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Periodic solutions of non-linear systems of differential equations. (**)

Introduction. Consider a system of differential equations of the form

$$\dot{\vec{x}} = A\vec{x} + \varepsilon\vec{q}(\vec{x};\, \varepsilon) \qquad (\dot{} = d/dt), \tag{1}$$

where A is a constant $n \cdot n$ matrix, $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{q} = (q_1, \ldots, q_n)$ and ε is a small parameter. Differential systems of this type have been studied by LINDSTEDT [**10**] (1), LIAPOUNOFF [**9**], POINCARÉ [**14**], MACMILLAN [**11**], DUFFING [**5**], KRYLOFF and BOGOLIUBOFF [**7**], BULGAKOV [**2**], and, among many others, more recently by CODDINGTON and LEVINSON [**4**]. If ε is a small real parameter and the equation $\dot{\vec{x}} = A\vec{x}$ has a periodic solution of period 2π, the problem of interest is to determine conditions for the existence of periodic solutions of (1) with a period close to 2π. According to the terminology of POINCARÉ, these periodic solutions are called limit cycles.

In the present paper, general theorems are given (§ **3**) assuring the existence of periodic solutions of system (1) when the functions q_j are analytic. A method of successive approximations for the periodic solutions of (1) is defined and the convergence of the method is proved. This method is a variant of POINCARÉ's method of casting out the secular terms in the solutions of (1) and is similar to the method used by L. CESARI [**3**] for linear equations with periodic coefficients. See, also, J. K. HALE [**6**]. By a direct examination of the approximation series, general sufficient conditions (§ **2**)

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(1) Numbers in brackets refer to Bibliography at the end of the paper.

are obtained for the existence of periodic solutions of (1) from which then the general theorems of § **3** (3.1.i, 3.3.i, 3.4.i, 3.5) are deduced. These theorems seem to be new (for $n > 1$) and, with exception of 3.1.i, seem not to be directly deducible from known results.

§ 1. – A method of successive approximations.

In the following, we shall denote by C_τ the family of all functions which are finite sums of functions of the form $f(x) = e^{\alpha x}\varphi(x)$, $-\infty < x < +\infty$, where α is any complex number and $\varphi(x)$ is any complex-valued function of the real variable x, periodic of period $T = 2\pi/\tau$, L-integrable in $[0, T]$. If $\varphi(x)$ has the FOURIER series,

$$\varphi(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\tau x},$$

then we shall denote the series

$$f(x) = e^{\alpha x}\varphi(x) \approx \sum_{n=-\infty}^{+\infty} c_n e^{(in\tau+\alpha)x}$$

as the series associated with $f(x)$. Moreover, we shall denote by *mean value* $m[f]$ of $f(x)$ the number $m[f] = 0$ if $in\tau + \alpha \neq 0$ for all n, $m[f] = c_n$ if $in\tau + \alpha = 0$ for some n.

We shall also make use of the following theorem: *If* $f(x) = e^{\alpha x}\varphi(x) \in C_\tau$ *and* $m[f] = 0$, *then there is one and only one primitive of* $f(x)$, *say* $\int e^{\alpha t}\varphi(t)\,dt$, *which belongs to* C_τ *and also such that* $m[\int e^{\alpha t}\varphi(t)\,dt] = 0$. *Moreover*

$$\int e^{\alpha t}\varphi(t)\,dt = e^{\alpha x}\psi(x) = e^{\alpha x}\sum_{n=-\infty}^{+\infty} c_n(in\tau+\alpha)^{-1}e^{in\tau x}.$$

For a proof of this theorem, essentially known, see J. K. HALE [**6**].

1.1 – Description of the method. Let us consider the system

$$\dot{\vec{y}} = A\vec{y} + \varepsilon\vec{q}(\vec{y};\varepsilon), \tag{1.1.1}$$

where $\varepsilon > 0$ is a small real parameter, $A = \operatorname{diag}(i\sigma, -i\sigma, \sigma_3, \dots \sigma_n)$, $\vec{y} = (y_1, \dots, y_n)$, $\vec{q} = (q_1, \dots, q_n)$, and $\sigma > 0$, $\sigma_3, \dots, \sigma_n$ are complex numbers such that $im\sigma + \sigma_\mu \neq 0$ $(\mu = 3, \dots, n;\ m = 0, \pm 1, \pm 2, \dots)$. Moreover, we assume that

$$q_j(\vec{y};\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k q_{jk}(\vec{y}) = \sum_{k=0}^{\infty}\varepsilon^k \sum_{m_1,\dots,m_n \geq 0} q_{jk}^{(m_1,\dots,m_n)} y_1^{m_1}\dots y_n^{m_n} \quad (j = 1, 2, \dots, n) \tag{1.1.2}$$

for $\varepsilon < \varepsilon_0$, $|y_j| < K$ $(j = 1, 2, \dots, n)$.

Let $2\beta = \min|im\sigma + \sigma_\mu|$ $(m = 0, \pm 1, \dots; \mu = 3, \dots, n)$. We know that $\beta > 0$. For each μ let I_μ be the imaginary part of σ_μ, let $I = \max |I_\mu|$, and M be an integer such that $m\sigma > 2i + 2\beta$ for all $m \geqslant M$. If we choose an arbitrary real number $\tau > \sigma/2$ such that $|\tau - \sigma| < \beta/M$, then for all $|m| \geqslant M$ we have $|im\tau + \sigma_\mu| \geqslant |m\tau + I_\mu| \geqslant (1/2)\,|m|\,\sigma - |I_\mu| > I + \beta - I = \beta > 0$. For all $|m| < M$ we have also $|im\tau + \sigma_\mu| = |im(\tau - \sigma) + (im\sigma + \sigma_\mu)| \geqslant$ $\geqslant 2\beta - m(\beta/M) > \beta > 0$. Thus, for all m and μ, we have

$$|im\tau + \sigma_\mu| \geqslant \beta > 0\,. \tag{1.1.3}$$

Let us replace σ by τ in (1.1.1) and consider the auxiliary equations

$$\dot{\vec{y}} = B\vec{y} + \varepsilon\vec{q}(\vec{y};\, \varepsilon) \tag{1.1.4}$$

where $B = \operatorname{diag}(i\tau, -i\tau, \sigma_3, \dots, \sigma_n)$.

By a convenient modification of the method of successive approximations, we shall determine a solution for the equation

$$\dot{\vec{y}} = [B - \varepsilon F(\tau, \varepsilon, a_1, a_2)]\vec{y} + \varepsilon\vec{q}(\vec{y};\, \varepsilon)\,. \tag{1.1.5}$$

where a_1, a_2 are constants and $F(\tau, \varepsilon, a_1, a_2) = \operatorname{diag}(f_1, f_2, 0, \dots, 0)$ and $f_k = f_k(\tau, \varepsilon, a_1, a_2)$ $(k = 1, 2)$. If it is possible to choose τ, a_1, a_2 in such a way that $B - \varepsilon F = A$, the solution of (1.1.5) will become a solution of (1.1.1).

Before proceeding, we shall introduce some notation which is similar to that introduced by S. LEFSCHETZ [**8**]. We shall denote by s_{mjk} the coefficient of ε^{m-1} $(m = 1, 2, \dots)$ when $y_1, \dots, y_n$ in $q_{jk}(\vec{y})$ are replaced by

$$y_\mu(t) = x_{0\mu}(t) + \varepsilon x_{1\mu}(t) + \varepsilon^2 x_{2\mu}(t) + \dots \qquad (\mu = 1, 2, \dots, n), \tag{1.1.6}$$

where each $x_{\nu\mu}(t)$ is independent of ε. It is clear that each $s_{mjk} =$ $= s_{mjk}(\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{m-1})$ $(j = 1, 2, \dots, n;\ k = 0, 1, 2, \dots;\ m = 1, 2, \dots)$ is a power series in the components of $\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{m-1}$ with coefficients which consist of positive integers and the coefficients in the expansion of $q_{jk}(\vec{y})$.

Moreover, we shall denote by s_{mj} the coefficient of ε^{m-1} $(m=1, 2, \dots)$ when $y_1, \dots, y_n$ in $q_j(\vec{y})$ are replaced by (1.1.6). From (1.1.2), we then have

$$s_{mj} = s_{mj0} + s_{m-1,j,1} + \dots + s_{1,j,m-1}\,. \tag{1.1.7}$$

In the following, we shall consider only the case where each s_{mjk}, and, thus, each s_{mj} is a power series in $e^{i\tau t}$, $e^{-i\tau t}$. Then each s_{mjk}, s_{mj} is contained in the class C_τ, and for convenience, we put

$$a_1 S_{m1k} = m[e^{-i\tau t} s_{m1k}]\,, \qquad a_2 S_{m2k} = m[e^{i\tau t} s_{m2k}], \tag{1.1.8}$$

and

$$(1.1.9)\qquad \begin{cases} a_1 S_{m1} = m[e^{-i\tau t} s_{m1}], \qquad a_2 S_{m2} = m[e^{i\tau t} s_{m2}], \\ S_m = \operatorname{diag}(S_{m1}, S_{m2}, 0, \ldots, 0) \ (m = 1, 2, \ldots), \end{cases}$$

where a_1, a_2 are constants different from zero. Since in the sequel a_1, a_2 will be known complex constants, the formulas (1.1.8) uniquely define the numbers S_{m1k}, S_{m2k} and the formulas (1.1.9), the numbers S_{m1}, S_{m2}.

We, therefore, have the result that

$$(1.1.10)\qquad S_{mk} = S_{mk0} + S_{m-1,k,1} + \ldots + S_{1,k,m-1} \qquad (k = 1, 2;\ m = 1, 2, \ldots).$$

It is to be noted that if s_{m1k} contains a term in $a_1 e^{i\tau t}$, then S_{m1k} is merely the coefficient of this term. If s_{m1k} does not contain a term in $a_1 e^{i\tau t}$, then $m[e^{-i\tau t} s_{m1k}] = 0$ and we take $S_{m1k} = 0$. Similarly, for S_{m2k}.

With these notations, we define our method of successive approximations as follows:

$$(1.1.11)\qquad \begin{cases} \vec{y}_0 = \vec{x}_0 = (a_1 e^{i\tau t}, a_2 e^{-i\tau t}, 0, \ldots, 0), \\ \dot{\vec{y}}_m \equiv B\vec{y}_m + \sum_{k=1}^{m} \varepsilon^k \vec{s}_k - (\sum_{k=1}^{m} \varepsilon^k S_k)\vec{y}_{m-1} \pmod{\varepsilon^{m+1}}, \ (m=1, 2, \ldots), \end{cases}$$

where

$$(1.1.12)\qquad \vec{y}_m = \vec{x}_0 + \varepsilon \vec{x}_1 + \ldots + \varepsilon^m \vec{x}_m,$$

a_1, a_2 are constants different from zero, and in the m^{th} approximation, we keep only terms in $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^m$. Note that $|a_1|$, $|a_2|$ must be less than K so that we may substitute in the $q_j(\vec{y})$.

