Rivista Mat. Univ. Parma 5, 255-268 (1954)

L. C. YOUNG (*)

A variational Algorithm. (**)

1. – Introduction. In one approach to the Calculus of Variations, adopted in the case of curves by CARATHÉODORY [**1**], and HARDY-LITTLEWOOD-POLYA [**2**], the principle of minimum is exhibited as a consequence of an apparently much stronger statement, which leads to the classical equations and conditions, and which we term the *homology principle*: it asserts the existence of a non-negative integrand $f+\varphi$ whose integral vanishes on some admissible locus — the desired minimizing curve or surface — and to which the integrand f of the given problem is equivalent (in the terminology of CARATHÉODORY), or, as we prefer to say (with DE RHAM [**3**]), *homologous*.

We shall widen slightly the scope of the homology principle by allowing certain discontinuous integrands. The purpose of this Note is to show that the homology principle is then equivalent to the principle of minimum in its natural formulation. In fact it suffices to show that the principle of minimum implies the homology principle, and this constitutes the algorithm in our title. Essentially this algorithm will be found to reduce to the HAHN-BANACH theorem. We shall treat explicitely only the case of parametric surfaces; that of curves is much simpler and is in part treated in [**4**].

Let us observe that the homology principle implies the principle of minimum in its strongest form. To this effect we agree to denote by (L, f) the integral of f on the surface L and to mean by the statement $f+\varphi$ is homologous to f that (L, φ) depends only on the boundary of L. Suppose now, in accordance with the homology principle, that a non-negative $f+\varphi$ is homologous to f and that $(L_0, f+\varphi)=0$ for a particular surface L_0. Then for

(*) Address: Department of Mathematics, University of Wisconsin, Madison 6, Wisconsin, U.S.A..

(**) Received October 25, 1954. This paper was sponsored by the Office of Ordnance Research, U.S. Army, under contract DA-11-022-ORD-1511.

every L with the same boundary as L_0 we have $(L, \varphi) = (L_0, \varphi)$ and hence $(L, f) - (L_0, f) = (L, f + \varphi) - (L_0, f + \varphi) = (L, f + \varphi) \geqslant 0$, so that the minimum of (L, f) is attained for $L = L_0$ (in the class of L with the same boundary as L_0) without any restriction as to the topological type of L.

For the validity of our algorithm, the parametric surfaces must therefore *not be subject to any restriction as to topological type.* It is unfortunate that unnatural restrictions as to the topological type occur in nearly all the literature, even in PLATEAU's problem whose solution has never been freed from them; indeed according to [**5**] a solution is not possible within the classical framework when the restrictions are removed, and the existence of a minimum is thus known only in terms of *generalized surfaces*, to which we shall return in a moment.

Actually the principle of minimum needs strengthening still further, except for surfaces situated in 3-space when this turns out to be immaterial: if we modify the brief analysis given above by taking for L a surface with the boundary of L_0 *taken twice*, we find that $(L, f) \geqslant 2(L_0, f)$, and that for each $k = 1, 2, \ldots$ and for each L with the boundary of L_0 taken k times, (L_0, f) is the minimum of $k^{-1}(L, f)$. Thus the problem of minimum that really concerns us is a very natural one in which we admit surfaces (of all topological types) whose boundaries are multiples of the given one and we compensate for this by dividing the surface integrals by the appropriate integers. We shall term it *problem* A.

It is necessary to specify in it more precisely what is meant by boundary, so as to ensure that, when L has boundary k times that of L_0, the relation $(L, \varphi) = k(L_0, \varphi)$ holds for every φ homologous to 0; this relation does not hold in general if we adopt the traditional definition of boundary as one or more FRÉCHET curves, unless we exclude the case of « heavy curves »; however we can ensure its validity without exception, simply by defining, in an abstract way, the boundary of a surface L to be the restriction of the linear functional (L, φ) to functions φ homologous to 0. The notion of boundary, that we are thus led to adopt, is the g-boundary defined in [**6**]; it extends automatically to generalized surfaces, which have to be introduced to give substance to the principle of minimum, but when the boundary of a generalized surface is *prescribed*, we are entitled to suppose that it is not too wildly different from that of some ordinary parametric surface; the precise class of boundaries to which we shall in fact confine ourselves constitutes a space Λ that we define in § **2.**

Another definition which will give us some trouble is the precise meaning attached to the surface integral (L, φ) when L is a generalized surface and φ a discontinuous function homologous to 0; the definition selected is slightly artificial, because φ is not in general everywhere defined, and partly, perhaps, because the nature of φ is not yet fully understood: we do not know at present whether some extension of $\varphi = \varphi(x, J)$ is linear in J; this result is actually true in 3-space, but the state of affairs in higher space is unclear on account of peculiar difficulties due to the non-linear character of the bi-vectors J.

It is for problem A with these further stipulations concerning the boundary and the precise interpretation of surface integration that we propose to show that the principle of minimum implies the homology principle.

