_{h=1}^{n} \varphi_{jh}(x;\lambda)\, y_h\,, \quad \text{a. e.,} \quad -\infty < x < +\infty \tag{3.1}$$

$$(j = 1, 2, \ldots, n),$$

where λ is a small parameter, each $\varrho_j(\lambda)$ is a continuous function in λ at $\lambda = 0$ with $\varrho_{j0} \not\equiv \varrho_{h0}$ (mod ωi), ($j \neq h$; $j, h = 1, 2, \ldots, n$), where $\varrho_{j0} = \varrho_j(0)$, and each function $\varphi_{jh}(x;\lambda)$ is periodic in x of period $T = 2\pi/\omega$, L-integrable with respect to x in $[0, T]$, continuous in λ at $\lambda = 0$, $\int_0^T \varphi_{jh}(x;\lambda)\,dx = 0$ and $|\varphi_{jh}(x;\lambda)| < \eta(x)$ for all $|\lambda| \leqslant \lambda_0$, $\lambda_0 > 0$, where $\eta(x)$ is L-integrable in $[0, T]$. Then, we necessarily have

$$\int_0^T |\varphi_{jh}(x;\lambda)|\,dx < M \qquad (j, h = 1, 2, \ldots, n), \tag{3.2}$$

for $|\lambda| \leqslant \lambda_0$, $\lambda_0 > 0$, and M is independent of λ. The algorithm in § 1 may be defined in exactly the same way by replacing the $\varrho_j(\lambda)$ by τ_j and the functions $\varphi_{jh}(x)$ in § 1 by the functions $\varphi_{jh}(x;\lambda)$ above. Since we have made the assumption (3.2), the proof of the convergence will be exactly the same.

Moreover, if we write the FOURIER series of $\varphi_{jh}(x;\lambda)$ as

$$\varphi_{jh}(x;\lambda) \sim \sum_{n=-\infty}^{+\infty} c_{jhn}(\lambda) e^{in\omega x} \qquad (j, h = 1, 2, \ldots, n),$$

then we see that the functions $d_{rr}^{(m)}$ in § 2 have exactly the same form except with c_{jhn} replaced by $c_{jhn}(\lambda)$. The proof to Theorem (2.1) can be extended in an obvious manner to the following theorem.

Theorem (3.1). *Consider the system*

$$y''_\mu + \sigma^2_\mu(\lambda)\, y_\mu + \lambda \sum_{\nu=1}^{n} \varphi_{\mu\nu}(x;\lambda)\, y_\nu = 0\,, \qquad \text{a. e.,} \quad -\infty < x < +\infty \tag{3.3}$$

$$(\mu = 1, 2, \ldots, n),$$

where λ is a real parameter, each $\sigma_\mu(\lambda)$ is a positive, continuous function in λ at $\lambda=0$ for $|\lambda| \leqslant \lambda_0$, $\lambda_0 > 0$, with $\sigma_\mu(0) = \sigma_{\mu 0}$, the functions $\varphi_{\mu\nu}(x;\lambda)$ are real functions, periodic in x of period $T = 2\pi/\omega$, L-integrable in $[0, T]$, continuous in λ at $\lambda = 0$, $\int_0^T \varphi_{\mu\nu}(x;\lambda)dx = 0$, and $|\varphi_{\mu\nu}(x;\lambda)| < \eta(x)$ for all $|\lambda| \leqslant \lambda_0$ and $\eta(x)$ is L-integrable in $[0, T]$. If either $\varphi_{\mu\nu}(x;\lambda) = \varphi_{\mu\nu}(-x;\lambda)$, $(\mu, \nu = 1, 2, \ldots, n)$, or $\varphi_{\mu\nu}(x;\lambda) = \varphi_{\nu\mu}(x;\lambda)$, $(\mu, \nu = 1, 2, \ldots, n)$, and if $m\omega \neq \sigma_{\mu 0} \pm \sigma_{\nu 0}$, $(\mu, \nu = 1, 2, \ldots, n;\ m = 0, 1, 2, \ldots)$, then, for $|\lambda|$ sufficiently small, the AC *solutions of* (3.3) *are bounded.*

II. – Suppose, in Theorem (3.1), that $\int_0^T \varphi_{\mu\nu}(x;\lambda)\,dx = m_{\mu\nu}(\lambda)$ is not necessarily zero for all μ, ν. By our assumptions on the functions $\varphi_{\mu\nu}(x;\lambda)$, we necessarily have $m_{\mu\nu}(\lambda)$ is continuous at $\lambda = 0$. If we let $A = \|\sigma^2_\mu(\lambda)\delta_{\mu\nu} + \lambda m_{\mu\nu}(\lambda)\|$, $(\mu, \nu = 1, 2, \ldots, n)$, $\Phi^*(x;\lambda) = \|\varphi^*_{\mu\nu}(x;\lambda)\| = \|\varphi_{\mu\nu}(x;\lambda) - m_{\mu\nu}(\lambda)\|$, then system (3.3) is transformed into the matrix equation

$$Y'' + A(\lambda)Y + \lambda\Phi^*(x;\lambda)Y = 0\,, \quad \text{a. e.,} \quad -\infty < x < +\infty, \tag{3.4}$$

where $\int_0^T \varphi^*_{\mu\nu}(x;\lambda)\,dx = 0$ for all $|\lambda| \leqslant \lambda_0$. Since $\sigma_1^2, \ldots, \sigma_n^2$ are distinct, positive functions of λ, the characteristic roots $\delta_1^2(\lambda), \ldots, \delta_n^2(\lambda)$ of the matrix $A(\lambda)$ are continuous functions of λ at $\lambda = 0$, and are positive and distinct for every λ, $|\lambda|$ sufficiently small. Thus, there is a non-singular matrix $P(\lambda)$, $\det P(\lambda) > \varepsilon > 0$ for all $|\lambda|$ sufficiently small, such that $P^{-1}AP = B(\lambda) = \operatorname{diag}(\delta_1^2(\lambda), \ldots, \delta_n^2(\lambda))$. Therefore, if we let $Y = PZ$, then the above system (3.4) becomes

$$Z'' + B(\lambda)\,Z + P^{-1}(\lambda)\Phi^*(x;\lambda)P(\lambda)\,Z = 0\,. \tag{3.5}$$