We make this choice of the zeroth approximation because if some of the other components are different from zero and of the form $e^{\sigma_j t}$, we introduce the small divisors in our successive approximations. Also, by defining the method of successive approximations in this manner, we are subtracting all those terms which would give rise to secular terms. We are subtracting terms other than the secular terms so that our solution will satisfy an equation of the form (1.1.5). These extra terms are also the reason for the congruence between polynomials in the definition.

If we replace $\vec{y}_m$ in (1.1.11) by its expression (1.1.12) and equate coefficients of powers of ε, remembering in the m^{th} approximation that only terms up through ε^m are considered, we have

$$(1.1.13)\qquad \begin{cases} \vec{x}_0 = (a_1 e^{i\tau t}, a_2 e^{-i\tau t}, 0, \ldots, 0), \\ \dot{\vec{x}}_m = B\vec{x}_m + \vec{s}_m - (S_1\vec{x}_{m-1} + S_2\vec{x}_{m-2} + \ldots + S_m\vec{x}_0) \qquad (m = 1, 2, \ldots) \end{cases}$$

and we may point out that these relations are equalities and not congruences as in (1.1.11).

If we let $e^{Bt} = \operatorname{diag}(e^{i\tau t}, e^{-i\tau t}, e^{\sigma_3 t}, \ldots, e^{\sigma_n t})$, $e^{-Bt} = \operatorname{diag}(e^{-i\tau t}, e^{i\tau t}, e^{-\sigma_3 t}, \ldots, e^{-\sigma_n t})$, then we have the following particular solution of (1.1.13):

$$(1.1.14)\qquad \begin{cases} \vec{x}_0 = (a_1 e^{i\tau t}, a_2 e^{-i\tau t}, 0, \ldots, 0), \\ \vec{x}_m = e^{Bt} \int e^{-Bt} \cdot [(\vec{s}_m - S_m \vec{x}_0) - (S_1 \vec{x}_{m-1} + \ldots + S_{m-1} \vec{x}_1)]\, dt, \quad (m = 1, 2, \ldots), \end{cases}$$

where the integrations are always performed so as to obtain the unique primitive of mean value zero. This is possible since we have already subtracted the secular terms, i.e., the integrand has mean value zero.

Since we are only performing the operations of additon, subtraction, and multiplication of terms of the form $a_1 e^{i\tau t}$, $a_2 e^{-i\tau t}$, and since we only integrate terms of the form

$$e^{\sigma t} \int e^{-\sigma t} e^{m i \tau t}\, dt = (m i \tau - \sigma)^{-1} e^{m i \tau t},$$

we necessarily have that s_{mj}, x_{mj} are power series in the zeroth approximation $x_{01} = a_1 e^{i\tau t}$, $x_{02} = a_2 e^{-i\tau t}$ for every $m = 1, 2, \ldots$ and $j = 1, 2, \ldots, n$. Therefore, we may write

$$(1.1.15)\qquad x_{mj} = \sum_{m_1, m_2 \geq 0} \sigma_{mj}^{(m_1, m_2)} x_{01}^{m_1} x_{02}^{m_2} = \sum_{m_1, m_2 \geq 0} \sigma_{mj}^{(m_1, m_2)} a_1^{m_1} a_2^{m_2} e^{(m_1 - m_2) i \tau t},$$

$$(1.1.16)\qquad s_{mj} = \sum_{m_1, m_2 \geq 0} \varrho_{mj}^{(m_1, m_2)} x_{01}^{m_1} x_{02}^{m_2} = \sum_{m_1, m_2 \geq 0} \varrho_{mj}^{(m_1, m_2)} a_1^{m_1} a_2^{m_2} e^{(m_1 - m_2) i \tau t}.$$

If we replace x_{kj}, s_{kj}, $(k = 0, 1, \ldots, m-1)$, in (1.1.14) by their corresponding expressions (1.1.15), (1.1.16) and integrate to obtain the unique primitive of mean value zero, remembering that $S_m \vec{x}_0$ is contained in $\vec{s}_m$, we obtain

$$(1.1.17)\qquad \begin{cases} x_{m1} = \sum_{k_1, k_2 \geq 0} [\varrho_{m1}^{(k_1, k_2)} - S_{11} \sigma_{m-1,1}^{(k_1, k_2)} - \ldots - S_{m-1,1} \sigma_{11}^{(k_1, k_2)}] \cdot [(k_1 - k_2 - 1) i \tau]^{-1} x_{01}^{k_1} x_{02}^{k_2}, \\ x_{m2} = \sum_{k_1, k_2 \geq 0} [\varrho_{m2}^{(k_1, k_2)} - S_{12} \sigma_{m-1,2}^{(k_1, k_2)} - \ldots - S_{m-1,2} \sigma_{12}^{(k_1, k_2)}] \cdot [(k_1 - k_2 + 1) i \tau]^{-1} x_{01}^{k_1} x_{02}^{k_2}, \\ x_{m\mu} = \sum_{k_1, k_2 \geq 0} [\varrho_{m\mu}^{(k_1, k_2)}] \cdot [(k_1 - k_2) i \tau - \sigma_\mu]^{-1} x_{01}^{k_1} x_{02}^{k_2} \qquad (\mu = 3, 4, \ldots, n). \end{cases}$$

1.2. – Majorants for the x_{mj}. In order to majorize the functions x_{mj}, we shall make use of the function

$$(1.2.1)\qquad \varphi(x) = \left(1 - \frac{x}{K}\right)^{-n} = 1 + \binom{n}{1} \frac{x}{K} + \binom{n+1}{2} \left(\frac{x}{K}\right)^2 + \ldots, \quad |x| < K.$$

Let x be replaced by $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ (x_j independent of ε) and order the result according to ascending powers of ε to obtain

$$\varphi(x) = \sum_{m=1}^{\infty} \varepsilon^{m-1} \psi_m(x_0, x_1, \dots, x_{m-1}), \tag{1.2.2}$$

where $\psi_m(x_0, x_1, \dots, x_{m-1})$ is a power series in $x_0, x_1, \dots, x_{m-1}$ whose coefficients are positive numbers. We may make the above substitutions for x provided that x_0, ε are small enough in order that $|x| < K$.

By assumption,

$$q_{jk}(\vec{y}) = \sum_{m_1, \dots, m_n \geq 0} q_{jk}^{(m_1, \dots, m_n)} y_1^{m_1} \dots y_n^{m_n}, \quad |y_j| < K, \quad (j = 1, 2, \dots, n; \ k = 0, 1, \dots),$$

where $q_{jk}^{(m_1, \dots, m_n)}$ are constants. Moreover, there exists a constant M such that

$$|q_{jk}^{(m_1, \dots, m_n)}| < M K^{(-m_1 - \dots - m_n)}$$

for all j, k. Thus, we have

$$|q_{jk}(\vec{y})| \leq \sum_{m_1, \dots, m_n \geq 0} |q_{jk}^{(m_1, \dots, m_n)}| \cdot |y_1|^{m_1} \dots |y_n|^{m_n} < M \left[\prod_{l=1}^{n} \left(1 - \frac{|y_l|}{K} \right) \right]^{-1},$$

for $|y_j| < K'$, $0 < K' < K$. Let

$$Q(y_1, \dots, y_n) = M \left[\prod_{l=1}^{n} \left(1 - \frac{y_l}{K} \right) \right]^{-1}, \tag{1.2.3}$$

and we then have

$$q_{jk}(\vec{y}) \ll Q(\vec{y}) \qquad (j = 1, 2, \dots, n; \ k = 0, 1, 2, \dots), \tag{1.2.4}$$

where we understand by $\ll$ that the coefficient of $y_1^{m_1} \cdot y_2^{m_2} \dots y_n^{m_n}$ in $q_{jk}(\vec{y})$ is in absolute value less than the corresponding coefficient of $y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}$ in $Q(\vec{y})$. Let, also, the effect on any number or function, of replacing each coefficient in q_j by the corresponding larger coefficient in Q be denoted by $\{\ \}$.

From (1.1.15), we have $x_{mj} = \sum_{m_1, m_2 \geq 0} \sigma_{mj}^{(m_1, m_2)} x_{01}^{m_1} x_{02}^{m_2}$ ($j = 1, 2, \dots, n$) and we shall denote by ξ_m a common upper bound for $\sum_{m_1, m_2 \geq 0} \{\sigma_{mj}^{(m_1, m_2)}\} |x_{01}|^{m_1} |x_{02}|^{m_2}$ ($j = 1, 2, \dots, n$), where, also, in the $\{\ \}$ everything is replaced by absolute value. Moreover, let $\vec{\xi}_m = (\xi_m, \dots, \xi_m)$. As a consequence, since the coefficients of the coordinates of $\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{m-1}$ in $\vec{s}_{mk}$ consist of positive integers and the coefficients in the q_{jk}, we have

$$|s_{mjk}(\vec{\xi}_0, \vec{\xi}_1, \dots, \vec{\xi}_{m-1})| < \{s_{mjk}(\vec{\xi}_0, \vec{\xi}_1, \dots, \vec{\xi}_{m-1})\}$$

$$(j = 1, 2, \dots, n; \ m = 1, 2, \dots; \ k = 0, 1, \dots).$$

Moreover, since $Q(y, ..., y) = M\varphi(y)$, we have, using (1.2.4) that

$$\{s_{mjk}(\vec{\xi}_0, \vec{\xi}_1, ..., \vec{\xi}_{m-1})\} = M\psi_m(\xi_0, \xi_1, ..., \xi_{m-1}) \tag{1.2.5}$$

and, thus,

$$|s_{mjk}(\vec{\xi}_0, \vec{\xi}_1, ..., \vec{\xi}_{m-1})| < M\,\psi_m(\xi_0, \xi_1, ..., \xi_{m-1}) \tag{1.2.6}$$

$$(j = 1, 2, ..., n;\ k = 0, 1, ...;\ m = 1, 2, ...).$$

Furthermore, from the definition (1.1.8), $a_1 e^{i\tau t} S_{m1k} = x_{01} S_{m1k}$, $a_2 e^{-i\tau t} S_{m2k} = x_{02} S_{m2k}$ are merely terms in s_{m1k}, s_{m2k}, respectively, and, as a consequence, we must also have $\xi_0\{S_{m1k}\}$, $\xi_0\{S_{m2k}\}$ are terms in $\{s_{m1k}(\vec{\xi}_0, ..., \vec{\xi}_{m-1})\}$, $\{s_{m2k}(\vec{\xi}_0, ..., \vec{\xi}_{m-1})\}$, respectively. Thus, $\xi_0\{S_{m1k}\} \leqslant \{s_{m1k}\}$, $\xi_0\{S_{m2k}\} \leqslant \{s_{m2k}\}$ and, moreover,

$$\{S_{m1k}\},\ \{S_{m2k}\} < (M/\xi_0)\,\psi_m(\xi_0, \xi_1, ..., \xi_{m-1}) \quad (k = 0, 1, 2, ...;\ m = 1, 2, ...). \tag{1.2.7}$$