We shall assume familiarity with [**6**] and reference should also be made to the first part of [7] in regard to the principle of minimum for generalized surfaces. On the other hand we do not use here the very complete and extensive form of the theory, given in [**8**], of generalized surfaces of finite topological types: it no longer applies now that we make no restriction as to topological type, even though it includes as special cases or corollaries, most of the previous variational existence theorems for parametric surfaces. To a large extent the present Note is a sequel to [**6**].

2. – Notation. We take over the notations and definitions of [**6**], with the following alterations and additions.

(i) We term toroid of radius a the Cartesian product of the sets $|x| < a$ and $|J| = 1$. The functions $f(x, J)$ which occur will not always be continuous, nor even everywhere defined; however it will be understood that each f is homogeneous in J, i.e. that for $t \geqslant 0$ the relation $f(x, tJ) = tf(x, J)$ holds whenever f is defined at (x, J); it will be supposed further that f is bounded on each toroid for the set of (x, J) for which it is defined; finally it will be assumed that, whenever $x(u, v)$ is linear in (u, v) and has a constant Jacobian $J \neq 0$, the composite function $f[x(u, v), J]$ is measurable in (u, v) and coincides with the derivative of its Lebesgue integral in (u, v) wherever this derivative exists [in particular $f[x(u, v), J]$ is then defined for almost all (u, v)].

(ii) We say polyhedron over the reals instead of polyhedron over R; otherwise the term polyhedron has its usual meaning. We term σ-polyhedron, a countable sum of polyhedra over the reals, situated in a bounded portion of the space of x, and for which the sum of the areas converges. We say generalized surface, instead of generalized parametric surface, for a non-negative linear functional of the continuous f.

(iii) If L is a generalized surface, its restriction to f of the form $f(x, J) = P(x) \cdot J$ of a scalar product of $P(x)$ and J, where $P(x)$ is a sum of continuous bi-vectors, is termed track of L. For such f, the expression (L, f) is a functional of P of the form $\int P(x) \cdot J(x)\, d\mu$, where μ is a measure vanishing outside some sphere, and where $J(x)$ is a sum of bi-vectors subject to $|J(x)| = 1$, defined almost everywhere with respect to μ; the quantities μ and $J(x)$ are unique.

We write

$$[\text{track } L, f] = \int f[x, J(x)]\, d\mu$$

for any function f for which this integral exists. We term intersection of the BOREL set E with the track $J(x)$, μ, or subtrack of L determined by E, the track $J(x)$, μ^* derived from the track $J(x)$, μ of L by writing $\mu^*=\mu$ for subsets of E, and $\mu^*=0$ for subsets of the complement. By a Lipschitzian track, we shall mean a subtrack of a classical parametric surface L with a Lipschitzian representation $x(u, v)$ on the unit square.

(iv) If α is a measure (1) in the space of generalized surface L and α vanishes outside a given class of these L, we say that a generalized surface L' possesses in terms of the given class the resolution defined by α, if it has the form

$$\int L\, d\alpha .$$

In particular if the given class consists of basic closed L, we speak of a basic-closed resolution; if the given class consists of micro-surfaces (2), we speak of a micro-resolution; if the given class consists of generalized surfaces whose tracks are Lipschitzian, we speak of a *Lipschitzian* resolution. The resolution is termed countable if α vanishes outside a countable set of L, unitary if the total measure defined by α is unity.

(v) A g-boundary λ will be termed polyhedrally approximable if there is a bounded sequence $\{\lambda_n\}$ of g-boundaries of polyhedra over the reals, such that $\|\lambda_n - \lambda\| \to 0$; we recall that $\|\lambda\|$ is the infimum of the areas of generalized surfaces with the g-boundary λ. We denote by Λ the space formed by the polyhedrally approximable λ with the norm $\|\lambda\|$, when we remove from the class of convergent sequences those not situated in bounded sets of x. We observe that $\lambda \in \Lambda$ if and only if there exists a σ-polyhedron whose g-boundary λ. Evidently Λ is a space whose elements include all admissible g-boundaries in the terminology of [**6**].

3. – Functions homologous to 0 and elements of the conjugate space to Λ. The function f will be termed homologous to 0 if $[\text{track } L, f] = 0$ for every closed polyhedron L; it is not assumed that f is continuous, in which case the

---

(1) The measure α is understood to be such that w^* open sets are measurable.

(2) A micro-surface is a generalized surface situated in a set consisting of a single point.

condition states that f is exact. The function f_1 is termed homologous to the function f_2 if the difference $f_1 - f_2$ is homologous to 0.

