Rivista Mat. Univ. Parma 5, 169-181 (1954)

ROBERT A. GAMBILL (*)

Stability criteria for linear differential systems with periodic coefficients. (**)

Introduction. In this paper we shall discuss questions of boundedness and stability for differential systems of the form

$$\ddot{y}_j + \sigma_j^2 y_j + \lambda \sum_{h=1}^{n} \varphi_{jh}(t) y_h = 0 \qquad (j = 1, 2, \ldots, n). \tag{1}$$

These systems, which contain the MATHIEU equation [see, e.g., **9**, **10**, **13**, **14**] (1)

$$\ddot{y} + (\sigma^2 + \lambda \cos 2t) y = 0\,, \tag{2}$$

as a particular case, have been studied by L. CESARI [**2**], J. K. HALE [**5**], W. HAACKE [**4**]. As in [**2**], we will suppose that (A) $\sigma_1, \ldots, \sigma_n$ are distinct positive numbers, (B) λ is a real parameter, (C) $\varphi_{jh}(t)$ are real functions, periodic of period $T = 2\pi/\omega$, $\int_0^T \varphi_{jh}(t)\, dt = 0$, $\varphi_{jh}(t) = \sum_{k=-\infty}^{+\infty} c_{jhk} e^{ik\omega t}$, and $\sum_{k=-\infty}^{+\infty} | c_{jhk} | < C$ $(j, h = 1, \ldots, n)$, (D) $m\omega \neq \sigma_j \pm \sigma_h$ $(j, h = 1, \ldots, n;\ m = 1, 2, \ldots)$. Condition (D) says that there is no resonance between the small periodic restoring forces $\lambda \sum_{h=1}^{n} \varphi_{jh}(t) y_h$ and the harmonic oscillations of the differential equations $\ddot{y}_j + \sigma_j^2 y_j = 0$, as noted in [**2**]. Using a variant of the POINCARÉ method of casting out the secular terms in the solution of (1) by successive approximations, L. CESARI [**2**] has proved that if (A), (B), (C), (D) are satisfied and either

---

(*) Address: U. S. Naval Ordnance Plant, Indianapolis, Indiana, U.S.A..

(**) Received 2 April, 1954. The present research has been made possible by a grant from the National Science Foundation [NSF Research Project 0101 at Purdue University under the direction of Professor LAMBERTO CESARI].

(1) Numbers in brackets refer to the Bibliography at the end of the paper.

(α) $\varphi_{jh}(t)=\varphi_{jh}(-t)$, $(j, h=1, \ldots, n)$, or (β) $\varphi_{jh}(t)=\varphi_{hj}(t)$ $(j, h=1, \ldots, n)$, then all solutions of (1) are bounded in $(-\infty, +\infty)$ for λ sufficiently small in absolute value.

In the present paper (§ 3), we will give a sufficient condition for the boundedness of the solutions of system (1), for $|\lambda|$ sufficiently small, which extends both (α) and (β). We will also discuss (§ 3), systems in which small periodic damping is present. In § 4 we will show that the solution $[y_j=0 \quad (j=1, \ldots, n)]$, with $\lambda=0$, is parametrically stable whenever the solutions of (1) are bounded. In § 5, we shall discuss systems of equations (1) with σ_j^2 replaced by $\sigma_j^2(\lambda)$, and $\varphi_{jh}(t)$ replaced by $\varphi_{jh}(t, \lambda)$, and we shall see that the condition that the functions $\varphi_{jh}(t, \lambda)$ have mean value zero can be relaxed so as to include essentially the cases considered by L. Cesari and W. Haacke (see remark at the end of § 5).

The material given in the next two sections will be used also in two later papers.

§ 1. Preliminary remarks.

Consider the system of differential equations

$$\dot{y}_j=\varrho_j y_j+\lambda \sum_{1}^{N}{}_h\, \psi_{jh}(t) y_h \qquad (j=1,2, \ldots, N), \tag{1.1}$$

where $\varrho_1, \varrho_2, \ldots, \varrho_N$ are distinct complex numbers, λ is a complex parameter, and $\psi_{jh}(t)$ are periodic complex valued functions of period $T=2\pi/\omega$, and such that $\int_0^T \psi_{jh}(t)\, dt=0$, $\psi_{jh}(t)=\sum_{-\infty}^{+\infty}{}_k\, \gamma_{jhk} e^{ik\omega t}$, $\sum_{-\infty}^{+\infty}{}_k |\gamma_{jhk}|<C$, $(j, h=1, \ldots, N)$, and suppose that $\varrho_j \not\equiv \varrho_h \pmod{\omega i}$, $j \neq h$, $(j, h=1, \ldots, N)$. A number τ is said to be a characteristic exponent provided there exists a solution y_0 of system (1.1) such that $y_0(t+T)=e^{\tau T} y_0(t)$ [3, 6], and it is known that there are N characteristic exponents $\tau_1, \ldots, \tau_N$, not necessarily distinct, of (1.1), which coincide with $\varrho_1, \varrho_2, \ldots, \varrho_N$ respectively when $\lambda=0$. Since the characteristic exponents are continuous functions of λ, and since $\varrho_j \not\equiv \varrho_h \pmod{\omega i}$, $j \neq h$, $(j, h=1, \ldots, N)$, then for $|\lambda|$ sufficiently small, we have: $\tau_j \not\equiv \tau_h \pmod{\omega i}$, $j \neq h$, $(j, h=1, \ldots, N)$ [2, 5]. In this case, there is a fundamental system of solutions of (1.1) of the form (α') $[y_{jh}(t, \lambda)=e^{\tau_h t} P_{jh}(t, \lambda) \ (j=1, \ldots, N)]$, $h=1, \ldots, N$, where each $P_{jh}(t, \lambda)$ is a periodic function of t, of period T. Furthermore, (β') the functions $P_{jh}(t, \lambda)$ are uniformly bounded in $(-\infty, +\infty)$ for all $|\lambda|$ sufficiently small, and (γ') $\det|P_{jh}(0, \lambda)|>C>0$ for an absolute constant C and the same values of λ [2]. These results have been obtained by a variant of POINCARÉ's method of casting out the secular terms, studied by L. CESARI [2]. For lack of space, we cannot give an account here of the method, and we shall refer to [2], or to the recent paper of J. K. HALE [5], where a summary of the method is given. We shall add here some more needed information, referring again to [2], or [5]. The characteristic exponents τ_j can be obtained as usual by the FLOQUET equation as soon as a fundamental system of solutions of (1.1) is known. However, for $|\lambda|$ sufficiently small, and under the conditions stated above, L. CESARI [2], has shown that the characteristic exponents τ_j are given by the following system of equations