Furthermore, from (1.1.7), (1.1.10), we see that

$$\left\{\begin{array}{ll} |s_{mj}| < M(\psi_m + \psi_{m-1} + ... + \psi_1) & (j = 1, 2, ..., n), \\ \{S_{m1}\},\ \{S_{m2}\} < (M/\xi_0)(\psi_m + \psi_{m-1} + ... + \psi_1) & (m = 1, 2, ...). \end{array}\right. \tag{1.2.8}$$

Using (1.1.3), (1.1.17) and the results above, we have

$$\begin{aligned} |x_{m1}| &\leqslant \beta^{-1}\Big[\sum_{k_1,k_2\geq 0} \{\varrho_{m1}^{(k_1,k_2)}\}\,|x_{01}|^{k_1}|x_{02}|^{k_2} + \{S_{11}\} \sum_{k_1,k_2\geq 0} \{\sigma_{m-1,1}^{(k_1,k_2)}\}\,|x_{01}|^{k_1}|x_{02}|^{k_2} + ... + \\ &\qquad + \{S_{m-1,1}\} \sum_{k_1,k_2\geq 0} \{\sigma_{11}^{(k_1,k_2)}\}\,|x_{01}|^{k_1}|x_{02}|^{k_2}\Big] \leqslant \\ &\leqslant \beta^{-1}[\{s_{m1}\} + \{S_{11}\}\xi_{m-1} + ... + \{S_{m-1,1}\}\xi_1] < \\ &< M\beta^{-1}(\psi_m + \psi_{m-1} + ... + \psi_1) + M(\beta\xi_0)^{-1}[\psi_1\xi_{m-1} + (\psi_2 + \psi_1)\xi_{m-2} + ... + \\ &\qquad + (\psi_{m-1} + ... + \psi_1)\xi_1]. \end{aligned}$$

Similarly, one finds that $|x_{m2}|$ is less than this same expression and

$$|x_{m\mu}| < M\beta^{-1}(\psi_m + \psi_{m-1} + ... + \psi_1) \qquad (\mu = 3, 4, ..., n)$$

Therefore, we may take

$$(1.2.9)\quad \begin{cases} \xi_0 = \xi_0, \qquad \xi_1 = M\beta^{-1}\psi_1 , \\ \xi_m = M\beta^{-1}(\psi_m + \dots + \psi_1) + M(\beta\xi_0)^{-1}[\psi_1\xi_{m-1} + (\psi_2 + \psi_1)\xi_{m-2} + \dots + \\ \qquad + (\psi_{m-1} + \dots + \psi_1)\xi_1] \qquad (m = 2, 3, \dots) \end{cases}$$

and it should be noted that these evaluations are independent of ε.

Finally, in order to show that the series

$$(1.2.10)\qquad \vec{y} = \vec{x}_0 + \varepsilon\vec{x}_1 + \varepsilon^2\vec{x}_2 + \dots$$

converges for ε sufficiently small, it remains only to show that the series

$$(1.2.11)\qquad \xi = \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \dots$$

converges for ε sufficiently small.

1.3. – Proof of the convergence. Consider the equation

$$(1.3.1)\quad F(\xi, \varepsilon) = \varepsilon M[\beta(1-\varepsilon)]^{-1}\varphi(\xi) + \varepsilon M[\beta(1-\varepsilon)\xi_0]^{-1}\varphi(\xi)\cdot(\xi-\xi_0) - (\xi-\xi_0) = 0,$$

where $\varphi(\xi)$ is defined by (1.2.1) and $\varepsilon > 0$ is at least smaller than one. If we can show that for ε sufficiently small, this equation has a unique solution $\xi(\varepsilon) = \sum_{k=0}^{\infty} b_k\varepsilon^k$ and the coefficients b_k satisfy the recurrence relation (1.2.9), then we will have shown that the method of successive approximations defined by (1.1.14) converges. This is the same type of reasoning used by LIAPOUNOFF [**9**].

From (1.3.1) we have $F(\xi_0, 0) = 0$, and, moreover, $\left.\dfrac{\partial F}{\partial \xi}\right|_{\xi_0,0} = -1 \neq 0$. Thus, from the theorem on implicit functions for complex variables [**13**], for ε sufficiently small, there exists a unique solution of (1.3.1) of the form

$$(1.3.2)\qquad \xi(\varepsilon) = \sum_{k=0}^{\infty} b_k\varepsilon^k ,$$

and such that $\xi(0) = \xi_0$, i.e. $b_0 = \xi_0$.

It remains only to show that the coefficients b_k in (1.3.2) satisfy the recurrence relation (1.2.9). From (1.2.2), we see that

$$(1.3.3)\qquad \varphi(\xi) = \varphi\Big(\sum_{k=0}^{\infty} b_k\varepsilon^k\Big) = \sum_{k=1}^{\infty} \varepsilon^{k-1}\psi_k(b_0, b_1, \dots, b_{k-1}) ,$$

provided that $b_0 = \xi_0$ and ε are small enough so that $|\xi| < K$. This may certainly be satisfied since ξ_0 may be taken to be $\max(|a_1 e^{i\tau t}|, |a_2 e^{-i\tau t}|) = \max(|a_1|, |a_2|)$. We also see from (1.3.3) that each $\psi_k(b_0, b_1, ..., b_{k-1})$ is a convergent series in $b_0, b_1, ..., b_{k-1}$. Moreover, since $\varepsilon < 1$, we may write

$$(1.3.4) \qquad (1-\varepsilon)^{-1} = 1 + \varepsilon + \varepsilon^2 + \dots .$$

If we replace ξ in (1.3.1) by its expression (1.3.2), make use of (1.3.3), (1.3.4), and equate the coefficients of powers of ε^k $(k = 0, 1, 2, ...)$ to zero, we get

$$b_0 = \xi_0 ,$$

$$M\beta^{-1}\psi_1(b_0) - b_1 = 0 ,$$

$$M\beta^{-1}(\psi_m + \psi_{m-1} + \dots + \psi_1) + M(\beta b_0)^{-1}[(\psi_1 b_{m-1} + \dots + \psi_{m-1} b_1) + (\psi_1 b_{m-2} + \dots + \psi_{m-2} b_1) + \dots + (\psi_1 b_1)] - b_m = 0 \qquad (m = 2, 3, ...)$$

or

$$M\beta^{-1}(\psi_m + \psi_{m-1} + \dots + \psi_1) + M(\beta b_0)^{-1}[\psi_1 b_{m-1} + (\psi_2 + \psi_1) b_{m-2} + \dots + (\psi_{m-1} + \psi_{m-2} + \dots + \psi_1) b_1] - b_m = 0 \qquad (m = 2, 3, ...)$$

which is precisely the recurrence relations((1.2.9).

Therefore, the series (1.2.11), with ξ_0 finite, converges for ε sufficiently small. As a consequence, for ε sufficiently small, the series (1.2.10) converges absolutely and uniformly for all t, $-\infty < t < +\infty$, since we saw before that ξ_0 may be taken as $\max(|a_1|, |a_2|)$.

Furthermore, we have already observed in (1.3.3), that $\sum_{k=1}^{\infty} \varepsilon^{k-1}\psi_k(\xi_0, \xi_1, ..., \xi_{k-1})$ converges for ε, ξ_0 small enough in order that ξ is such that $|\xi| < K$. Moreover, since $\varepsilon < 1$, we have that the series

$$[\sum_{k=1}^{\infty} \varepsilon^{k-1}\psi_k](1-\varepsilon)^{-1} = \sum_{k=1}^{\infty} \varepsilon^{k-1}(\psi_k + \psi_{k-1} + \dots + \psi_1)$$

is convergent for ε, ξ_0 such that $|\xi| < K$. Therefore, from (1.2.8) and WEIERSTRASS' test, we see that the series

$$(1.3.5) \qquad f_j(\tau, \varepsilon, a_1, a_2) \equiv S_{1j} + \varepsilon S_{2j} + \varepsilon^2 S_{3j} + \dots \qquad (j = 1, 2),$$

and the series

$$s_{1k} + \varepsilon s_{2k} + \varepsilon^2 s_{3k} + \dots \qquad (k = 1, 2, ..., n)$$

converge absolutely and uniformly for all t, $-\infty < t < +\infty$. Moreover,

(1.3.6) $$\sum_{m=1}^{\infty} \varepsilon^{m-1} s_{mk} = q_k(\vec{y}; \varepsilon) \qquad (k = 1, 2, \ldots, n).$$

Making use of the first component of $\vec{x}_m$ in (1.1.14), we have

$$\sum_{m=0}^{\infty} \varepsilon^m \frac{dx_{m1}}{dt} = i\tau \sum_{m=0}^{\infty} \varepsilon^m x_{m1} + \sum_{m=1}^{\infty} \varepsilon^m s_{m1} - \sum_{m=1}^{\infty} \varepsilon^m (S_{11} x_{m-1,1} + \ldots + S_{m1} x_{01})$$

and since the right side of this equation is absolutely and uniformly convergent, we have that $\sum_{m=0}^{\infty} \varepsilon^m \frac{dx_{m1}}{dt} = \frac{d}{dt} \sum_{m=1}^{\infty} \varepsilon^m x_{m1} = \dot{y}_1$. Therefore, if in the above expression we make use of (1.3.5), (1.3.6) and the fact that the last sum is a product, we get the final result

$$\dot{y}_1 = [i\tau - \varepsilon f_1(\tau, \varepsilon, a_1, a_2)] y_1 + \varepsilon q_1(\vec{y}; \varepsilon) .$$

The same reasoning may be applied to the other components of $\vec{x}_m$ in (1.1.14) and we obtain a solution to equation (1.1.5).

It remains only to show that the equations

$$\begin{cases} i\tau - \varepsilon f_1(\tau, \varepsilon, a_1, a_2) = i\sigma , \\ -i\tau - \varepsilon f_2(\tau, \varepsilon, a_1, a_2) = -i\sigma , \end{cases}$$

have a real solution τ for some a_1, a_2. This implies first of all that $f_1 = \bar{f}_2$ at least for the solution of these equations. Later, we shall show that this relation is satisfied in very general cases.

§ 2. – Further considerations on the method.

2.1. – Consistency of the equations for τ. Consider the system

(2.1.1) $$\begin{cases} \ddot{x}_j + \alpha_j \dot{x}_j + \sigma_j^2 x_j = \varepsilon q_j(\vec{x}, \dot{\vec{x}}; \varepsilon) & (j = 1, 2, \ldots, n) , \\ \dot{x}_\mu + \beta_\mu x_\mu = \varepsilon q_\mu(\vec{x}, \dot{\vec{x}}; \varepsilon) & (\mu = n+1, \ldots, N), \end{cases}$$

where $\varepsilon > 0$, $\sigma_j > 0$, $\alpha_j \geqslant 0$, $\beta_\mu > 0$, $\vec{x} = (x_1, \ldots, x_N)$, $\dot{\vec{x}} = (\dot{x}_1, \ldots, \dot{x}_n)$ and each q_k is analytic for $\varepsilon < \varepsilon_0$, $|x_k|$, $|\dot{x}_k| < A$, $(k = 1, 2, \ldots, N)$. Moreover, we assume that $4\sigma_j^2 - \alpha_j^2 > 0$ $(j = 1, 2, \ldots, n)$. Notice that the vector $\vec{x}$ has N coordinates and $\dot{\vec{x}}$ has n coordinates. We make this choice in order to simplify the notation.