(3.1) *Given any $f(x, J)$ homologous to 0, there is a linear functional $g(\lambda)$ on Λ such that $g(\lambda) = [\text{track } L, f]$ for every σ-polyhedron L with the g-boundary $\lambda \in \Lambda$. Conversely, given any linear functional $g(\lambda)$ on Λ, there is an $f(x, J)$ homologous to 0 such that $g(\lambda) = [\text{track } L, f]$ for every σ-polyhedron L with the g-boundary $\lambda \in \Lambda$.*

A corresponding, but slightly different, theorem for generalized surfaces (4.2) instead of σ-polyhedra, will be given in the next §. We shall also give a similarly modified version (4.1) of the following corollary:

(3.2) *If f is homologous to a continuous function and the σ-polyhedron L is translated through a vector ξ, then the quantity $[\text{track } L, f]$ varies continuously with ξ.*

Proof of (3.1). We remark in the first place that, given an f homologous to 0, the quantity $[\text{track } L, f]$ clearly takes the same value for any two polyhedra L with the same polygonal boundary λ, and therefore defines a function $g^*(\lambda)$ of a polygonal boundary λ; and that, conversely, given the linear functional $g(\lambda)$ on Λ, the equation

$$(*) \qquad g^*(\lambda) = [\text{track } L, f]$$

will likewise be satisfied for every polyhedron L with λ as its polygonal boundary, if we choose $g^*(\lambda) = g(\lambda)$ for polygonal boundaries λ and if we define f by making $f(x, J)$, in each plane parallel to J, coincide for almost all x of that plane with the derivative of the function of intervals in that plane, obtained from $g(\lambda)$ by selecting for λ the boundary of a variable oriented square of that plane. We note that the equation $(*)$ implies both that $g^*(\lambda)$ is additive and that f is homologous to 0.

The definition of $g^*(\lambda)$ and the validity of $(*)$ can now be extended to the case in which λ is the boundary of a polyhedron L over the reals. It is again sufficient to verify that the right hand side of $(*)$ depends only on λ, or what amounts to the same, that this right hand side vanishes when $\lambda = 0$, a fact which can be deduced at once from Lemma (3.3) of [**6**]. In its new range, the functional $g^*(\lambda)$ is again additive on account of $(*)$; it is also uniformly continuous in the metric defined by the norm $\|\lambda\|$ since f is bounded on each toroid.

The definition of $g^*(\lambda)$ and the validity of $(*)$ now extend by uniform continuity to the case in which $\lambda \in \Lambda$ and L is any σ-polyhedron with λ as its g-boundary. From the validity of $(*)$ it follows that $g^*(\lambda)$ is a linear functional and that the latter is uniquely determined by its restriction to polygonal boundaries λ. From its existence and from its unicity, respectively, it follows that we may set $g^*(\lambda) = g(\lambda)$ in the two parts of our theorem, and this completes the proof.

Proof of (3.2). We may clearly suppose that L is situated in the sphere $|x| \leqslant 1$ and that f satisfies the inequality $|f| \leqslant 1$ in the toroid of radius 2; moreover, we shall limit ourselves to translations by vectors ξ such that $|\xi| \leqslant 1$ and we shall write $[\text{track } L, f]^*$ for the value taken by $[\text{track } L, f]$ after L has been translated through ξ. We choose $\varepsilon > 0$ and we have to show that for small ξ,

$$|[\text{track } L, f]^* - [\text{track } L, f]| < \varepsilon .$$

If L is a polygon over the reals, the assertion is verified at once. In the general case, we write $L = L' + L''$, where L' is a polygon over the reals and where L'' is a σ-polygon of area $< \varepsilon/3$. When L is replaced by L' the difference to be estimated will, for small ξ, be $< \varepsilon/3$; when L is replaced by L'' the difference concerns two terms each of which is $< \varepsilon/3$ and is in consequence $< 2\varepsilon/3$. By combining the inequalities for L' and L'', we obtain the desired inequality for L and this completes the proof.

4. – Surface integration. Let L be a generalized surface whose g-boundary λ belongs to Λ, and let f be homologous to a continuous function. Let $\bar{f}(x, J)$ denote the mean value in of $f(x+h\xi, J)$, where $h > 0$, the mean value being taken over the unit cube of centre 0 of the vector ξ. Evidently $\bar{f}$ is continuous in x and linear in J, and hence continuous in (x, J). We define

$$(L, f) = \lim_{h \to 0} (L, \bar{f}) .$$

(4.1) (i) *The quantity (L, f) exists whenever L is a generalized surface whose g-boundary belongs to Λ and f is homologous to a continuous function.* (ii) *If L is translated through a vector ξ, (L, f) varies continuously with ξ.*

(4.2) *Given any f homologous to 0, there is a linear functional $g(\lambda)$ on Λ such that $g(\lambda) = (L, f)$ for every generalized surface L with the g-boundary $\lambda \in \Lambda$. Conversely, given any linear functional $g(\lambda)$ on Λ, there is an f homologous to 0, such that $g(\lambda) = (L, f)$ for every generalized surface L with the g-boundary $\lambda \in \Lambda$.*

(4.3) *If f is homologous to 0 and L' is a σ-polyhedron with the same g-boundary as the generalized surface L then*

$$(L, f) = [\text{track } L', f] .$$

Proofs. Since (4.1) evidently holds for a continuous integrand, we may suppose in it that f is homologous to 0. We denote by L' a σ-polyhedron with the same g-boundary as L, and we note that our hypotheses are now the same as in (4.3). We observe further that for each $h > 0$ the function $\bar{f}$ is exact, and hence that $(L, \bar{f}) = (L', \bar{f}) = [\text{track } L', \bar{f}] \to [\text{track } L', f]$ as $h \to 0$ by (3.2). This proves (4.1) (i) and (4.3); (4.1) (ii) and (4.2) now follow from (3.2) and (3.1).