$$(1.2)\qquad \begin{cases} \tau_1 - \lambda\, d_1(\tau_1, \tau_2, \ldots, \tau_N, \lambda) = \varrho_1 \\ \tau_2 - \lambda\, d_2(\tau_1, \tau_2, \ldots \tau_N, \lambda) = \varrho_2 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \tau_N - \lambda\, d_N(\tau_1, \tau_2, \ldots, \tau_N, \lambda) = \varrho_N, \end{cases}$$

where an explicit expression for the functions d_j is given below. Each $d_j = d_j(\tau_1, \ldots, \tau_N, \lambda)$ is a holomorphic function of λ. The complex variables τ_j are supposed to belong to certain small neighborhoods C_j of ϱ_j $(j = 1, \ldots, N)$. For $\varrho_j \not\equiv \varrho_h \pmod{\omega i}$ and $|\lambda|$ sufficiently small, system (1.2) has a unique solution $\tau_1, \ldots, \tau_N$ with $\tau_j \in C_j$ and $\tau_j \not\equiv \tau_h \pmod{\omega i}$, $j \neq h$, $(j, h = 1, \ldots, N)$. This solution can be obtained by the standard method of successive approximations: $\tau_j = \lim_{n\to\infty} \tau_j^{(n)}$, $\tau_j^{(0)} = \varrho_j$, $\tau_j^{(n+1)} = \varrho_j + \lambda d_j(\tau_1^{(n)}, \tau_2^{(n)}, \ldots, \tau_N^{(n)}, \lambda)$ $(n = 0, 1, \ldots;$ $j = 1, \ldots, N)$ [2]. Then we have $\tau_j = \varrho_j + \lambda f_j(\varrho_1, \ldots, \varrho_N, \lambda)$ $(j = 1, \ldots, N)$, where the f_j are holomorphic functions of λ. The function d_j is given by the following formula: $d_j = \lim_{m\to\infty} d_j^{(m)}$, where

$$(1.3)\qquad d_j^{(m)} = \lambda \sum_{0}^{m-2}{}_p\, \lambda^p \sum_{1}^{N}{}_{t_1, \ldots, t_{p+1}} \sum_{0}{}_{k_1 + \ldots + k_{p+2}} \gamma_{jt_1k_1} \gamma^{(m-1)}_{t_1t_2k_2} \cdots \gamma^{(m-p)}_{t_pt_{p+1}k_{p+1}} \gamma_{t_{p+1}j\,k_{p+2}} \cdot$$
$$\cdot \{[-\tau_{t_1} + i(k_2 + \ldots + k_{p+2})\omega + \tau_j][-\tau_{t_2} + i(k_3 + \ldots + k_{p+2})\omega + \tau_j] \cdot$$
$$\cdot \ldots [-\tau_{t_{p+1}} + ik_{p+2}\omega + \tau_j]\}^{-1},$$

and $d_j^{(0)} = d_j^{(1)} = 0$.

The numbers γ_{ihk} are the FOURIER coefficients of the functions $\psi_{jh}(t)$ and $\gamma_{jhk}^{(m)} = -d_j^{(m)}$ if $h = j$, $k = 0$; $\gamma_{jhk}^{(m)} = \gamma_{jhk}$ if $k \neq 0$. Finally, in (1.3) the convention is made that those summands for which the denominator vanishes are excluded.

Since the periodic functions $\psi_{jh}(t)$ have mean value zero, the characteristic exponents τ_j are related to the characteristic roots ϱ_j by the formula $\tau_1 + \ldots + \tau_N = \varrho_1 + \ldots + \varrho_N$ [2].

An explicit expression for a fundamental system of solutions of (1.1) is given in [2] in the following form: $y_{jh}(t) = \lim_{m \to \infty} y_{jh}^{(m)}(t)$, where

$$(1.4) \qquad y_{jh}^{(m)} = \delta_{jh} e^{\tau_h t} + \sum_{1}^{m}{}_{p} \lambda^p \sum_{1}^{N}{}_{t_1, \dots, t_{p-1}} \sum_{-\infty}^{+\infty}{}_{k_1, \dots, k_p} \gamma_{j t_1 k_1}^{(m)} \gamma_{t_1 t_2 k_2}^{(m-1)} \dots \gamma_{t_{p-2} t_{p-1} k_{p-1}}^{(m-p+2)} \gamma_{t_{p-1} h k_p} \cdot$$

$$\cdot e^{[i(k_1 + \dots + k_p)\omega + \tau_h]t} \{[-\tau_j + i(k_1 + \dots + k_p)\omega + \tau_h][-\tau_{t_1} + i(k_2 + \dots + k_p)\omega + \tau_h] \cdot$$

$$\cdot \dots [-\tau_{p-1} + i k_p \omega + \tau_h]\}^{-1} .$$

The symbols $\gamma_{jhk}^{(m)}$ are defined above and the convention is made in (1.4) that those summands for which the denominators vanish are excluded.

§ 2. – Notations.

Let us write system (1) of the Introduction in the form

$$(2.1) \qquad \ddot{Y} + AY + \lambda \Phi Y = 0 ,$$

where $Y = \mathrm{col}\,(y_1, \dots, y_n)$, $A = \mathrm{diag}\,(\sigma_1^2, \dots, \sigma_n^2)$, $\Phi = \|\varphi_{jh}(t)\|$, and suppose that conditions (A), (B), (C), (D) are satisfied. Since in the following, we shall consider systems (2.1) with different matrices Φ, say Φ_1, Φ_2, ..., then often, in order to simplify the notation, we shall refer to them as systems (2.1) relative to the matrices Φ_1, Φ_2, ...