We shall designate the roots of the equation $\varrho^2 + \alpha_j \varrho + \sigma_j^2 = 0$ by ϱ_{j1}, ϱ_{j2}. We then have

(2.1.2) $$\begin{cases} \varrho_{j1} = -(\alpha_j/2) + (i/2)\gamma_j , & \gamma_j = (4\sigma_j^2 - \alpha_j^2)^{1/2} > 0, \\ \varrho_{j2} = \bar{\varrho}_{j1} , & (j = 1, 2, \ldots, n) . \end{cases}$$

If we introduce new variables $z_1, \ldots, z_{N+n}$ defined by

$$\text{(2.1.3)} \quad \begin{cases} z_{2j-1} = -\varrho_{j2}x_j + \dot{x}_j\,, & x_j = (2i\gamma_j)^{-1}(z_{2j-1} + z_{2j}), \\ z_{2j} = \varrho_{j1}x_j - \dot{x}_j\,, & \dot{x}_j = (2i\gamma_j)^{-1}(\varrho_{j1}z_{2j-1} + \varrho_{j2}z_{2j}), \\ & (j = 1, 2, \ldots, n)\,, \\ z_{n+\mu} = x_\mu & (\mu = n+1, \ldots, N)\,, \end{cases}$$

the system (2.1.1) is transformed into the canonical system

$$\text{(2.1.4)} \quad \begin{cases} \dot{z}_{2j-1} = \varrho_{j1}z_{2j-1} + \varepsilon q_j[(2i\gamma_\nu)^{-1}(z_{2\nu-1}+z_{2\nu}), z_{2n+1}, \ldots, z_{n+N}, (2i\gamma_\nu)^{-1}(\varrho_{\nu1}z_{2\nu-1} + \varrho_{\nu2}z_{2\nu})], \\ \dot{z}_{2j} = \varrho_{j2}z_{2j} - \varepsilon q_j[\ldots]\,, \qquad (j = 1, 2, \ldots, n)\,, \\ \dot{z}_{n+\mu} = -\beta_\mu z_{n+\mu} + \varepsilon q_\mu[\ldots] \qquad (\mu = n+1, \ldots, N)\,, \end{cases}$$

where in the arguments for q_j the symbols $(2i\gamma_\nu)^{-1}(z_{2\nu-1}+z_{2\nu})$, $(2i\gamma_\nu)^{-1}(\varrho_{\nu1}z_{2\nu-1}+$ $+\varrho_{\nu2}z_{2\nu})$ denote $2n$ fo the arguments obtained by letting $\nu = 1, 2, \ldots, n$. Moreover, $[\ldots]$ denotes that the arguments of the functions are the same as those of the first function.

If in (2.1.1) we have $\alpha_j = 0$ for some j, say $j = 1$, then $\varrho_{11} = i\sigma_1$, $\varrho_{12} = -i\sigma_1$ and system (2.1.4) is a special case of the system considered in § **1**. Therefore, we replace σ_1 by τ and consider the auxiliary system

$$\text{(2.1.5)} \quad \begin{cases} \dot{z}_1 = i\tau z_1 + \varepsilon q_1[\ldots]\,, \\ \dot{z}_2 = -i\tau z_2 - \varepsilon q_1[\ldots]\,, \\ \dot{z}_{2j-1} = \varrho_{j1}z_{2j-1} + \varepsilon q_j[\ldots]\,, \\ \dot{z}_{2j} = \varrho_{j2}z_{2j} - \varepsilon q_j[\ldots]\,, \qquad (j = 2, 3, \ldots, n), \\ \dot{z}_{n+\mu} = -\beta_\mu z_{n+\mu} + \varepsilon q_\mu[\ldots] \qquad (\mu = n+1, \ldots, N), \end{cases}$$

where the arguments are the same as in the preceding system.

If we assume that

$$\text{(2.1.6)} \qquad im\sigma_1 + \varrho_{jk} \neq 0 \qquad (j=2, 3, \ldots, n;\ k=1, 2;\ m=0, \pm 1, \pm 2, \ldots),$$

then we may apply the preceding algorithm to system (2.1.5), and we take as the zeroth approximation

$$\text{(2.1.7)} \qquad \vec{z}_0 = \vec{x}_0 = (ae^{i\tau t}, -\bar{a}e^{-i\tau t}, 0, \ldots, 0)\,.$$

(In terms of the preceding notation, we have let $a_1 = a$, $a_2 = -\bar{a}$.) We shall denote the m^{th} approximation by $\vec{z}_m = \vec{x}_0 + \varepsilon\vec{x}_1 + \dots + \varepsilon^m\vec{x}_m$.

It is obvious that

$$s_{r,2j-1} = -s_{r,2j} \qquad (j = 1, 2, \dots, n). \tag{2.1.8}$$

Lemma 2.1.i. If we apply the preceding algorithm to system (2.1.5) with zeroth approximation given by (2.1.7), we have

$$x_{r,2j-1} = -\bar{x}_{r,2j}, \quad x_{r,n+\mu} = \bar{x}_{r,n+\mu}, \quad s_{r,l} = \bar{s}_{r,l}, \quad S_{r1} = \bar{S}_{r2},$$

for $r = 0, 1, 2, \dots$; $j = 1, 2, \dots, n$; $\mu = n+1, \dots, N$; $l = 1, 2, \dots, n+N$.

Proof. We shall first prove by induction that $x_{r,2j-1} = -\bar{x}_{r,2j}$, $x_{r,n+\mu} = \bar{x}_{r,n+\mu}$ for all r, j, μ. From (2.1.7), we see that the assertion is true for $r=0$. Assume assertion is true for $r = 0, 1, \dots, m-1$. Then $x_{r,2j-1} + x_{r,2j} = x_{r,2j-1} - \bar{x}_{r,2j-1}$ is purely imaginary, and $\varrho_{j1}x_{r,2j-1} + \varrho_{j2}x_{r,2j} = \varrho_{j1}x_{r,2j-1} - \overline{\varrho_{j1}x_{r,2j-1}}$ is purely imaginary. As a consequence, since

$$z_{r,2j-1} + z_{r,2j} = \sum_{l=0}^{r} \varepsilon^l (x_{l,2j-1} + x_{l,2j}),$$

$$\varrho_{j1}z_{r,2j-1} + \varrho_{j2}z_{r,2j} = \sum_{l=0}^{r} \varepsilon^l (\varrho_{j1}x_{l,2j-1} + \varrho_{j2}x_{l,2j}),$$

we have $\overline{z_{r,2j-1} + z_{r,2j}} = -(z_{r,2j-1} + z_{r,2j})$; $\overline{(\varrho_{j1}z_{r,2j-1} + \varrho_{j2}z_{r,2j})} = -(\varrho_{j1}z_{r,2j-1} + \varrho_{j2}z_{r,2j})$, and, therefore,

$$\begin{aligned} &\bar{q}_k[(2i\gamma_\nu)^{-1}(z_{r,2\nu-1} + z_{r,2\nu}), z_{r,2n+1}, \dots, z_{r,n+N}, (2i\gamma_\nu)^{-1}(\varrho_{\nu1}, z_{r,2\nu-1} + \varrho_{\nu2}z_{r,2\nu}); \varepsilon] = \\ &\qquad = + q_k[(2i\gamma_\nu)^{-1}(z_{r,2\nu-1} + z_{r,2\nu}), \dots] \qquad (k = 1, 2, \dots, N), \end{aligned} \tag{2.1.9}$$

or, q_k is real for every k.

From (2.1.5) we see that $s_{r,2j-1}$, $s_{r,2j}$, $s_{r,n+\mu}$ are merely the coefficients of ε^{r-1} in $q_j[(2i\gamma_\nu)^{-1}(z_{r,2\nu-1} + z_{r,2\nu}), \dots]$, $-q_j[\dots]$, $q_{n+\mu}[\dots]$, respectively, and, thus, from (2.1.9), we have that

$$s_{rk} = \bar{s}_{rk} \qquad (r = 1, 2, \dots, m;\ k = 1, 2, \dots, N). \tag{2.1.10}$$

We know, also, from (2.1.7) and (1.1.9) that $S_{r1}ae^{i\tau t}$ and $S_{r2}(-\bar{a}e^{-i\tau t})$ are terms in s_{r1} and s_{r2} respectively. Moreover, since $s_{r1} = -s_{r2}$, we have that $S_{r1}ae^{i\tau t} + S_{r2}\bar{a}e^{-i\tau t}$ is a term in s_{r1}. Therefore, since s_{r1} is a power series in $e^{i\tau t}$, $e^{-i\tau t}$ with constant coefficients and $s_{r1} = \bar{s}_{r1}$, we have $S_{r1}ae^{i\tau t} + S_{r2}\bar{a}e^{-i\tau t} =$

$= \bar{S}_{r1}\bar{a}e^{-i\tau t} + \bar{S}_{r2}ae^{i\tau t}$. As a consequence, since $a \neq 0$, we have

$$S_{r1} = \bar{S}_{r2} \qquad (r = 1, 2, ..., m). \tag{2.1.11}$$

Therefore, if we use all the previous results in (1.1.14), we obtain

$$\begin{aligned}\bar{x}_{m1} &= e^{-i\tau t}\int e^{i\tau t}[\bar{s}_{m1} - (\bar{S}_{11}\bar{x}_{m-1,1} + ... + \bar{S}_{m1}\bar{x}_{01})]\,dt = \\ &= e^{-i\tau t}\int e^{i\tau t}[s_{m1} + (S_{12}x_{m-1,2} + ... + S_{m2}x_{02})]\,dt = \\ &= -e^{-i\tau t}\int e^{i\tau t}[s_{m2} - (S_{12}x_{m-1,2} + ... + S_{m2}x_{02})]dt = -x_{m2}\,,\end{aligned}$$

$$\bar{x}_{m,2j-1} = e^{\bar{\varrho}_{j1}t}\int e^{-\bar{\varrho}_{j1}t}\bar{s}_{m,2j-1}\,dt = -e^{\varrho_{j2}t}\int e^{-\varrho_{j2}t}s_{m,2j}\,dt = -x_{m,2j} \qquad (j = 2, ..., n),$$

$$\bar{x}_{m,n+\mu} = e^{\beta_\mu t}\int e^{-\beta_\mu t}\bar{s}_{m,n+\mu}\,dt = x_{m,n+\mu} \qquad (\mu = n+1, ..., N),$$

and the induction on the x_{rk} is completed. Therefore, (2.1.10) and (2.1.11) also hold for all r and the lemma is proved.