5. – The Algorithm. We now prove:

(5.1) *Let f_0 be homologous to a continuous function and let $\lambda_0 \in \Lambda$. Suppose further that L_0 is a generalized surface with g-boundary λ_0 such that $(L_0, f_0) \leqslant (L, f_0)$ for all generalized surfaces L with the same g-boundary λ_0. Then there exists a non-negative f_1 homologous to f_0, such that $(L_0, f_1) = 0$.*

Proof. Let $q(\lambda)$ denote the infimum of (L, f_0) for all those L whose g-boundary is $\lambda \in \Lambda$. From our hypotheses it clearly follows that $q(0) = 0$. As in [**6**], p. 466, we find that $q(\lambda)$ is convex and homogeneous and that there exists a linear $g(\lambda) \leqslant q(\lambda)$ such that $g(\lambda_0) = q(\lambda_0) = (L_0, f_0)$. By (4.2) there exists an f homologous to 0 such that $g(\lambda) = (L, f)$ for every L with g-boundary $\lambda \in \Lambda$. Writing $f_1 = f_0 - f$ we deduce that f_1 is homologous to f_0, that $(L_0, f_1) = g(\lambda_0) - g(\lambda_0) = 0$, and that $(L, f_1) \geqslant q(\lambda) - g(\lambda) \geqslant 0$ for every L with g-boundary $\lambda \in \Lambda$. Choosing for L any square, we have by (4.3), $[\text{track } L, f_1] \geqslant 0$. Thus the double integral of $f_1(x, J)$ on every square in a plane parallel to J is non-negative, and so f_1 is non-negative. This completes the proof.

6. – An extension of (4.3). It is of some importance, particularly with regard to the derivation of local conditions for a minimum from our algorithm, to eliminate the reference to σ-polyhedra from the definition of Λ and from (4.3) as follows:

(6.1) *In order that $\lambda' \in \Lambda$ it is necessary and sufficient that λ' be the g-boundary of a generalized surface L' which possesses a Lipschitzian resolution $\int L\,d\alpha$. Moreover, if L' has such a resolution, and if f is homologous to 0, we then have*

$$(L', f) = \int [\text{track } L, f]\,d\alpha\,.$$

There are strong reasons for believing that the first part of (6.1) can be strengthened, and that in order that a generalized surface L' possess a g-boundary $\lambda' \in \Lambda$ it is *necessary* (as well as sufficient) that L' be the sum of a singular generalized surface and a Lipschitzian resolution $\int L\,d\alpha$. We shall doubtless return to this important question in the future.

We shall derive (6.1) from the Appendix and from the first of the following lemmas, which will be derived in its turn from the second.

(6.2) *Let* λ *be the g-boundary of a generalized surface* L *whose track is Lipschitzian, and let* f *be homologous to* 0. *Then* $\lambda \in \Lambda$ *and* $(L, f) = [\text{track } L, f]$.

(6.3) *Let* $x(u, v)$ *be a Lipschitzian parametric representation with Jacobian* $J(u, v)$, *let* Δ *denote a variable parallelogram in the* (u, v)*-square of definition, and let* λ_Δ *be the rectifiable g-boundary defined by restricting* $x(u, v)$ *to the oriented perimeter of* Δ. *Further let* f *be homologous to* 0 *and let* $g(\lambda)$ *be the associated linear functional on* Λ *in accordance with* (3.1). *Then* $g(\lambda_\Delta)$ *is absolutely continuous in* Δ *and has in* Δ *the derivative* $f[u(u, v), J(u, v)]$ (¹) *for almost all* (u, v) (²).

Reduction of (6.1) to (6.2). The condition in (6.1) is necessary, since every σ-polyhedron possesses a Lipschitzian resolution; its sufficiency is clearly implied by (6.2) together with (A, 3) of the Appendix.

Reduction of (6.2) to (6.3). We shall verify successively that (6.3) implies the conclusion of (6.2) in each of the following cases:

(i) L is a Lipschitzian parametric surface; its g-boundary is then rectifiable and so belongs to Λ; moreover by (4.2), (L, f) is the value of $g(\lambda_\Delta)$ when Δ is the (u, v)-square of definition, and (6.2) implies that this value coincides with the integral of $f[x(u, v), J(u, v)]$ and therefore with $[\text{track } L, f]$ by (B, 4) of the Appendix.