If we make the transformation

$$(2.2) \qquad y_j = -\frac{1}{2}(z_{2j-1} + z_{2j}) , \qquad \dot{y}_j = -\frac{i}{2}\sigma_j(z_{2j-1} - z_{2j}) , \qquad (j = 1, \dots, n) ,$$

then (2.1) becomes

$$(2.3) \qquad \begin{cases} \dot{z}_{2j-1} = i\sigma_j z_{2j-1} + \dfrac{\lambda i}{2\sigma_j} \sum_{1}^{n}{}_{h} \varphi_{jh}(t)[z_{2h-1} + z_{2h}] \\ \dot{z}_{2j} = -i\sigma_j z_{2j} - \dfrac{\lambda i}{2\sigma_j} \sum_{1}^{n}{}_{h} \varphi_{jh}(t)[z_{2h-1} + z_{2h}] \end{cases} \qquad (j = 1, \dots, n),$$

which is a system of the form (1.1) with $N = 2n$, where $\varrho_1 = i\sigma_1$, $\varrho_2 = -i\sigma_1$, ..., $\varrho_{2n-1} = i\sigma_n$, $\varrho_{2n} = -i\sigma_n$. The corresponding system of equations (1.2) for the determination of the characteristic exponents is then a system of $N = 2n$ equations in $N = 2n$ unknowns. Since we want to prove that for certain matrices Φ, the characteristic exponents of system (2.1) relative to Φ are purely imaginary and two by two complex conjugate, it is convenient to replace $\tau_1, \tau_2, \ldots, \tau_{2n}$ in (1.3) by new variables $i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n$, where the τ_j are real and belong to certain small neighborhoods e_j of σ_j $(j = 1, \ldots, n)$ [2]. To do this, we shall make in (2.3) and (1.3) the substitutions

$$j = 2j' - 2 + u, \quad h = 2h' - 2 + v, \quad (j', h' = 1, \ldots, n;\ u, v = 1, 2). \tag{2.4}$$

It follows from (1.1) with $N = 2n$ and (2.3) that

$$\gamma'_{jhk} = (-1)^{u-1} \frac{i}{2\sigma_{j'}} c_{j'h'k}. \tag{2.5}$$

If we put

$$d_{1,j'}^{(m)} = -2i\sigma_{j'} d_{2j'-1}^{(m)}, \qquad d_{2,j'}^{(m)} = 2i\sigma_{j'} d_{2j'}^{(m)}, \qquad (j' = 1, \ldots, n), \tag{2.6}$$

and

$$\begin{cases} c_{j'h'k}^{(u,v,m)} = -d_{1,j'}^{(m)}, & u = v = 1,\ h' = j',\ k = 0, \\ c_{j'h'k}^{(u,v,m)} = -d_{2,j'}^{(m)}, & u = v = 2,\ h' = j',\ k = 0, \\ c_{j'h'k}^{(u,v,m)} = c_{j'h'k}, & k \neq 0, \end{cases} \tag{2.7}$$

then

$$\gamma_{jhk}^{(m)} = (-1)^{u-1} \frac{i}{2\sigma_{j'}} c_{j'h'k}^{(u,v,m)}. \tag{2.8}$$

With the above notations, (1.3) with $N = 2n$ can be reduced to the convenient forms [cf. 2]

$$\begin{aligned} d_{1,j}^{(m)} = \frac{\lambda}{2} \sum_{0}^{m-2}{}_{p} \left(\frac{\lambda}{2}\right)^{p} \sum_{1}^{n}{}_{t_1,\ldots,t_{p+1}} \sum_{0}{}_{k_1+\ldots+k_{p+2}} \sum_{1}^{2}{}_{u_1,\ldots,u_{p+1}} (-1)^{u_1+\ldots+u_{p+1}-p-1} c_{jt_1k_1} c_{t_1t_2k_2}^{(u_1,u_2,m-1)} \ldots \\ \ldots c_{t_pt_{p+1}k_{p+1}}^{(u_p,u_{p+1},m-p)} c_{t_{p+1}jk_{p+2}} \{\sigma_{t_1} \ldots \sigma_{t_{p+1}} [(-1)^{u_1}\tau_{t_1} + (k_2 + \ldots + k_{p+2})\omega + \tau_j] \cdot \\ \cdot [(-1)^{u_2}\tau_{t_2} + (k_3 + \ldots + k_{p+2})\omega + \tau_j] \ldots [(-1)^{u_{p+1}}\tau_{t_{p+1}} + k_{p+2}\omega + \tau_j]\}^{-1} \\ (j = 1, 2, \ldots, n), \end{aligned} \tag{2.9}$$

$$\begin{aligned} d_{2,j}^{(m)} = \frac{\lambda}{2} \sum_{0}^{m-2}{}_{p} \left(\frac{\lambda}{2}\right)^{p} \sum_{1}^{n}{}_{t_1,\ldots,t_{p+1}} \sum_{0}{}_{k_1+\ldots+k_{p+2}} \sum_{1}^{2}{}_{u_1,\ldots,u_{p+1}} (-1)^{u_1+\ldots+u_{p+1}-p-1} c_{jt_1k_1} c_{t_1t_2k_2}^{(u_1,u_2,m-1)} \ldots \\ \ldots c_{t_pt_{p+1}k_{p+1}}^{(u_p,u_{p+1},m-p)} c_{t_{p+1}jk_{p+2}} \{\sigma_{t_1} \ldots \sigma_{t_{p+1}} [(-1)^{u_1}\tau_{t_1} + (k_2 + \ldots + k_{p+2})\omega - \tau_j] \cdot \\ \cdot [(-1)^{u_2}\tau_{t_2} + (k_3 + \ldots + k_{p+2})\omega - \tau_j] \ldots [(-1)^{u_{p+1}}\tau_{t_{p+1}} + k_{p+2}\omega - \tau_j]\}^{-1} \\ (j = 1, 2, \ldots, n), \end{aligned} \tag{2.10}$$

with the convention in (2.9), (2.10) that those summands for which the denominators vanish are excluded. The system (1.2) with $N = 2n$ then takes the form

$$(2.11)\quad \begin{cases} i\tau_1 = i\sigma_1 + \dfrac{\lambda i}{2\sigma_1} d_{1,1}(i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n, \lambda) \\ -i\tau_1 = -i\sigma_1 - \dfrac{\lambda i}{2\sigma_1} d_{2,1}(i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n, \lambda) \\ \cdots\cdots\cdots\cdots \\ i\tau_n = i\sigma_n + \dfrac{\lambda i}{2\sigma_n} d_{1,n}(i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n, \lambda) \\ -i\tau_n = -i\sigma_n - \dfrac{\lambda i}{2\sigma_n} d_{2,n}(i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n, \lambda), \end{cases}$$