From (1.3.5), we have that

$$f_j(\tau, \varepsilon, a) = \sum_{m=1}^{\infty} \varepsilon^{m-1}S_{mj} \qquad (j = 1, 2),$$

and, thus, from the preceding lemma, we necessarily have $f_1(\tau, \varepsilon, a) = \overline{f_2}(\tau, \varepsilon, a)$ for all real numbers τ, ε, a. Finally, we have the following lemma.

Lemma 2.1.ii. By applying the preceding algorithm to system (2.1.5) with zeroth approximation given by (2.1.7), we have $\overline{f_1}(\tau, \varepsilon, a) = f_2(\tau, \varepsilon, a)$ for all real numbers τ, ε, a, and the equations

$$\begin{cases} i\tau - \varepsilon f_1(\tau, \varepsilon, a) = i\sigma_1\,, \\ -i\tau - \varepsilon f_2(\tau, \varepsilon, a) = -i\sigma_1 \end{cases}$$

are equivalent.

2.2. – Existence of periodic solutions. We saw in the preceding section that the equations for τ were consistent and, therefore, in order to find a periodic solution to (2.1.1), it remains only to find a real solution τ to the equation

$$i\tau - \varepsilon f_1(\tau, \varepsilon, a) = i\sigma_1 \tag{2.2.1}$$

for some value of a.

A closer inspection of the function $f_1(\tau, \varepsilon, a)$ shows that it does not depend on the complex number a, but on $|a|^2$; for, S_{r1} is the coefficient of x_{01} in s_{r1};

i.e. S_{r1} will arise from terms of the form $x_{01}^{\mu+1}x_{02}^{\mu} = a^{\mu}(-\bar{a})^{\mu}x_{01} = |a|^{2\mu}x_{01}$. Therefore, if we let

$$\text{(2.2.2)} \qquad \begin{cases} h(\tau, \varepsilon, |a|^2) = I(f_1) = I(S_{11}) + \varepsilon I(S_{21}) + \dots, \\ g(\tau, \varepsilon, |a|^2) = R(f_1) = R(S_{11}) + \varepsilon R(S_{21}) + \dots, \end{cases}$$

then equation (2.2.1) will have a solution if the two equations

$$\text{(2.2.3)} \qquad \begin{cases} H(\tau, \varepsilon, |a|^2) \equiv \tau - \varepsilon h(\tau, \varepsilon, |a|^2) - \sigma_1 = 0, \\ g(\tau, \varepsilon, |a|^2) = 0, \end{cases}$$

have a solution.

If $R[f_1(\tau, \varepsilon, a)] = g(\tau, \varepsilon, |a|^2) = 0$ for every a, then $H(\sigma_1, 0, |a|^2) = 0$, $\dfrac{\partial H(\sigma_1, 0, |a|^2)}{\partial \tau} = 1 \neq 0$ and the theorem of implicit functions implies that there exists a unique solution to equations (2.2.3) of the form

$$\tau = \sigma_1 + \sum_{k=1}^{\infty} b_k \varepsilon^k,$$

where the b_k are constants depending on σ_1, $|a|^2$.

Now, suppose that

$$\text{(2.2.4)} \qquad R(S_{11}) = 0, \qquad \frac{\partial R(S_{11})}{\partial(|a|^2)} \neq 0$$

for some $|a|^2 = c_0^2 > 0$. Then $H(\sigma_1, 0, c_0^2) = 0$, $g(\sigma_1, 0, c_0^2) = 0$, and

$$\frac{\partial H(\sigma_1, 0, c_0^2)}{\partial \tau} = 1, \qquad \frac{\partial H(\sigma_1, 0, c_0^2)}{\partial(|a|^2)} = 0, \qquad \frac{\partial g(\sigma_1, 0, c_0^2)}{\partial \tau} = 0,$$

$$\frac{\partial g(\sigma_1, 0, c_0^2)}{\partial(|a|^2)} = \frac{\partial R(S_{11})}{\partial(|a|^2)} \neq 0.$$

Thus, the Jacobian of equations (2.2.3) at the point $(\sigma_1, 0, c_0^2)$ is different from zero, and by the theorem of implicit functions, there exists a unique solution to equations (2.2.3) of the form

$$\text{(2.2.5)} \qquad \begin{cases} \tau = \sigma_1 + \sum_{k=1}^{\infty} b_k \varepsilon^k, \\ |a|^2 = c_0^2 + \sum_{k=1}^{\infty} c_k \varepsilon^k, \end{cases}$$

where b_k, c_k are constants depending on σ_1.

Thus, let us assume that equations (2.2.3) are satisfied for some τ, $|a|^2$, and try to find the behavior of the corresponding solution of (2.1.1). If we let $a = |a|e^{i\varphi}$, and use (1.1.15), we observe that the following identity holds

$$(2i\gamma_j)^{-1}(x_{m,2j-1} + x_{m,2j}) = \sum_{m_1,m_2\geq 0} (2i\gamma_j)^{-1}(\sigma_{m,2j-1}^{(m_1,m_2)} + \sigma_{m,2j}^{(m_1,m_2)})a^{m_1}(-\bar{a})^{m_2}e^{(m_1-m_2)i\tau t} =$$
$$= \sum_{l\geq 0}[(2i)^{-1}B_{mj}^{(l)}e^{il(\tau t+\varphi)} + (2i)^{-1}B_{mj}^{(-l)}e^{-il(\tau t+\varphi)}],$$

where $B_{mj}^{(l)}$, $B_{mj}^{(-l)}$ are constants. Moreover, we have also observed in lemma (2.1.i) that $(2i\gamma_j)^{-1}(x_{m,2j-1}+x_{m,2j})$ is real. Therefore, we must have

$$e^{-il\varphi}(2i)^{-1}B_{mj}^{(-l)} = \overline{e^{il\varphi}(2i)^{-1}B_{mj}^{(l)}} = -e^{-il\varphi}(2i)^{-1}\overline{B_{mj}^{(l)}},$$

or $B_{mj}^{(-l)} = -\overline{B_{mj}^{(l)}}$. Therefore, if we let $B_{mj}^{(l)} = \gamma_{mj}^{(l)} + i\delta_{mj}^{(l)}$, where $\gamma_{mj}^{(l)}$, $\delta_{mj}^{(l)}$ are real, we have

$$(2.2.6)\qquad (2i\gamma_j)^{-1}(x_{m,2j-1} + x_{m,2j}) = \sum_{l\geq 0}[\gamma_{mj}^{(l)}\sin l(\tau t+\varphi) + \delta_{mj}^{(l)}\cos l(\tau t+\varphi)] = W_{mj}(\tau t+\varphi),$$

where $W_{mj}(\tau t+\varphi)$ $(j = 1, 2, ..., n;\ m = 0, 1, ...)$ is periodic of period $2\pi/\tau$ in t. Similarly, we have $x_{m,\mu} = W_{m\mu}(\tau t+\varphi)$ $(\mu = n+1,..., N)$. Moreover,

$$(2i\sigma_1)^{-1}(x_{01} + x_{02}) = (2i\sigma_1)^{-1}|a|(e^{i(\tau t+\varphi)} - e^{-i(\tau t+\varphi)}) = |a|\sigma_1^{-1}\sin(\tau t+\varphi),$$

$$x_{0k} = 0 \qquad (k = 3, 4, ..., n+N).$$

As a consequence, since $z_j = x_{0j} + \varepsilon x_{1j} + \varepsilon^2 x_{2j} + ...$, we have, upon applying (2.1.3), (2.1.7) and (2.2.6), that

$$\begin{cases} x_1 = |a|\sigma_1^{-1}\sin(\tau t+\varphi) + \varepsilon W_1(\tau t+\varphi;\ \varepsilon), \\ x_k = \varepsilon W_k(\tau t+\varphi;\ \varepsilon) \qquad (k = 2, 3, ..., N), \end{cases}$$

where each $W_k(\tau t+\varphi;\ \varepsilon)$ $(k = 1, 2, ..., N)$ is periodic of period 2π in $\tau t+\varphi$, φ an arbitrary constant. Therefore, we have the following theorem.

Theorem 2.2.i. *If in (2.1.1), $\alpha_j = 0$ for some j, say $j = 1$, and if the equation*

$$i\tau - \varepsilon f_1(\tau, \varepsilon, a) = i\sigma_1$$

has a real solution

$$\tau = \tau(\varepsilon) = \sigma_1 + \sum_{k=1}^{\infty} b_k\varepsilon^k$$

for some value of a, *then there exists a solution*

$$\begin{cases} x_1 = |a|\sigma_1^{-1} \sin(\tau t + \varphi) + \varepsilon W_1(\tau t + \varphi; \varepsilon), \\ x_k = \varepsilon W_k(\tau t + \varphi; \varepsilon) \qquad (k = 2, 3, \ldots, N), \end{cases}$$

of system (2.1.1) *periodic of period* 2π *in* $\tau t + \varphi$, *where* φ *is an arbitrary constant. The above equation will have a real solution* τ *if either*

$$(\alpha) \qquad R[f_1(\tau, \varepsilon, a)] = 0 \qquad \textit{for all } \tau, \varepsilon, a,$$

or

$$(\beta) \qquad R(S_{11}) = 0, \qquad \frac{\partial R(S_{11})}{\partial(|a|^2)} \neq 0 \qquad \textit{for some } a.$$

Remark. We shall show in the Appendix that the sufficient condition (β) is precisely the condition given by CODDINGTON and LEVINSON [4] for systems of the type (2.1.1). It should be remarked that here the condition is derived from the method of successive approximations. For one second order differential equation, see KRYLOFF and BOGOLIUBOFF [7]. The condition (α) given in the present paper is not contained in the quoted papers.