If the AC solutions Z of this equation are bounded, for $|\lambda|$ sufficiently small, then, since $P(\lambda)$ is continuous at $\lambda = 0$, we have $Y = PZ$ is also bounded for $|\lambda|$ sufficiently small. Also, if Φ^* is even, then $P^{-1}\Phi^*P$ is even. If $\varphi_{\mu\nu}(x;\lambda) = \varphi_{\nu\mu}(x;\lambda)$, then $P(\lambda)$ is an orthogonal matrix and, thus, $P^{-1}\Phi^*P$ is symmetric. Finally, since (3.5) is a special case of a system satisfying theorem (3.1), we may state the following theorem.

Theorem (3.2). *The preceding Theorem* (3.1) *is true even if we do not assume that* $\int_0^T \varphi_{\mu\nu}(x; \lambda)\, dx = 0$ *for all* $|\lambda| \leqslant \lambda_0$ *and all* $\mu, \nu = 1, 2, \ldots, n$.

For a more detailed discussion of the proof of this Theorem, see the paper of R. A. Gambill: « *Stability criteria for linear differential systems with periodic coefficients*, Rivista Mat. Univ. Parma **5**, 169-181 (1954) ».

Bibliography.

1. R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill, New York 1953.
2. D. Caligo, *Un criterio sufficiente di stabilità per le soluzioni dei sistemi di equazioni integrali lineari e sue applicazioni*, Atti 2° Congr. Un. Mat. Ital. 177-185 (1940).
3. C. Carathéodory, *Vorlesungen über reelle Funktionen*, Teubner, Berlin 1918.
4. L. Cesari: (a) *Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici*, Atti Accad. Italia, Mem. Cl. Sci. Fis. Mat. Nat. (6) **11**, 633-692 (1940); (b) *Un nuovo criterio di stabilità per le soluzioni delle equazioni differenziali lineari*. Ann. Scuola Norm. Super. Pisa (2) **9**, 163-186 (1940).
5. L. Cesari and J. K. Hale, *Second order linear differential systems with periodic L-integrable coefficients*, Rivista Mat. Univ. Parma **5**, 55-61 (1954).
6. H. Cohen, *The stability equation with periodic coefficients*, Quart. Appl. Math. **10**, 266-270 (1952).
7. G. Duffing, *Erzewungene Schwingungen bei veränderlichen Eigenfrequenz*, F. Vieweg u. Sohn, Braunschweig 1918.
8. G. Floquet, *Sur les équations différentielles linéaires à coefficients périodiques*, Ann. Sci. École Norm. Sup. (2) **12**, 47-89 (1883).
9. W. Haacke, *Über die Stabilität eines Systems von gewöhnlichen linearen Differentialgleichungen zweiter Ordnung mit periodischen Koeffizienten, die von Parametern abhängen* (I und II), Math. Z.: **56**, 65-79 (1952); **57**, 34-35 (1952).
10. J. K. Hale, *Evaluations concerning products of exponential and periodic functions*, Rivista Mat. Univ. Parma **5**, 63-81 (1954).
11. E. W. Hobson, *The theory of functions of a real variable*, Vol. II, Harren Press, Washington 1950.
12. S. Lefschetz, *Lectures on differential equations*, Ann. Math. Studies, n. **14**, Princeton Univ. Press, 1948.
13. A. Liapounoff, *Problème générale de la stabilité du mouvement*, Ann. Fac. Sci. Univ. Toulouse (2) **9**, 203-475 (1907); Ann. Math. Studies, n. **17**, Princeton Univ. Press, 1949 (Reprint).

14. M. LINDSTEDT, *Beitrag zur Integration der Differentialgleichungen der Störungstheorie*, Mem. Acad. Sci. St. Pétersbourg (7) **31**, ...

15. N. W. MCLACHLAN, *Theory and application of Mathieu functions*, Clarendon Press, Oxford 1947.

16. E. MEISSNER, *Über Schüttelerscheinungen in Systemen mit periodisch veränderlicher Elastizität*, Schweiz. Bauzeitung **17**, ... (1918).

17. N. MINORSKY, *Introduction to non-linear mechanics*, J. W. Edwards, AnnArbor 1947.

18. É. PICARD, *Traité d'Analyse*, Tome II, Gauthier-Villars, Paris 1905.

19. J. H. POINCARÉ, *Méthodes nouvelles de la mécanique céleste*, Tome II (cf. Ch. 17), Gauthier-Villars, Paris 1893.

20. M. J. O. STRUTT, *Lamésche Mathieusche, und verwandte Functionen in Physik und Technik*, Ergebnisse für Mathematik, J. Springer, Berlin 1932.

21. C. TAAM, *Linear differential equations with small perturbations*, Duke Math. J. **20**, 13-25 (1953).