where $d_{u,j} = \lim_{m\to\infty} d_{u,j}^{(m)}$ $(u = 1, 2;\ j = 1, \dots, n)$. It is proved by CESARI [2] that $d_{1,j} = \bar{d}_{2,j}$ $(j = 1, \dots, n)$, for every differential system (2.1) which satisfies conditions (A), (B), (C), (D). Thus, if the functions $d_{1,j}$ $(j = 1, \dots, n)$, are real for all real numbers $\tau_j \in C_j$ $(j = 1, \dots, n)$ and all real λ sufficiently small in absolute value, then $d_{1,j} = d_{2,j}$ $(j = 1, \dots, n)$ and the equations (2.11) are consistant and reduce to n equations in the n unknowns $\tau_1, \dots, \tau_n$. By putting $d_j^*(\tau_1, \dots, \tau_n, \lambda) = d_{1,j}(i\tau_1, -i\tau_1, \dots, i_n\tau_n, -i\tau_n, \lambda)$, we have thus for the determination of the characteristic exponents $i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n$, the system

$$(2.12)\qquad \tau_j = \sigma_j + \frac{\lambda}{2\sigma_j} d_j^*(\tau_1, \dots, \tau_n, \lambda) \qquad (j = 1, 2, \dots, n).$$

As shown by CESARI [2], whenever the functions $d_{1,j}$, with $\tau_j \in C_j$, and $|\lambda|$ sufficiently small, are real, then system (2.12) has a unique real solution $\tau_1, \dots, \tau_n$, with $\tau_j \in C_j$, and the numbers $i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n$ are the characteristic exponents of (2.1). Since $\tau_1, \dots, \tau_n$ are continuous functions of λ, it follows from D that $\tau_j \pm \tau_h \neq m\omega$ $(j, h = 1, \dots, n;\ m = 1, 2, \dots)$ and thus the $2n$ characteristic exponents are incongruent mod ωi. As a consequence, both systems (2.3) and (2.1) have a fundamental system of bounded solutions. We conclude that whenever the functions $d_{1,j}$ are real, then all solutions of (2.1) are bounded in $(-\infty, +\infty)$ for λ sufficiently small in absolute value. In particular, in the case (α) or (β) of the Introduction, the functions $d_{1,j}$ are real [2].

Remark. It has been proved recently by J. K. HALE [5] that the condition of absolute convergence of the FOURIER series of $\varphi_{jh}(t)$ can be replaced by the condition that each $\varphi_{jh}(t)$ is L-integrable in $(0, T)$.

§ 3. – Sufficient conditions for stability.

3.1. – Coupling of systems. Let $\{\Phi\}$ be the class of all $n \cdot n$ matrices $\Phi = \|\alpha_{ih}\|$ of the form $\Phi = \Phi_0 + \Psi$, where $\Phi_0 = \mathrm{diag}\,(\Phi_1, \Phi_2, \ldots, \Phi_k)$ is the direct sum of $n_j \cdot n_j$ matrices Φ_j with $n_1 + \ldots + n_k = n$, and the elements α_{ih} of the matrix Ψ which are on or above [on or below] Φ_0 are all zero. Thus, $\alpha_{ih} = 0$ if $n_1 + \ldots + n_{j-1} + 1 \leqq i \leqq n_1 + \ldots + n_j$, $n_1 + \ldots + n_j + 1 \leqq h$, $[n_1 + \ldots + n_{j-1} + 1 \leqq h \leqq n_1 + \ldots + n_j$, $n_1 + \ldots + n_j + 1 \leqq i]$, $(j = 1, \ldots k;\ n_0 = 0)$. The elements α_{ih} of the matrix Φ are assumed real. It is well known that $\{\Phi\}$ is a group.

Lemma I. *If the system*

$$\ddot{Y} + AY + \lambda\Phi Y = 0 \tag{3.1.1}$$

satisfies conditions (A), (B), (C), (D) *and if* $\Phi \in \{\Phi\}$, *then the functions* $d_{1,i}$ $(i = 1, 2, \ldots, n)$, *relative to* (3.1.1) *are identical to the functions* $d_{1,i}$, $(i = 1, 2, \ldots, n)$, *relative to the system*

$$\ddot{Y} + AY + \lambda\Phi_0 Y = 0 . \tag{3.1.2}$$

Proof. Consider the function $d_{1,i}^{(m)}$ relative to (3.1.1) and assume that $n_1 + \ldots + n_j + 1 \leqq i \leqq n_1 + \ldots + n_{j+1}$. In order that $c_{it_1k_1} \neq 0$, we must have $1 \leqq t_1 \leqq n_1 + \ldots + n_{j+1}$. Then in order that $c_{t_1t_2k_2}^{(u_1,u_2,m-1)} \neq 0$, $(u_1, u_2 = 1, 2)$, we must have $1 \leqq t_1 \leqq n_1 + \ldots + n_{j+1}$, etc.. Therefore, $1 \leqq t_1, t_2, \ldots, t_{p+1} \leqq n_1 + \ldots + n_{j+1}$. On the other hand, in order that $c_{t_{p+1},i,k_{p+2}} \neq 0$, we must have $n_1 + \ldots + n_j + 1 \leqq t_{p+1}$. Then in order that $c_{t_pt_{p+1}k_{p+1}}^{(u_p,u_{p+1},m\ \ p)} \neq 0$, $(u_p, u_{p+1} = 1, 2)$, we must have $n_1 + \ldots + n_j + 1 \leqq t_p$, etc.. It follows that $n_1 + \ldots + n_j + 1 \leqq t_1, t_2, \ldots, t_{p+1} \leqq n_1 + \ldots + n_{j+1}$.

Hence

$$d_{1,i}^{(m)} = \frac{\lambda}{2} \sum_0^{m-2}{}_p \left(\frac{\lambda}{2}\right)^p \sum_{n_1+\ldots+n_j+1}^{n_1+\ldots+n_{j+1}}{}_{t_1,\ldots,t_{p+1}} \sum_0 {}_{k_1+\ldots+k_{p+2}} \sum_1^2 {}_{u_1,\ldots,u_{p+1}} (-1)^{u_1+\ldots+u_{p+1}-p-1} c_{it_1k_1} c_{t_1t_2k_2}^{(u_1,u_2,m-1)} \ldots$$

$$\ldots c_{t_pt_{p+1}k_{p+1}}^{(u_p,u_{p+1},m-p)} c_{t_{p+1},i,k_{p+2}} \{\ldots\}^{-1} .$$

But this is the function $d_{1,i}^{(m)}$ of system (3.1.2) relative to the matrix Φ_j of Φ_0, and since the above discussion is independent of m, the proof is completed.