2.3. – Explicit expression for S_{11}. In order to find an explicit expression for S_{11}, let us write

$$q_1(\vec{x}, \dot{\vec{x}}; \varepsilon) = A(x_1, \dot{x}_1) + B(x_1, \dot{x}_1) + q_1^*(\vec{x}, \dot{\vec{x}}; \varepsilon), \tag{2.3.1}$$

where

$A(-x_1, \dot{x}_1) = A(x_1, \dot{x}_1)$, $B(-x_1, \dot{x}_1) = -B(x_1, \dot{x}_1)$, $q_1^*(x_1, 0, \ldots, 0, \dot{x}_1, 0, \ldots, 0; 0) = 0$,

and A, B, q_1^* are analytic for $\varepsilon < \varepsilon_0$, $|x_j|$, $|\dot{x}_j| < A$. Then by definition and (2.3.1), we have

$$s_{11} = q_1[(2i\sigma_1)^{-1}(x_{01} + x_{02}), 0, \ldots, 0, 2^{-1}(x_{01} - x_{02}), 0, \ldots, 0; 0] =$$
$$= A[(2i\sigma_1)^{-1}(x_{01} + x_{02}), 2^{-1}(x_{01} + x_{02})] + B[\ldots],$$

and

$$\begin{aligned} aS_{11} = m[e^{-i\tau t}s_{11}] = m[e^{-i\tau t}A((2i\sigma_1)^{-1}(x_{01} + x_{02}), 2^{-1}(x_{01} - x_{02}))] + \\ + m[e^{-i\tau t}B(\ldots)]. \end{aligned} \tag{2.3.2}$$

If, as before, we let $a = |a|e^{i\varphi}$, then $(2i\sigma_1)^{-1}(x_{01} + x_{02}) = |a|\sigma_1^{-1}\sin(\tau t + \varphi)$

$2^{-1}(x_{01}-x_{02})=|a|\cos(\tau t+\varphi)$ and (2.3.2) becomes

$$S_{11}=|a|^{-1}m[e^{-i(\tau t+\varphi)}A(|a|\sigma_1^{-1}\sin(\tau t+\varphi),\ |a|\cos(\tau t+\varphi))]+$$
$$+|a|^{-1}m[e^{-i(\tau t+\varphi)}B(\ldots)]=(|a|T)^{-1}\int_0^T A(\ldots)e^{-i(\tau t+\varphi)}\,dt+(|a|T)^{-1}\int_0^T B(\ldots)e^{-i(\tau t+\varphi)}\,dt,$$

where $T=2\pi/\tau$, and the symbol $(\ldots)$ is to mean that the arguments of all the functions are the same, i.e. $|a|\sigma_1^{-1}\sin(\tau t+\varphi)$, $|a|\cos(\tau t+\varphi)$. If we replace $\tau t+\varphi$ by t, we have

$$\begin{aligned}S_{11}&=(|a|2\pi)^{-1}\Big[\int_0^{2\pi}A(|a|\sigma_1^{-1}\sin t,\ |a|\cos t)e^{-it}\,dt+\int_0^{2\pi}B(\ldots)e^{-it}\,dt\Big]=\\&=(2\pi|a|)^{-1}\Big[\int_0^{2\pi}A(|a|\sigma_1^{-1}\sin t,\ |a|\cos t)\cos t\,dt-i\int_0^{2\pi}B(\ldots)\sin t\,dt\Big],\end{aligned}\tag{2.3.3}$$

since $A(|a|\sigma_1^{-1}\sin t,\ |a|\cos t)$ is an even function of t, and $B(|a|\sigma_1^{-1}\sin t,\ |a|\cos t)$ is an odd function of t.

Thus, we have the result that

$$\begin{aligned}R(S_{11})&=(2\pi|a|)^{-1}\int_0^{2\pi}q_1(|a|\sigma_1^{-1}\sin t,0,\ldots,0,|a|\cos t,0,\ldots,0;0)\cos t\,dt\\&=(2\pi|a|)^{-1}\int_0^{2\pi}A(|a|\sigma_1^{-1}\sin t,\ |a|\cos t)\cos t\,dt,\end{aligned}\tag{2.3.4}$$

where A is given by (2.3.1).

One may also obtain the explicit expression for $R(S_{11})$ by first observing that the coefficient of x_{01} in the expansion of $B[(2i\sigma_1)^{-1}(x_{01}+x_{02}),\ 2^{-1}(x_{01}-x_{02})]$ is purely imaginary since B is odd in the first argument. Then, from (2.3.2), $R(S_{11})$ is merely the coefficient of x_{01} in the expansion of A, i.e.,

$$\begin{aligned}&A[(2i\sigma_1)^{-1}(x_{01}+x_{02}),\ 2^{-1}(x_{01}-x_{02})]=\\&=R(S_{11})a\,e^{i\tau t}+\sum_{k=2}^{\infty}D_k a^k e^{ik\tau t}+\sum_{k=0}^{\infty}D_{-k}(-\bar a)^k e^{-i\tau k t},\end{aligned}\tag{2.3.5}$$

where D_k, D_{-k} are constants depending on $|a|^2$.

§ 3. – Systems with periodic solutions.

3.1. – A first general statement. In equation (2.3.5), we observed that $R(S_{11})$ is merely the coefficient of x_{01} in the expansion of $A[(2i\sigma_1)^{-1}(x_{01}+x_{02}), 2^{-1}(x_{01}-x_{02})]$, where A is defined by (2.3.1). Let us consider the special case where

$$A(x_1, \dot{x}_1) = \varphi(x_1)\dot{x}_1 = \dot{x}_1 \sum_{k=0}^{\infty} a_k x_1^{2k}.$$

Then

$$(3.1.1) \qquad \begin{aligned} &A[(2i\sigma_1)^{-1}(x_{01}+x_{02}), 2^{-1}(x_{01}-x_{02})] = \\ &\qquad = 2^{-1}(x_{01}-x_{02}) \sum_{k=0}^{\infty} (-1)^k (2\sigma_1)^{-2k}(x_{01}+x_{02})^{2k}. \end{aligned}$$

Moreover,

$$\begin{aligned} (x_{01}-x_{02})(x_{01}+x_{02})^{2k} &= \left[x_{01}^{2k} + \binom{2k}{1} x_{01}^{2k-1}x_{02} + \ldots + \binom{2k}{k-1} x_{01}^{2k-k+1}x_{02}^{k-1} + \right. \\ &\qquad \left. + \binom{2k}{k} x_{01}^{2k-k}x_{02}^{k} + \ldots + x_{02}^{2k}\right](x_{01}-x_{02}) = \\ &= \ldots + \left[\binom{2k}{k} - \binom{2k}{k-1}\right] x_{01}^{k+1}x_{02}^{k} + \ldots = \ldots + \binom{2k}{k}\frac{(-1)^k |a|^{2k}}{k+1} x_{01} + \ldots . \end{aligned}$$

Thus, if we substitute this expression in (3.1.1) and keep only the coefficient of x_{01}, we have

$$(3.1.2) \qquad R(S_{11}) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{a_k}{k+1}\binom{2k}{k}\left(\frac{|a|}{2\sigma_1}\right)^{2k}.$$

Theorem 3.1.i. *Consider the system of equations*

$$(3.1.3) \qquad \begin{cases} \ddot{x}_1 + \sigma_1^2 x_1 = \varepsilon f(x_1)\dot{x}_1 + \varepsilon g(x_1, \dot{x}_1) + \varepsilon h(\vec{x}, \dot{\vec{x}}; \varepsilon), \\ \ddot{x}_j + \alpha_j \dot{x}_j + \sigma_j^2 x_j = \varepsilon q_j(\vec{x}, \dot{\vec{x}}; \varepsilon) \qquad (j = 2, 3, \ldots, n), \\ \dot{x}_\mu + \beta_\mu x_\mu = \varepsilon q_\mu(\vec{x}, \dot{\vec{x}}; \varepsilon) \qquad (\mu = n+1, \ldots, N), \end{cases}$$

where $\vec{x} = (x_1, \ldots, x_N)$, $\dot{\vec{x}} = (\dot{x}_1, \ldots, \dot{x}_n)$, $\varepsilon > 0$, $\sigma_j > 0$, $\alpha_j \geqslant 0$, $\beta_\mu > 0$, *and* $4\sigma_j^2 - \alpha_j^2 > 0$, $im\sigma_1 - (\alpha_j/2) + (i/2)\sqrt{4\sigma_j^2 - \alpha_j^2} \neq 0$ $(m = 0, \pm 1, \pm 2, \ldots; j = 1, 2, \ldots, n;$

$\mu = n + 1, \ldots, N$). *Moreover, assume that*

$$f(x_1) = \sum_{k=0}^{m} a_k x_1^{2k}, \qquad a_0 a_m < 0,$$

$$g(-x_1, \dot{x}_1) = -g(x_1, \dot{x}_1), \qquad h(x_1, 0, \ldots, 0, \dot{x}_1, 0, \ldots, 0; 0) = 0$$

and $g(x_1, \dot{x}_1)$, $h(\vec{x}, \dot{\vec{x}}; \varepsilon)$, $q_k(\vec{x}, \dot{\vec{x}}; \varepsilon)$ $(k = 2, 3, \ldots, N)$ *are analytic for* $\varepsilon < \varepsilon_0$, $|x_k|$, $|\dot{x}_j| < A$. *Then* $R(S_{11})$ *as a polynomial in* $|a|^2$ *has a zero at* $|a|^2 = c_0^2 > 0$. *If such a zero is simple, there will be a periodic solution to equations* (3.1.3) *of the form*

$$\text{(3.1.4)} \qquad \begin{cases} x_1 = |a|\sigma_1^{-1} \sin(\tau t + \varphi) + \varepsilon W_1(\tau t + \varphi; \varepsilon), \\ x_k = \varepsilon W_k(\tau t + \varphi; \varepsilon) \qquad (k = 2, 3, \ldots, N), \end{cases}$$

periodic of period 2π *in* $\tau t + \varphi$, φ *an arbitrary constant, and the numbers* τ *and* $|a|$ *are given by*

$$\tau = \sigma_1 + \sum_{k=1}^{\infty} b_k \varepsilon^k, \qquad |a|^2 = c_0^2 + \sum_{k=1}^{\infty} c_k \varepsilon^k,$$

where b_k *and* c_k *are constants depending on* σ_1, c_0^2.

Proof. From (3.1.3), and (2.3.1), we see that $A(x_1, \dot{x}_1) = \dot{x}_1 f(x_1)$, and from the expression for $f(x_1)$ and (3.1.2), we have

$$\text{(3.1.5)} \qquad R(S_{11}) = \frac{1}{2} \sum_{k=0}^{m} \frac{a_k}{k+1} \binom{2k}{k} \left(\frac{|a|}{2\sigma_1}\right)^{2k},$$

and $a_0 \dfrac{a_m}{m+1}\dbinom{2m}{m} < 0$. Therefore, the function $R(S_{11})$ has a positive zero. If it has a simple zero, then from theorem (2.2.i), we prove Theorem (3.1.i).

Corollary. If, in (3.1.3), $\alpha_j = 0$ $(j = 1, 2, \ldots, n)$, there may be n such limit cycles (3.1.4).

Example 1. Suppose that in the system (3.1.3) the functions g, h, q_j, q_μ are arbitrary functions satisfying the conditions stated in the Theorem (3.1.i) and that the polynomial

$$f(x_1) = a_0 + a_1 x_1^2 + \ldots + a_m x_m^{2k}, \qquad a_0 a_m < 0,$$

has exactly one variation in sign. Then the sequence of numbers a_0, a_1, $2a_2$, ..., $(m+1)^{-1}\binom{2m}{m} a_m$ has exactly one variation in sign and, thus, by DESCARTES rule of signs, the function (3.1.5) has exactly one positive zero. As a consequence, there is only one periodic solution to (3.1.3) of the type described.