(ii) L has the same track as a Lipschitzian parametric surface L'; then its g-boundary λ, and therefore also the value of (L, f), are unaltered by replacing L by L'; this requires $\lambda \in \Lambda$ and $(L, f) = [\text{track } L', f] = [\text{track } L, f]$.

(iii) The track of L is the intersection of a set E with the track of a Lipschitzian parametric surface $x(u, v)$, where E is the image under $x(u, v)$ of a finite sum of (u, v) intervals; in this case our conclusions clearly follow by addition from those of the preceding case.

(iv) Here and in the remaining cases the hypotheses are as in case (iii) except those concerning E; we now suppose E to be an open set; in this case, the open set of (u, v) for which $x(u, v)$ lies in E can be approximated by a finite sum of (u, v) intervals, and our conclusions follow by passage to the limit from those of case (iii).

(¹) The derivative in this statement is taken for regular sequences of Δ.

(²) What is remarkable is that, according to (6.3), the function $f(x, J)$ originally defined for almost every x of each plane parallel to J, is now found to have been defined for almost every pair $x = x(u, v)$, $J = J(u, v)$ on each Lipschitzian surface.

(v) E is now a closed set; this case is dealt with by passing to the complement.

(vi) E is a countable sum of closed sets; this case is dealt with by passage to the limit from case (v).

(vii) E is a BOREL set; this is the general case and reduces at once to case (vi) by removing from E a subset whose-measure is 0.

Proof of (6.3). Since λ_Δ is the rectifiable g-boundary of a Lipschitzian parametric surface of area $\iint_\Delta |J(u, v)| \, du \, dv$, there exists a σ-polyhedron with at most double this area, whose g-boundary is λ_Δ; this implies that $|g(\lambda_\Delta)| \leqslant 2M \iint_\Delta |J(u, v)| \, du \, dv$, where M is the supremum of $|f(x, J)|$ in some fixed toroid, and therefore that $g(\lambda_\Delta)$ is absolutely continuous Δ and that its derivative is 0 at almost all the points (u, v) at which $J(u, v) = 0$.

To complete the proof, it suffices to show that, at every (u_0, v_0) for which $J(u_0, v_0) \neq 0$, for which further $x(u, v)$ is differentiable, and for which $g(\lambda_\Delta)$ has a derivative $D(u_0, v_0)$, the latter has the value $f[x(u_0, v_0), J(u_0, v_0)]$.

We write $x^*(u, v)$ for the linear function which agrees with $x(u, v)$ at (u_0, v_0) and has the same partial derivatives at that point, and we note that

$$(*) \qquad |x^*(u, v) - x(u, v)| < \varepsilon \varrho ,$$

where ϱ is the distance of (u, v) from (u_0, v_0) and $\varepsilon \to 0$ as $\varrho \to 0$. We write λ_Δ^* for the g-boundary of the parallelogram described by $x^*(u, v)$ on Δ. From $(*)$ it follows that $\lambda_\Delta^* - \lambda_\Delta$ is the g-boundary of a parametric surface of area $< 2K\varepsilon|\Delta|$, where K is a constant exceeding the relevant values of $|x_u|$ and $|x_v|$. By continuity $g[(\lambda_\Delta^* - \lambda_\Delta)/|\Delta|]$ therefore tends to 0 with the area $|\Delta|$; by linearity of g this implies that the derivative of $g(\lambda_\Delta^*) - g(\lambda_\Delta)$ is 0 and therefore that at (u_0, v_0) $g(\lambda_\Delta^*)$ has the derivative $D(u_0, v_0)$.

However, now that we know that the derivative of $g(\lambda_\Delta^*)$ exists, we can calculate it in a different way by choosing Δ so that λ_Δ^* is the perimeter of a square, and by observing that $\|\lambda_\Delta^*\| = |\Delta| \cdot |J(u_0, v_0)|$. We thus find that the value of this derivative is

$$|J(u_0, v_0)| \cdot \lim g(\lambda_\Delta^*)/\|\lambda_\Delta^*\| ,$$

which may be written simply $|J(u_0, v_0)| \cdot f(x_0, J_0)$, where x_0 is $x(u_0, v_0)$ and J_0 is the bi-vector of unit length parallel to $J(u_0, v_0)$. Hence $D(u_0, v_0) = |J(u_0, v_0)| \cdot f(x_0, J_0)$, which reduces to $f[x_0, J(u_0, v_0)]$ by homogenuity of f.

Appendix [1].

Part A. – Consider the space whose elements are differences $\mathcal{L}$ of generalized surfaces with the w^* topology, and define the g-boundary of $\mathcal{L}$ and the symbol $(\mathcal{L}, f)$ as differences of corresponding expressions. We write $\mathcal{K}$ and $\mathcal{K}_a$, respectively, for the sets of $\mathcal{L}$ whose g-boundaries satisfy $\lambda \in \Lambda$ and $\|\lambda\| < a$, where $a > 0$. We write further $\mathcal{G}$ for the closed set consisting of the $\mathcal{L}$ which are generalized surfaces.