From Lemma I, it follows as a corollary,

Theorem I. *In the conditions of Lemma* I, *if each matrix* Φ_i $(i = 1, 2, \ldots, k)$ *of* Φ_0 *satisfies either of the conditions* (α) *the elements of* Φ_i *are even*, (β) Φ_i *is a* *symmetric matrix, then all solutions of* (3.1.1) *are bounded in* $(-\infty, +\infty)$ *for* λ *sufficiently small in absolute value.*

Proof. By Lemma I, the functions $d_{1,i}$ relative to (3.1.1) are the same as the functions $d_{1,i}$ relative to (3.1.2), $(i=1, \ldots, n)$. But $d_{1,i}$ relative to (3.1.2) is real, hence $d_{1,i}$ relative to (3.1.1) is real. Thereby Theorem I is proved.

3.2. – Systems with small damping. Consider the differential system

$$\ddot{Y} + AY + \lambda\Phi Y + \lambda\Phi^*\dot{Y} = 0\,, \tag{3.2.1}$$

where conditions (A), (B), (C), (D) of the Introduction are satisfied, and let Φ^* also satisfy (C), i.e. $\varphi^*_{jh}(t) = \sum_{k=-\infty}^{+\infty} c^*_{jhk}\, e^{ik\omega t}$, $\sum_{k=-\infty}^{+\infty} |c^*_{jhk}| < C$, φ^*_{jh} periodic of period $T = 2\pi/\omega$, $c^*_{jh0} = 0$ $(j, h = 1, \ldots, n)$.

Theorem II. *If* $\Phi(t) = \Phi(-t)$, *and* $\Phi^*(t) = -\Phi^*(-t)$, *then for* $|\lambda|$ *sufficiently small, all solutions of* (3.2.1) *are bounded in* $(-\infty, +\infty)$.

Proof. Using the transformation (2.2), the system (3.2.1) is reduced to the following system, of $2n$ first order differential equations,

$$\begin{cases} \dot{z}_{2j-1} = i\sigma_j z_{2j-1} + \dfrac{\lambda i}{2\sigma_j} \sum_{h=1}^{n} [(\varphi_{jh} + i\sigma_h\varphi^*_{jh}) z_{2h-1} + (\varphi_{jh} - i\sigma_h\varphi^*_{jh}) z_{2h}] \\ \dot{z}_{2j} = -i\sigma_j z_{2j} - \dfrac{\lambda i}{2\sigma_j} \sum_{h=1}^{n} [(\varphi_{jh} + i\sigma_h\varphi^*_{jh}) z_{2h-1} + (\varphi_{jh} - i\sigma_h\varphi^*_{jh}) z_{2h}] \end{cases} \tag{3.2.2}$$

$$(j = 1, \ldots, n)\,,$$

which is analogous to (2.3). This system is of the form (1.1) with $N = 2n$, and $\varrho_1 = i\sigma_1$, $\varrho_2 = -i\sigma_1, \ldots, \varrho_{2n-1} = i\sigma_n$, $\varrho_{2n} = -i\sigma_n$. Here too, as in § **2**, the corresponding system (1.2) for the determination of the characteristic exponents is a system of $N = 2n$ equations in $N = 2n$ unknowns. In order to prove that the characteristic exponents of system (3.2.1) are purely imaginary and two by two complex conjugate, let us first replace, as in § **2**, the $2n$ unknowns τ_1, $\tau_2, \ldots, \tau_{2n}$ in system (1.2) by $i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n$, where the new variables $\tau_1, \tau_2, \ldots, \tau_n$, are supposed to be real, and where τ_j belongs to a convenient small neighborhood c_j of σ_j, $(j = 1, \ldots, n)$. Using the substitutions (2.4), it follows from (1.1) with $N = 2n$, and (3.2.2), that

$$\gamma_{jhk} = (-1)^{u-1} \frac{i}{2\sigma_{j'}} \{c_{j'h'k} + (-1)^{v-1} i\sigma_{h'} c^*_{j'h'k}\}\,. \tag{3.2.3}$$

To simplify the notation, put

(3.2.4) $$\widehat{c}_{j'h'kv} = c_{j'h'k} + (-1)^{v-1}\, i\sigma_{h'}\, c^*_{j'h'k}\,.$$

If we put $\widetilde{d}^{(m)}_{1,j'} = -2i\sigma_{j'}\, d^{(m)}_{2j'-1}$, $\widetilde{d}^{(m)}_{2,j'} = 2i\sigma_{j'}\, d^{(m)}_{2j'}$ and $\widehat{c}^{(u,v,m)}_{j'h'kv} = -\widetilde{d}^{(m)}_{1,j'}$ if $u = v = 1$, $h = j$, $k = 0$; $\widehat{c}^{(u,v,m)}_{j'h'kv} = -\widetilde{d}^{(m)}_{2,j}$ if $u = v = 1$, $h = j$, $k = 0$; $\widehat{c}^{(u,v,m)}_{j'h'kv} = \widehat{c}_{j'h'kv}$ if $k \neq 0$, then

(3.2.5) $$\gamma^{(m)}_{jhk} = (-1)^{u-1}\frac{i}{2\sigma_{j'}}\,\widehat{c}^{(u,v,m)}_{i'j'kv}\,.$$

With the above notation, (1.3) with $N = 2n$ is reduced to the forms

(3.2.6) $$\widetilde{d}^{(m)}_{1,j} = \frac{\lambda}{2}\sum_0^{m-2}{}_p\left(\frac{\lambda}{2}\right)^p \sum_1^n{}_{t_1,\dots,t_{p+1}} \sum_0{}_{k_1+\dots+k_{p+2}} \sum_1^2{}_{u_1,\dots,u_{p+1}} (-1)^{u_1+\dots+u_{p+1}-p-1}\widehat{c}_{jt_1k_1u_1}\,\widehat{c}^{(u_1,u_2,m-1)}_{t_1t_2k_2u_2}\dots$$
$$\dots\widehat{c}^{(u_p,u_{p+1},m-p)}_{t_pt_{p+1}k_{p+1}u_{p+1}}\,\widehat{c}_{t_{p+1},j,k_{p+2},1}\{\sigma_{t_1}\dots\sigma_{t_{p+1}}[(-1)^{u_1}\tau_{t_1} + (k_2+\dots+k_{p+2})\omega + \tau_j]\cdot$$
$$\cdot[(-1)^{u_2}\tau_{t_2} + (k_3+\dots+k_{p+1})\omega + \tau_j]\dots[(-1)^{u_{p+1}}\tau_{p+1} + k_{p+2}\omega + \tau_j]\}^{-1},$$