Example 2. Let us consider a more specific example where $f(x_1)=1-x_1^2$. Then $A(x_1, \dot{x}_1)=(1-x_1^2)\dot{x}_1$ and

$$R(S_{11})=\frac{1}{2}\left[1-\left(\frac{|a|}{2\sigma_1}\right)^2\right]=0$$

for $|a|=2\sigma_1$. Then there is a periodic solution to (3.1.3) of the form (3.1.4) with

$$\tau=\sigma_1+\sum_{k=1}^{\infty} b_k\varepsilon^k, \qquad |a|=2\sigma_1+\sum_{k=1}^{\infty} c_k\varepsilon^k .$$

Note that this system contains the VAN DER POL equation $\ddot{x}-\varepsilon(1-x^2)\dot{x}+x=0$ as a special case.

Example 3. As a particular case of Example 1, consider the system

$$\begin{cases} \ddot{x}+x-\varepsilon(1-x^2-y^2)\dot{x}=\varepsilon f(x, y, \dot{y}), \\ \ddot{y}+2y-\varepsilon(1-x^2-y^2)\dot{y}=\varepsilon g(x, \dot{x}, y), \end{cases}$$

where $f(-x, y, \dot{y})=-f(x, y, \dot{y})$, $g(x, \dot{x}, -y)=-g(x, \dot{x}, y)$ are any analytic functions in x, $\dot{x}$, y, $\dot{y}$. There are two periodic solutions to this equation given by

$$x=|a_1|\sin(\tau_1 t+\varphi)+\varepsilon W_1(\tau_1 t+\varphi;\ \varepsilon), \qquad y=\varepsilon W_2(\tau_1 t+\varphi;\ \varepsilon),$$
$$\tau_1=1+o(\varepsilon), \qquad |a_1|=2+o(\varepsilon),$$

and

$$x=\varepsilon W_3(\tau t+\varphi;\ \varepsilon), \qquad y=|a_2|2^{-1/2}\sin(\tau_2 t+\varphi)+\varepsilon W_4(\tau_2 t+\varphi;\ \varepsilon),$$
$$\tau_2=2^{1/2}+o(\varepsilon), \qquad |a_2|=2^{3/2}+o(\varepsilon).$$

Moreover, by applying the conditions given in MINORSKY ([**12**], p. 158), we see that both of these solutions are stable in the sense of LIAPOUNOFF.

3.2. – Some preliminary formulas. Consider the system of equations

$$\ddot{x}_j+\sigma_j^2 x_j=\varepsilon q_j(\vec{x}, \dot{\vec{x}};\ \varepsilon) \qquad (j=1, 2, \ldots, n), \tag{3.2.1}$$

where $\vec{x}=(x_1, \ldots, x_n)$, $\dot{\vec{x}}=(\dot{x}_1, \ldots, \dot{x}_n)$, and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are distinct real numbers such that $m\sigma_j + \sigma_k \neq 0$ $(j \neq k;\ j, k = 1, 2, \ldots, n;\ m = 0, \pm 1, \pm 2, \ldots)$, and each function q_j is analytic for $\varepsilon < \varepsilon_0$, $|x_j|$, $|\dot{x}_j| < A$, $(j = 1, 2, \ldots, n)$.

System (3.2.1) is clearly a special case of system (2.1.1). From (2.1.2), we have $\varrho_{j1} = i\sigma_j$, $\varrho_{j2} = -i\sigma_j$ and equations (2.1.4) become

$$\text{(3.2.2)} \quad \begin{cases} \dot{z}_{2j-1} = i\sigma_j z_{2j-1} + \varepsilon q_j[(2i\sigma_1)^{-1}(z_1 + z_2), \ldots, (2i\sigma_n)^{-1}(z_{2n-1} + z_{2n}), \\ \qquad\qquad 2^{-1}(z_1 - z_2), \ldots, 2^{-1}(z_{2n-1} - z_{2n});\ \varepsilon], \\ \dot{z}_{2j} = -i\sigma_j z_{2j} - \varepsilon q_j[\ldots], \qquad (j = 1, 2, \ldots, n), \end{cases}$$

where [...] denotes that the arguments are the same as those of the preceding function.

Since we have assumed that each $q_j(\vec{x}, \dot{\vec{x}}; \varepsilon)$ is analytic, we have

$$q_j(\vec{x}, \dot{\vec{x}}; \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k q_{jk}(\vec{x}, \dot{\vec{x}}) = \sum_{k=0}^{\infty} \varepsilon^k \sum_{(\nu+\mu)\geq 0} C_{jk}^{(\nu+\mu)} x_1^{\nu_1} \ldots x_n^{\nu_n} \dot{x}_1^{\mu_1} \ldots \dot{x}_n^{\mu_n},$$

where $(\nu + \mu) = (\nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_n)$ and the $C_{jk}^{(\nu+\mu)}$ are constants. If we let $2^{-\nu_1 - \ldots - \nu_n - \mu_1 - \ldots - \mu_n} C_{jk}^{(\nu+\mu)} = D_{jk}^{(\nu+\mu)}$, then we may write (3.2.2) as

$$\text{(3.2.3)} \quad \begin{cases} \dot{z}_{2j-1} = i\sigma_j z_{2j-1} + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{(\nu+\mu)\geq 0} D_{jk}^{(\nu+\mu)} \prod_{l=1}^{n} (i\sigma_l)^{-\nu_l} (z_{2l-1} + z_{2l})^{\nu_l} (z_{2l-1} - z_{2l})^{\mu_l}, \\ \dot{z}_{2j} = -i\sigma_j z_{2j} - \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{(\nu+\mu)\geq 0} D_{jk}^{(\nu+\mu)} \prod_{l=1}^{n} (i\sigma_l)^{-\nu_l} (z_{2l-1} + z_{2l})^{\nu_l} (z_{2l-1} - z_{2l})^{\mu_l}, \\ \qquad (j = 1, 2, \ldots, n). \end{cases}$$

Let us replace σ_1 by τ and consider the auxiliary system

$$\text{(3.2.4)} \quad \begin{cases} \dot{z}_1 = i\tau z_1 + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{(\nu+\mu)\geq 0} D_{1k}^{(\nu+\mu)} \prod_{l=1}^{n} (i\sigma_l)^{-\nu_l} (z_{2l-1} + z_{2l})^{\nu_l} (z_{2l-1} - z_{2l})^{\mu_l}, \\ \dot{z}_2 = -i\tau z_2 - \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{(\nu+\mu)\geq 0} (\ldots), \\ \dot{z}_{2j-1} = i\sigma_j z_{2j-1} + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{(\nu+\mu)\geq 0} D_{jk}^{(\nu+\mu)} \prod_{l=1}^{n} (\ldots), \\ \dot{z}_{2j} = -i\sigma_j z_{2j} - \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \sum_{(\nu+\mu)\geq 0} (\ldots), \qquad (j = 2, 3, \ldots, n). \end{cases}$$

We shall again take the zeroth approximation to be

$$\text{(3.2.5)} \qquad \vec{z}_0 = \vec{x}_0 = (ae^{i\tau t}, -\bar{a}e^{-i\tau t}, 0, \ldots, 0)$$

and shall denote the m^{th} approximation by

$$\vec{z}_m = \vec{x}_0 + \varepsilon \vec{x}_1 + \dots + \varepsilon^m \vec{x}_m \qquad (m = 1, 2, \dots).$$

We, therefore, have

$$(3.2.6) \qquad \begin{cases} z_{s,2j-1} + z_{s,2j} = \sum_{p=0}^{s} \varepsilon^p (x_{p,2j-1} + x_{p,2j}), \\ z_{s,2j-1} - z_{s,2j} = \sum_{p=0}^{s} \varepsilon^p (x_{p,2j-1} - x_{p,2j}) \end{cases}$$

and

$$(3.2.7) \qquad \begin{cases} (z_{s,2l-1} + z_{s,2l})^{\nu_l} = \sum_{\alpha_1,\dots,\alpha_{\nu_l}=0}^{s} \varepsilon^{\alpha_1+\dots+\alpha_{\nu_l}} \prod_{\delta=1}^{\nu_l} (x_{\alpha_\delta,2l-1} + x_{\alpha_\delta,2l}), \\ (z_{s,2l-1} - z_{s,2l})^{\mu_l} = \sum_{\beta_1,\dots,\beta_{\mu_l}=0}^{s} \varepsilon^{\beta_1+\dots+\beta_{\mu_l}} \prod_{\lambda=1}^{\mu_l} (x_{\beta_\lambda,2l-1} - x_{\beta_\lambda,2l}). \end{cases}$$

From the definition in § **1**, we known that $s_{r,2j-1,k}$ is the coefficient of ε^{r-1} in q_{jk} when $z_{2l-1}+z_{2l}$, $z_{2l-1}-z_{2l}$ are replaced by their corresponding expressions (3.2.6). Thus, if we substitute (3.2.7) in (3.2.4) and keep only terms in ε^{r-1}, we obtain

$$(3.2.8) \qquad \begin{cases} s_{r,2j-1,k} = \sum D^{(\nu+\mu)}_{2j-1,k} \prod_{l=1}^{n} (i\sigma_l)^{-\nu_l} \cdot \\ \qquad \cdot \left[\sum \prod_{\delta=1}^{\nu_l} (x_{\alpha_\delta,2l-1} + x_{\alpha_\delta,2l}) \cdot \prod_{\lambda=1}^{\mu_l} (x_{\beta_\lambda,2l-1} - x_{\beta_\lambda,2l}) \right], \end{cases}$$

where the last sum is to be taken over all values of $\alpha_1, \dots, \alpha_{\nu_l}$ and $\beta_1, \dots, \alpha_{\mu_l}$ such that

$$(3.2.9) \qquad \alpha_1 + \dots + \alpha_{\nu_l} + \beta_1 + \dots + \beta_{\mu_l} = r - 1,$$

and the first sum is taken over all values of $\nu_1, \dots, \nu_n, \mu_1, \dots, \mu_n$ which make (3.2.9) possible.

3.3. – A second general statement. Let us suppose in (3.2.1) that

$$(3.3.1) \qquad q_j(\vec{x}, -\dot{\vec{x}}; \varepsilon) = q_j(\vec{x}, \dot{\vec{x}}; \varepsilon) \qquad (j = 1, 2, \dots, n),$$

Then each of the numbers $\mu_1, \dots, \mu_n$ in (3.2.4) is even. If we use the same notation as in (1.1.15), (1.1.16), i.e.

$$x_{mj} = \sum_{m_1, m_2 \geq 0} \sigma_{mj}^{(m_1,m_2)} x_{01}^{m_1} x_{02}^{m_2} ,$$

$$s_{mj} = \sum_{m_1, m_2 \geq 0} \varrho_{mj}^{(m_1,m_2)} x_{01}^{m_1} x_{02}^{m_2} ,$$

we can prove the following Lemma.