(A, 1) *$\mathcal{K}_a$, $\mathcal{K}$ are Borel sets.*

Proof. Denote by $\{\mathcal{L}_n\}$ the sequence of differences of a countable dense subset of the set of closed generalized surfaces, by f_0 the function $f_0(x, J) = |J|$ and by $\{f_n\}$ a dense sequence of continuous f. Further let E_{pqr} denote, for positive integers p, q, r, the empty set of $\mathcal{L}$ unless $\mathcal{L}_{p+q}$ is a difference of generalized surfaces of areas $< r$ situated in the sphere $|x| < r$, in which case it is the open set of $\mathcal{L}$ for which

$$(\mathcal{L} + \mathcal{L}_{p+q}, f_0) < a + 1/p \quad \text{and} \quad (\mathcal{L} + \mathcal{L}_{p+q}, f_n) > -1/p \quad \text{for} \quad 1 \leqslant n < r\,.$$

We find that $\mathcal{K}_a = \Sigma_r \Pi_p \Sigma_q E_{pqr}$, which is a BOREL set.

Again, denote by L_n the sequence of the polyhedra over the reals whose vertices and coefficients are rational, and write E_{pq} for the BOREL set, obtained by the translation L_p of $\mathcal{K}_a$ for $a = 1/q$. We find that $\mathcal{K} = \Pi_q \Sigma_p E_{pq}$, which is a BOREL set.

(A, 2) *Given $\varepsilon > 0$, there esists an expression of the set $\mathcal{K}\mathcal{G}$ as a countable sum of disjoint Borel sets G_n such that $\|\lambda' - \lambda''\| < \varepsilon$ whenever λ', λ'' are g-boundaries of two elements of a same G_n.*

Proof. Let $\{\lambda_n\}$ be a dense sequence in Λ consisting of g-boundaries of polyhedra L_n over the reals, whose vertices and coefficients are rational. We choose $a = \varepsilon/2$ and write E_n for the translation by L_n of $\mathcal{K}_a$; the sets G_n defined by writing $G_1 = E_1 \mathcal{K}\mathcal{G}$, $G_n = E_n \mathcal{K}\mathcal{G} - G_{n-1}$ then satisfy our requirements.

(A, 3) *Let L' be a generalized surface possessing a resolution $\int L\,d\alpha$ in terms of generalized surfaces L whose g-boundaries belong to Λ. Then* (i), *the g-boundary λ' of L' belongs to Λ. Moreover* (ii), *if f is homologous to a continuous function, $(L', f) = \int (L, f)\,d\alpha$.*

---

[1] This Appendix deals with auxiliary material depending unavoidedly on routine considerations of measurability and the like, some of which are not short.

Proof. We may suppose (by completeness of Λ) that α is finite and hence (by multiplying by a suitable constant) that $\alpha \leqslant 1$. Given $\varepsilon > 0$, we define G_n as in (A, 2) and we select an element $\lambda_n \in G_n$; we observe, moreover, that if λ'' is the sum of a sufficiently large number of terms of the series $\sum \lambda_n \alpha(G_n)$, we have $\|\lambda'' - \lambda'\| < 2\varepsilon$. Clearly $\lambda'' \in \Lambda$; it follows by making $\varepsilon \to 0$ that $\lambda' \in \Lambda$ as asserted in (i).

Again, with the meaning attached to $\bar{f}$ in § 4, we have $(L', f) = \lim_h (L', \bar{f}) = \lim_h \int (L, \bar{f})\, d\alpha$. Further, since L has g-boundary in Λ, the relation $(L, \bar{f}) \to (L, f)$ holds by definition of (L, f); moreover, if $M = \operatorname{Sup} |f(x, J)|$ in a suitable toroid, we find that $|(L, \bar{f})| \leqslant M \cdot (L, f_0)$ where $f_0(x, J) = |J|$, so that $|(L, \bar{f})|$ remains below a quantity independent of h which is integrable in α. We may therefore take limits in h under the integral sign for a sequence of h, and this gives $(L', f) = \int \lim_h (L, \bar{f})\, d\alpha = \int (L, f)\, d\alpha$, which establishes (ii), and so completes the proof.

Part B. – The following propositions state, in effect, that a Lipschitzian surface has almost no non-tangential intersections with itself, with another Lipschitzian surface, and with a surface which possesses a Lipschitzian resolution.

(B, 1) *Let $x(u, v)$ be Lipschitzian, let $J(u, v)$ denote its Jacobian, and let E be the set of (u, v) such that there exists (u', v') distinct from (u, v) for which*

(i) $x(u', v') = x(u, v)$,

(ii) $\dfrac{J(u, v)}{|J(u, v)|} \neq \pm \dfrac{J(u', v')}{|J(u', v')|}$ (or else one side at least is undefined).