(3.2.7) $$\widetilde{d}^{(m)}_{2,j} = \frac{\lambda}{2}\sum_0^{m-2}{}_p\left(\frac{\lambda}{2}\right)^p \sum_1^n{}_{t_1,\dots,t_{p+1}} \sum_0{}_{k_1+\dots+k_{p+2}} \sum_1^2{}_{u_1,\dots,u_{p+1}} (-1)^{u_1+\dots+u_{p+1}-p-1}\,\widehat{c}_{jt_1k_1u_1}\,\widehat{c}^{(u_1,u_2,m-1)}_{t_1t_2k_2u_2}\dots$$
$$\dots\widehat{c}^{(u_p,u_{p+1},m-p)}_{t_pt_{p+1}k_{p+1}u_{p+1}}\,\widehat{c}_{t_{p+1},j,k_{p+2},2}\,\{\sigma_{t_1}\dots\sigma_{t_{p+1}}[(-1)^{u_1}\tau_{t_1} + (k_2+\dots+k_{p+2})\omega - \tau_j]\cdot$$
$$\cdot[(-1)^{u_2}\tau_{t_2} + (k_3+\dots+k_{p+2})\omega - \tau_j]\dots[(-1)^{u_{p+1}}\tau_{t_{p+1}} + k_{p+2}\omega - \tau_j]\}^{-1},$$

with the convention in (3.2.6) and (3.2.7) that those summands for which the denominators vanish are excluded.

We will show now that $\widetilde{d}^{(m)}_{1,j} = \overline{\widetilde{d}}^{(m)}_{2,j}$ $(m = 0, 1, 2, \dots;\ j = 1, \dots, n)$. This is certainly true for $m = 0, 1$, for $\widetilde{d}^{(0)}_{1,j} = \widetilde{d}^{(0)}_{2,j} = \widetilde{d}^{(1)}_{1,j} = \widetilde{d}^{(1)}_{2,j} = 0$. Assume that $\widetilde{d}^{(n)}_{1,j} = \overline{\widetilde{d}}^{(n)}_{2,j}$ for $n = 0, 1, \dots, m-1$, and let us prove that

(3.2.8) $$\widehat{c}^{(u,v,n)}_{j,h,-k,v} = \overline{\widehat{c}}^{(3-u,3-v,n)}_{j,h,k,3-v}\,.$$

Indeed, if $u = v$, $h = j$, $k = 0$, then (3.2.8) is reduced to $\widetilde{d}^{(n)}_{u,j} = \overline{\widetilde{d}}^{(n)}_{3-u,j}$ $(u = 1, 2)$, which is assumed true: if $k \neq 0$, then $\widehat{c}_{j,h,-k,v} = c_{j,h,-k} + (-1)^{v-1}i\sigma_h c^*_{j,h,-k} = \overline{c}_{jhk} - (-1)^{3-v-1}i\sigma_h\overline{c}^*_{jhk} = \overline{\widehat{c}}_{jhk,3-v}$.

In 3.2.7), replace $k_1, k_2, \dots, k_{p+2}$ by $-k_1, -k_2, \dots, -k_{p+2}$ and use (3.2.8). In the so obtained formula, replace everywhere $3 - u_s$ by u_s $(s = 1, 2, \dots, p+1)$. The result is $\overline{\widetilde{d}}^{(m)}_{1,j}$. Hence $\widetilde{d}^{(m)}_{1,j} = \overline{\widetilde{d}}^{(m)}_{2,j}$ and the induction is completed. Since the elements of $\Phi(t)$ are even and the elements of $\Phi^*(t)$ are odd, it follows c_{jhk} is real and c^*_{jhk} is purely imaginary, $j, h = 1, \dots, n$; $k = -\infty, \dots, +\infty$. Hence by (3.2.4), $\widehat{c}_{jhkv}$ is real. Therefore, $\widetilde{d}^{(m)}_{1,j}$ is real for every m, for all real $\tau_j \in c_j$ $(j = 1, \dots, n)$, and for all real λ sufficiently small in absolute value. Hence $d_{1,j}$ is real, and thereby Theorem II is proved.

§ 4. – Stability of the solutions of (2.1).

Definition. Given is a differential system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, t, \lambda) \qquad (i = 1, \ldots, n) \tag{4.1}$$

containing a parameter λ, and a solution $X_{\lambda_0} = [x_{01}(t, \lambda_0), \ldots, x_{0n}(t, \lambda_0)]$ of (4.1) for $\lambda = \lambda_0$. We say that X_{λ_0} is parametrically stable provided (a) X_{λ_0} exists in $t_0 \leqq t < +\infty$ or $(-\infty < t \leqq t_0)$, (b) given $\varepsilon > 0$, there exists a $\delta > 0$, such that for all λ with $|\lambda - \lambda_0| < \delta$, the solution $X_\lambda = [x_1(t, \lambda), \ldots, x_n(t, \lambda)]$ with $|x_i(t_0, \lambda) - x_{0i}(t_0, \lambda_0)| < \delta$, $(i = 1, \ldots, n)$, exists for all $t \geqslant t_0$ (or $t \leqq t_0$), (c) $|x_i(t, \lambda) - x_{0i}(t, \lambda_0)| < \varepsilon$, $(i = 1, 2, \ldots, n)$, for all $t \geqslant t_0$ (or $t \leqslant t_0$). If the relations above are satisfied both for $t \geqslant t_0$ and $t \leqslant t_0$ then we say that X_{λ_0} is parametrically stable in both directions.

Let us observe that parametric stability implies stability in the sense of LIAPOUNOFF for the system with $\lambda = \lambda_0$ constant [7].

Concerning systems of the form (2.1) and (3.2.1), the following statement holds. Under the conditions of Theorem I or II, the solution $[y_i = 0, (i = 1, \ldots, n)]$ with $\lambda = 0$, of system (2.1) or (3.2.1) is parametrically stable in both directions. Since the systems in question are linear, conditions (a), (b) are obviously satisfied. Condition (c) is a consequence of the existence of the fundamental system of solutions (α'), for every $|\lambda|$ sufficiently small, satisfying (β') and (γ').

§ 5. - Some generalizations.

W. HAACKE [4], has discussed differential systems of the form

$$\ddot{y}_j + \sigma_j^2 y_j + \lambda \sum_{0}^{\infty}{}_r \lambda^r \sum_{1}^{n}{}_h \varphi_{jhr}(t)\, y_h = 0 \qquad (j = 1, \ldots, n), \tag{5.1}$$

under the conditions that the functions φ_{jhr} are even, continuous in $(-\infty, +\infty)$, periodic of period $2\pi/\omega$, and that the series $\sum_{0}^{\infty}{}_r \lambda^r \varphi_{jhr}$ converges absolutely for $|\lambda|$ sufficiently small, and x, ω real. HAACKE has proved that if $\sigma_1, \ldots, \sigma_n$ are positive numbers and if $m\omega \neq \sigma_j \pm \sigma_h$ $(j, h = 1, \ldots, n;\ m = 1, 2, \ldots)$, then for $|\lambda|$ sufficiently small, (λ real), all solutions of (5.1) are bounded in $(-\infty, +\infty)$. Cases where the functions φ_{jhr} are not necessarily even but satisfy conditions like (β) of the introduction are not considered by W. HAACKE. We shall consider below (II) systems somewhat more general than (5.1).

I. Consider the differential system

$$\dot{y}_j + \varrho_j(\lambda)\, y_j + \lambda \sum_{1}^{N} {}_h\, \psi_{jh}(t, \lambda)\, y_h = 0 \qquad (j = 1, 2, \ldots, N), \tag{5.2}$$

where each $\varrho_j(\lambda)$ is a continuous complex valued function of the complex parameter λ for $|\lambda|$ sufficiently small, and $\varrho_j(0) = \varrho_j^0$ with $\varrho_j^0 \not\equiv \varrho_h^0 \pmod{\omega i}$, $j \neq h$, $(j, h = 1, \ldots, N)$, each $\psi_{jh}(t, \lambda)$ is a periodic function of t of period $2\pi/\omega$, with $\psi_{jh}(t, \lambda) = \sum_{-\infty}^{+\infty}{}_k\, \gamma_{jhk}(\lambda) e^{ik\omega t}$, $\gamma_{jh0}(\lambda) = 0$, $\sum_{-\infty}^{+\infty}{}_k |\gamma_{jhk}(\lambda)| < C$ (C independent of λ), for all $|\lambda|$ sufficiently small. With the above conditions, the results which L. CESARI [2] has obtained for systems of the form (1.1) may be obtained for systems of the form (5.2) by replacing everywhere ϱ_j by $\varrho_j(\lambda)$, and γ_{jhk} by $\gamma_{jhk}(\lambda)$. In particular, consider the system

$$\ddot{y}_j + \sigma_j^2(\lambda)\, y_j + \lambda \sum_{1}^{n} {}_h\, \varphi_{jh}(t, \lambda)\, y_h = 0 \qquad (j = 1, \ldots, n), \tag{5.3}$$

where (A′) each $\sigma_j(\lambda)$ is a positive continuous function of the real parameter λ, for $|\lambda|$ sufficiently small, and $\sigma_j(0) = \sigma_j^0$ with $\sigma_j^0 \pm \sigma_h^0 \neq m\omega$, $(j, h = 1, \ldots, n;$ $m = 1, 2, \ldots)$, and (B′) each $\varphi_{jh}(t, \lambda)$ is a real valued functions of t, λ, periodic in t of period $T = 2\pi/\omega$, $\varphi_{jh}(t, \lambda) = \sum_{-\infty}^{+\infty}{}_h\, c_{jhk}(\lambda) e^{ik\omega t}$, $\int_0^T \varphi_{jh}(t, \lambda)\, dt = 0$, $\sum_{-\infty}^{+\infty}{}_k |c_{jhk}(\lambda)| < C$ (C independent of λ), for $|\lambda|$ sufficiently small. We can now state a boundedness theorem for systems (5.3) corresponding to Theorem I of § **3**.

Theorem III. *Consider the differential system* (5.3) *which satisfies conditions* (A′), (B′). *If* $\Phi(t, \lambda) = \|\varphi_{jh}(t, \lambda)\| \in \{\Phi\}$, *and if each block* Φ_j *of* Φ_0 *satisfies either* (α) $\Phi_j(t, \lambda)$ *is even in* t, *or* (β) $\Phi_j(t, \lambda)$ *is a symmetric matrix, then for* $|\lambda|$ *sufficiently small, all solutions of* (5.3) *are bounded in* $(-\infty, +\infty)$.

II. Consider the differential system

$$\ddot{y}_j + \sigma_j^2\, y_j + \lambda \sum_{1}^{n} {}_h\, \varphi_{jh}(t, \lambda)\, y_h = 0 \qquad (j = 1, \ldots, n), \tag{5.4}$$

where (A″) $\sigma_1, \ldots, \sigma_n$ are distinct positive numbers, (B″) λ is a real parameter, (C″) each $\varphi_{jh}(t, \lambda)$ is a real function of t, λ ,continuous in λ, periodic in t of period $2\pi/\omega$, $\varphi_{jh}(t, \lambda) = \sum_{-\infty}^{+\infty}{}_k\, c_{jhk}(\lambda) e^{ik\omega t}$, $\sum_{-\infty}^{+\infty}{}_k |c_{jhk}(\lambda)| < C$ (C independent of λ), for $|\lambda|$ sufficiently small, (D″) $\sigma_j \pm \sigma_h \neq m\omega$, $(j, h = 1, \ldots, n;\ m = 1, 2, \ldots)$. We have not assumed here that the functions φ_{jh} have mean value zero. Let us write system (5.4) in the form

$$\ddot{y}_j + \sigma_j^2 y_j + \lambda \sum_{h}^{n} {}_1 c_{jh0}(\lambda)\, y_h + \lambda \sum_{h}^{n} {}_1 \varphi_{jh}^{*}(t, \lambda)\, y_h = 0 \qquad (j = 1, \dots, n), \tag{5.5}$$

where $\varphi_{jh}^{*}(t, \lambda) = \varphi_{jh}(t, \lambda) - c_{jh0}(\lambda)$. Hence the functions $\varphi_{jh}^{*}(t, \lambda)$ have mean value zero. Consider the equation

$$A(\varrho, \lambda) = \det |(\sigma_j^2 - \varrho)\, \delta_{jh} + \lambda c_{jh0}(\lambda)| = 0 \qquad (h, j = 1, \dots, n), \quad \delta_{jh} = \begin{cases} 1, & j = h \\ 0, & j \neq h. \end{cases} \tag{5.6}$$