Then

$$\iint_E |J(u, v)|\, du\, dv = 0 .$$

(B, 2) *Let $x(u, v)$, $x^*(u, v)$ be Lipschitzian and let $J(u, v)$, $J^*(u, v)$ be their Jacobians. Let E be the set of (u, v) such that there exists (u^*, v^*) for which*

(i) $x^*(u^*, v^*) = x(u, v)$,

(ii) $\dfrac{J(u, v)}{|J(u, v)|} \neq \pm \dfrac{J^*(u^*, v^*)}{|J^*(u^*, v^*)|}$ (or else one side at least is undefined).

Then

$$\iint_E |J(u, v)|\, du\, dv = 0 .$$

(B, 3) *Let* $J(x)$, μ *be a Lipschitzian track and let* $J'(x)$, μ' *be the track of a generalized surface which possesses a Lipschitzian resolution. Then the set of* x *for which* $J(x) \neq \pm J'(x)$ *is the sum of a set of* μ*-measure* 0 *and a set of* μ'*-measure* 0.

The following is a Corollary of (B, 1):

(B, 4) *Let* L *be a parametric surface with a Lipschitzian representation* $x(u, v)$ *whose Jacobian is* $J(u, v)$. *Then*

$$[\text{track } L, f] = \iint f[x(u, v), J(u, v)]\, du\, dv$$

for every f *subject to* $f(x, -J) = -f(x, J)$ *for which the double integral on the right hand side exists.*

Proof of (B, 1). Since the area of the Lipschitzian surface defined by $x(u, v)$ coincides with its BANACH area [**9**], the set of points x at which the multiplicity is infinite, occupies two-dimensional measure 0. Hence there is a BOREL subset E_0 of E, on whose complement $\iint |J(u, v)|\, du\, dv = 0$, such that for each (u, v) of the set E_0 there are at most a finite number of (u', v') for which $x(u, v) = x(u', v')$ and each such (u', v') belongs to E_0. From the theory of Analytic sets [**10**], it follows that E_0 is the sum of a countable system of disjoint BOREL sets E_n on each of which $x(u, v)$ never takes a value twice. Without loss of generality, we may suppose that, in each E_n, $x(u, v)$ is differentiable and $J(u, v)$ does not vanish. This amounts to excluding countably many sets whose x-images are of two-dimensional measure 0, and over which therefore $\iint |J(u, v)|\, du\, dv = 0$. To establish (B, 1) it suffices to show that each E_n has plane measure 0; since the order of the sets E_n is immaterial, it suffices to show that E_1 has measure 0; further it clearly suffices to show that a BOREL subset E_{1n} of E_1 in which the values of $x(u, v)$ are assumed also at points (u', v') of E_n, has measure 0, and we need only prove this for $n = 2$.

Let A_1 be the set E_{12}, let A be the set of values assumed in it by $x(u, v)$, and let A_2 be the subset of E_2 at which $x(u, v)$ takes values in A. Then A and A_2 are likewise BOREL sets. It follows easily in two stages, that there exists for each $\varepsilon > 0$ a closed subset B of A, which is the x-image of a closed subset B_1 of A_1 and also that of a closed subset B_2 od A_2, such that $A - B$ has two-dimensional measure $< \varepsilon$. We denote by C_1 the set of points of density of B_1, by C_2 the set of points of density of B_2, and by C_1^*, C_2^* the sets of values of $x(u, v)$ for $(u, v) \in C_1$ and $(u, v) \in C_2$ respectively.

If $x \in C_1^*$, and (u, v) is the corresponding point of C_1, $J(u, v)$ is tangent to B at x. Consequently the corresponding point of B_2 cannot be a point of density. Thus C_1^* and C_2^* are disjoint. Hence B is the set of values of $x(u, v)$ in the sets $B_1 - C_1$ and $B_2 - C_2$ which are of measure 0, and is thus of two-dimensional measure 0. Consequently A has two-dimensional measure $< \varepsilon$, where ε is at our disposal, and therefore two-dimensional measure 0. This requires $\iint |J(u, v)| \, du \, dv$ to vanish in A_1, and therefore A_1 to have measure 0 since $J(u, v) \neq 0$ in A_1, which completes the proof.

Proof of (B, 2). Since those intersections of the two surfaces which are at the same time non-tangential multiple self-intersections for one of them, are dealt with in (B, 1), the proof is now an easy adaptation of the relevant parts of the same argument.