$A(\varrho, 0) = 0$ has the n distinct positive roots $\varrho_j = \sigma_j^2$, $(j = 1, \dots, n)$. Hence for $|\lambda|$ sufficiently small, $A(\varrho, \lambda) = 0$ has n distinct positive roots $\varrho_j = \sigma_j^{*^2}(\lambda)$ $(j = 1, \dots, n)$ [**8**]. Let A_0 represent the matrix whose determinant has the elements of $A(0, \lambda)$. Since the characteristic roots of A_0 are distinct, A_0 is similiar to a diagonal matrix $B = \operatorname{diag}(\sigma_1^{*^2}(\lambda), \dots, \sigma_n^{*^2}(\lambda))$, i.e. there exists a non-singular real matrix $P = P(\lambda)$, which can be constructed so that $\det|P(\lambda)| = 1 + 0(\lambda)$, such that $P^{-1}AP = B$. If we put $Y = (y_1, \dots, y_n)$, $Z = (z_1, \dots, z_n)$, $\Phi^* = \|\varphi_{jh}^{*}(t, \lambda)\|$, and $Y = PZ$, then (5.5) becomes

$$\ddot{Z} + BZ + \lambda P^{-1} \Phi^* P Z = 0\,. \tag{5.7}$$

System (5.7) satisfies conditions (A′), (B′) imposed on system (5.3). If in (5.4), $\varphi_{jh}(t, \lambda) = \varphi_{jh}(-t, \lambda)$, then $P^{-1}\Phi^*(t, \lambda)P = P^{-1}\Phi^*(-t, \lambda)P$. If in (5.4), $\varphi_{jh}(t, \lambda) = \varphi_{hj}(t, \lambda)$, then A_0 is symmetric, and it is orthogonally similiar to B. In this case, $P^{-1}\Phi^* P$ is also symmetric since Φ^* is symmetric [**11**]. We can now state the following theorem:

Theorem IV. *Consider the differential system* (5.4) *which satisfies conditions* (A″), (B″), (C″), (D″). *If either* (α) $\varphi_{jh}(t, \lambda) = \varphi_{jh}(-t, \lambda)$, $(j, h = 1, \dots, n)$, *or* (β) $\varphi_{jh}(t, \lambda) = \varphi_{hj}(t, \lambda)$, $(j, h = 1, \dots, n)$, *then for* $|\lambda|$ *sufficiently small, all solutions of* (5.4) *are bounded in* $(-\infty, +\infty)$.

Proof. System (5.7) satisfies the conditions of Theorem III in the special case when Φ_0 consists of one $n \cdot n$ block. Hence all solutions of (5.7) are bounded in $(-\infty, +\infty)$ for $|\lambda|$ sufficiently small. Since $Y = PZ$ satisfies (5.4), and each element of P is bounded for $|\lambda|$ sufficiently small, it follows that all solutions of (5.4) are bounded in $(-\infty, +\infty)$ for $|\lambda|$ sufficiently small.

Remark. The condition in II, that the functions $\varphi_{jh}(t, \lambda)$ have absolutely convergent Fourier series, is not essential. It is only necessary to assume that each $\varphi_{jh}(t, \lambda)$ is L-integrable in $[0, T]$, and that $|\varphi_{jh}(t, \lambda)| < M(t)$, for all $|\lambda|$ sufficiently small, $M(t)$ L-integrable in $[0, T]$. In this case, system (5.4) includes the system (5.1) considered by W. Haacke [**4**].

Bibliography.

1. R. BELLMAN, *A survey of the theory of the boundedness, stability, and asymptotic behavior of solutions of linear and non-linear differential and difference equations.* Office of Naval Research, 1949.

2. L. CESARI: *Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici*, Accad. Italia, Mem. Cl. Sci. Fis. Mat. Nat. **11**, 633-692 (1940).

3. G. FLOQUET, *Sur les équations différentielles linéaires à coefficients périodiques*, Ann. Sci. École Norm. Sup. (2) **12**, 47-49 (1883).

4. W. HAACKE, *Über die Stabilität eines Systems von gewöhnlichen linearen Differentialgleichungen zweiter Ordnung mit periodischen Koeffizienten, die von Parametern abhängen* (I und II), Math. Z.: **56**, 65-79 (1952); **57**, 34-45 (1952).

5. J. K. HALE, *On boundedness of the solutions of linear differential systems with periodic coefficients*, Rivista Mat. Univ. Parma **5**, 137-167 (1954).

6. S. LEFSCHETZ, *Lectures on differential equations*, Ann. Math. Studies, No. **14**, Princeton University Press, 1948.

7. A. LIAPOUNOFF, *Problème générale de la stabilité du mouvement*, Ann. Toulouse (2) **9**, 203-475 (1907); Ann. Math. Studies, No. **17**, Princeton University Press, 1948 (Reprint).

8. Morris MARDEN, *The geometry of the zeros of a polynomial in a complex variable*, Mathematical Surveys, No. **3**, Amer. Math. Soc., New York 1949.

9. N. W. MCLACHLAN, *Theory and application of Mathieu functions*, Clarendon Press, Oxford 1947.

10. N. MINORSKY, *Introduction to non-linear mechanics*, J. W. Edwards, Ann Arbor 1947.

11. Sam PERLIS, *Theory of matrices*, Addison-Wesley Press, Cambridge 1952 (cf. p. 186).

12. J. H. POINCARÉ, *Méthodes nouvelles de la mécanique céleste*, Vol. II, Gauthier-Villars, Paris 1883 (cf. Ch. 17).

13. J. J. STOKER, *Non-linear vibrations*, Interscience, New York 1950.

14. M. J. O. STRUTT, *Lamésche, Mathieusche, und verwandte Funktionen in Physik und Technik*, Ergebnisse für Mathematik, J. Springer, Berlin 1932.