Proof of (B, 3). We write for brevity j, j' for $J(x)$, $J'(x)$. By hypothesis, the track of L is a subtrack of a parametric surface with a Lipschitzian representation $x(u, v)$ and this subtrack is determined by intersection with a Borel set Q in x-space. We may suppose that Q is contained in the set of values of $x(u, v)$. Since the complement of Q has μ-measure 0, we need only prove that the subset of Q in which we do not have $j = \pm j'$ has μ'-measure 0, and this will be the case if

$$\text{(I)} \qquad \int_Q [j^2 j'^2 - (jj')^2] \, d\mu' = 0 \, .$$

To establish this relation, let $L' = \int L^* \, d\alpha$ be our Lipschitzian resolution, and let j^*, μ^* define the track of L^*, where j^* is written for $J^*(x)$. From the resolution of L' we derive that

$$\text{(II)} \qquad \int P(x) J'(x) \, d\mu' = \int d\alpha \int P(x) J^*(x) \, d\mu^*$$

for continuous bi-vectors $P(x)$, and hence also for $P(x)$ bounded and measurable (B), and in particular for the sum of bi-vectors

$$P(x) = j^2 j' - (jj') j \quad \text{in} \quad Q \, ,$$

and $P(x) = 0$ outside Q. By (B, 2) we have $J^*(x) = \pm J(x)$ in Q except in a subset of μ^*-measure 0. Hence in Q, except in a subset of μ^*-measure 0, we have

$$P(x) J^*(x) = \pm P(x) J(x) = \pm [j^2 (j'j) - (jj') j^2] = 0$$

so that by (II) $\int P(x) J'(x) \, d\mu' = 0$, which is (I). This completes the proof.

Proof of (B, 4). From (B, 1) and from the theory of Analytic sets [**10**], it follows that there is a countable decomposition of the (u, v) interval into disjoint Borel sets $\{W_n\}$ and a set W_0 such that $\iint |J(u, v)| \, du \, dv$ vanishes on W_0, and that, in each W_n, $x(u, v)$ has a Jacobian $J(u, v) \neq 0$ and takes different values at different points, that further, in $\sum W_n$, $x(u, v)$ takes a same value at most a finite set of (u, v), and that finally the relations $(u, v) \in \sum W_n$, $(u', v') \in \sum W_n$ and $x(u, v) = x(u', v')$ together imply $J(u, v) =$

$= J(u', v')$. Writing Q_n for the x-image of W_n and $J_n(x)$ for the unit bi-vector parallel to $J(u, v)$ when $(u, v) \in W_n$ and $x = x(u, v)$, we have

$$\iint f[x(u, v), J(u, v)]\, du\, dv = \sum \iint_{Q_n} f[x, J_n(x)]\, d\nu ,$$

where ν denotes two-dimensional measure.

Now by (B, 1) the various $J(x)$ defined at a same point x reduce to two which we may write $\theta_n J(x)$, where $\theta_n = 1$ for N_1 values of n and $\theta_n = -1$ for N_2 values of n. We may suppose the notation such that $N_1 \geqslant N_2$, and this specifies $J(x)$ in $\sum Q_n$; we define further $\mu = \iint (N_1 - N_2)\, d\nu$ in $\sum Q_n$ and $\mu = 0$ elsewhere, so that $J(x)$ is defined except in μ-measure 0. Writing $Q_n(x)$ for the characteristic function of Q_n, we find that

$$\sum \iint_{Q_n} f[x, J_n(x)]\, d\mu = \sum_n \iint f[x, J(x)]\, \theta_n Q_n(x)\, d\nu =$$
$$= \iint f[x, J(x)] \{ \sum_n \theta_n Q_n(x) \} d\nu =$$
$$= \iint f[x, J(x)] \cdot (N_1 - N_2)\, d\nu = \int f[x, J(x)]\, d\mu$$

and therefore that

$$\iint f[x(u, v), J(u, v)]\, du\, dv = \int f[x, J(x)]\, d\mu .$$

This last formula shows incidentally, by restricting f to be linear in J and continuous, that $J(x)$, μ is the track of L, and therefore that the right hand side is [track L, f]. This completes the proof.

References.

1. C. Carathéodory, *Variationsrechnung*, Leipzig 1935. (Cf. § 227.)
2. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge 1934. (Cf. Chap. VII.)
3. G. de Rham and K. Kodaira, *Harmonic Integrals*, Institute for Advanced Study, Princeton, N.J., 1950.
4. W. Damköhler und E. Hopf, *Über einige Eigenschaften von Kurvenintegralen und über die Äquivalenz von indefiniten mit definiten Variationsproblemen*, Math. Ann. **120**, 12-20 (1947).
5. W. H. Fleming, *An example in the problem of least area*, submitted to the Proc. Math. Soc..
6. W. H. Fleming and L. C. Young, *A generalized notion of boundary*, Trans. Amer. Math. Soc. **76**, 457-484 (1954).
7. L. C. Young, *Surfaces paramétriques généralisées*, Bull. Soc. Math. France **79**, 59-84 (1951).
8. L. C. Young, *Generalized surfaces of finite topological type*, Mem. Amer. Math. Soc. (to appear).
9. L. C. Young, *On area and length*, Fund. Math. **35**, 275-302 (1948).
10. N. Lusin, *Leçons sur les ensembles analytiques*, Paris 1930. (Cf. Chap. IV.)