Lemma 3.3.i. If the preceding algorithm is applied to system (3.2.4), satisfying (3.3.1), and with zeroth approximation given by (3.2.5), we have

$$\sigma_{m,2j-1}^{(m_1,m_2)} = \sigma_{m,2j}^{(m_2,m_1)} , \qquad \varrho_{mk}^{(m_1,m_2)} = \varrho_{mk}^{(m_2,m_1)} , \qquad S_{m1} = - S_{m2}$$

for every $m_1,\ m_2 \geqslant 0$; $m = 0, 1, 2, \ldots$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, 2n$.

Proof. (By induction.) From (3.2.5), we see that $\sigma_{0,2j-1}^{(1,0)} = \sigma_{0,2j}^{(0,1)}$ $(j = 1, 2, \ldots, n)$. Assume that $\sigma_{p,2j-1}^{(m_1,m_2)} = \sigma_{p,2j}^{(m_2,m_1)}$ for all $m_1,\ m_2 \geqslant 0$, and $p=0, 1, \ldots, m-1$. Then

$$x_{p,2j-1} = \sum_{m_1,m_2 \geq 0} \sigma_{p,2j-1}^{(m_1,m_2)} x_{01}^{m_1} x_{02}^{m_2} ,$$

$$x_{p,2j} = \sum_{m_1,m_2 \geq 0} \sigma_{p,2j}^{(m_1,m_2)} x_{01}^{m_1} x_{02}^{m_2} = \sum_{m_1,m_2 \geq 0} \sigma_{p,2j-1}^{(m_1,m_2)} x_{01}^{m_2} x_{02}^{m_1} ,$$

for $j = 1, 2, \ldots, n$; $p = 0, 1, \ldots, m-1$, and thus,

$$\text{(3.3.2)} \qquad \begin{cases} x_{p,2j-1} + x_{p,2j} = \sum\limits_{m_1,m_2 \geq 0} \sigma_{p,2j-1}^{(m_1,m_2)} (x_{01}^{m_1} x_{02}^{m_2} + x_{01}^{m_2} x_{02}^{m_1}) , \\ x_{p,2j-1} - x_{p,2j} = \sum\limits_{m_1,m_2 \geq 0} \sigma_{p,2j-1}^{(m_1,m_2)} (x_{01}^{m_1} x_{02}^{m_2} - x_{01}^{m_2} x_{02}^{m_1}) . \end{cases}$$

If we substitute this expression (3.3.2) in expression (3.2.8) for $s_{r,2j-1,k}$ we see that the last sum in (3.2.8) consists of a sum of terms of the form

$$\text{(3.3.3)} \qquad \prod_{\delta=1}^{\nu_l} (x_{01}^{m_{1\delta}} x_{02}^{m_{2\delta}} + x_{01}^{m_{2\delta}} x_{02}^{m_{1\delta}}) \prod_{\lambda=1}^{\mu_l} (x_{01}^{n_{1\lambda}} x_{02}^{n_{2\lambda}} - x_{01}^{n_{2\lambda}} x_{02}^{n_{1\lambda}}) ,$$

where $x_{01}^{m_{1\delta}} x_{02}^{m_{2\delta}} + x_{01}^{m_{2\delta}} x_{02}^{m_{1\delta}}$, $x_{01}^{n_{1\lambda}} x_{02}^{n_{2\lambda}} - x_{01}^{n_{2\lambda}} x_{02}^{n_{1\lambda}}$ are terms in $x_{\alpha_\delta,2l-1} + x_{\alpha_\delta,2l}$, $x_{\beta_\lambda,2l-1} - x_{\beta_\lambda,2l}$, respectively.

Let us consider any term in the development of (3.3.3), say $x_{01}^{\theta_1} x_{02}^{\theta_2}$. Then, the corresponding term $x_{01}^{\theta_2} x_{02}^{\theta_1}$ will also be in the development of (3.3.3) and the coefficients of these two terms will differ by a factor $(+1)^{\nu_l}(-1)^{\mu_l}=1$ since, by assumption, μ_l is even. This means the coefficients are the same for each term of this form. As a consequence, since

$$s_{r,2j-1} = s_{r,2j-1,0} + s_{r-1,2j-1,1} + \dots + s_{1,2j-1,r-1},$$

we have that the coefficient of $x_{01}^{\theta_1} x_{02}^{\theta_2}$ in $s_{r,2j-1}$ is the same as the coefficient of $x_{01}^{\theta_2} x_{02}^{\theta_1}$ in $s_{r,2j-1}$, i.e.

$$\varrho_{r,2j-1}^{(m_1,m_2)} = \varrho_{r,2j-1}^{(m_2,m_1)} \qquad (j = 1, 2, \dots, n;\ r = 1, 2, \dots, m;\ m_1, m_2 \geqslant 0). \tag{3.3.4}$$

In particular, $\varrho_{r1}^{(\theta+1,0)} = \varrho_{r1}^{(0,\theta+1)}$ $(r = 1, 2, \dots, m)$. But, from the definition (1.1.9), whe have that $S_{r1}x_{01}$, $S_{r2}x_{02}$ are terms in s_{r1}, s_{r2}, respectively, and since $s_{r1} = -s_{r2}$, we must have that $S_{r1}x_{01} - S_{r2}x_{02}$ is a term in s_{r1}. As a consequence, since $\varrho_{r1}^{(\theta+1,0)} = \varrho_{r1}^{(0,\theta+1)}$, we have

$$S_{r1} = - S_{r2} \qquad (r = 1, 2, \dots, m). \tag{3.3.5}$$

If we substitute (3.3.5) in (1.1.17) (note that σ_μ is replaced by $i\sigma_\mu$), we have

$$\begin{aligned}
x_{m1} &= \sum_{m_1,m_2 \geq 0} [\varrho_{m1}^{(m_1,m_2)} - S_{11}\sigma_{m-1,1}^{(m_1,m_2)} - \dots - S_{m-1,1}\sigma_{11}^{(m_1,m_2)}][(m_1 - m_2 - 1)i\tau]^{-1} x_{01}^{m_1} x_{02}^{m_2},\\
x_{m2} &= \sum_{m_1,m_2 \geq 0} [\varrho_{m2}^{(m_1,m_2)} + S_{11}\sigma_{m-1,2}^{(m_1,m_2)} + \dots + S_{m-1,1}\sigma_{12}^{(m_1,m_2)}][(m_1 - m_2 + 1)i\tau]^{-1} x_{01}^{m_1} x_{02}^{m_2},\\
x_{m,2j-1} &= \sum_{m_1,m_2 \geq 0} \varrho_{m,2j-1}^{(m_1,m_2)}[(m_1 - m_2)i\tau - i\sigma_j]^{-1} x_{01}^{m_1} x_{02}^{m_2},\\
x_{m,2j} &= \sum_{m_1,m_2 \geq 0} \varrho_{m,2j}^{(m_1,m_2)}[(m_1 - m_2)i\tau + i\sigma_j]^{-1} x_{01}^{m_1} x_{02}^{m_2}, \qquad (j = 2, 3, \dots, n).
\end{aligned}$$

Moreover, since $s_{r,2j-1} = -s_{r,2j}$, we have $\varrho_{m2}^{(m_1,m_2)} = -\varrho_{m1}^{(m_1,m_2)}$, and if we make use of (3.3.4) and our assumption on the $\sigma_{s,j}^{(m_1,m_2)}$ $(s = 0, 1, \dots, m)$, we have

$$\begin{aligned}
x_{m,2} &= \sum_{m_1,m_2 \geq 0} [\varrho_{m1}^{(m_1,m_2)} - S_{11}\sigma_{m-1,1}^{(m_1,m_2)} - \dots - S_{m-1,1}\sigma_{11}^{(m_1,m_2)}][(m_1 - m_2 - 1)i\tau]^{-1} x_{01}^{m_2} x_{02}^{m_1},\\
x_{m,2j} &= \sum_{m_1,m_2 \geq 0} \varrho_{m,2j-1}^{(m_1,m_2)}[(m_1 - m_2)i\tau - i\sigma_j]^{-1} x_{01}^{m_2} x_{02}^{m_1} \qquad (j = 2, 3, \dots, n).
\end{aligned}$$

Thus, $\sigma_{m,2j-1}^{(m_1,m_2)} = \sigma_{m,2j}^{(m_2,m_1)}$, and the induction on the x_{mj} is completed. As a consequence, (3.3.4) and (3.3.5) also hold for all r and the Lemma is proved.

Theorem 3.3.i. *Consider the system*

$$\ddot{x}_j + \sigma_j^2 x_j = \varepsilon q_j(\vec{x}, \dot{\vec{x}}; \varepsilon) \qquad (j = 1, 2, \dots, n), \tag{3.3.6}$$

where $\sigma_1, \ldots, \sigma_n$ *are distinct positive numbers such that* $m\sigma_j + \sigma_k \neq 0$ $(j \neq k;\ j = 1, 2, \ldots, n;\ \ m = 0, \pm 1, \pm 2, \ldots)$, *and each* $q_j(\vec{x}, \dot{\vec{x}}; \varepsilon)$ *is analytic for* $|x_j|,\ |\dot{x}_j| < A$, $\varepsilon < \varepsilon_0$, *and, also,* $q_j(\vec{x}, -\dot{\vec{x}}; \varepsilon) = q_j(\vec{x}, \dot{\vec{x}}; \varepsilon)$ $(j = 1, 2, \ldots, n)$.

Then, for ε *sufficiently small and every complex number* a, $|a| < A$, *there exists a solution to the equations* (3.3.6) *of the form*

$$\text{(3.3.7)} \qquad \begin{cases} x_j = |a|\sigma_j^{-1}\sin(\tau t + \varphi) + \varepsilon W_j(\tau t + \varphi; \varepsilon)\,, \\ x_k = \varepsilon W_k(\tau t + \varphi; \varepsilon) \qquad\qquad (k \neq j;\ \ k = 1, 2, \ldots, n), \end{cases}$$

where all $W_j(\tau t + \varphi; \varepsilon)$ $(j = 1, 2, \ldots, n)$ *are analytic functions of* ε *with coefficients which are periodic of period* 2π *in* $\tau t + \varphi$, φ *an arbitrary constant, and* τ *is defined by*

$$\tau = \sigma_j + \sum_{k=1}^{\infty} b_k \varepsilon^k\,,$$

where the b_k *are constants depending on* σ_j, a. *As* j *takes on the values* $1, 2, \ldots, n$, *there exist* n *such systems* (3.3.7) *of periodic solutions, each depending upon the two parameters* $|a| < A$, φ *arbitrary. (Thus,* n *two-manifolds of periodic solutions.)*

Proof. Combining Lemmas (2.1.i) and the preceding Lemma, we have $S_{r1} = \bar{S}_{r2}$, $S_{r1} = -S_{r2}$, and, thus, $\bar{S}_{r1} = -S_{r1}$, or S_{r1}, S_{r2} are purely imaginary for every r. Furthermore, using (1.3.5), we have that $f_1(\tau, \varepsilon, a)$ is purely imaginary for every a, $|a| < A$. Therefore, using (α) of Theorem (2.2.i), we find a solution of the form (3.3.7) for $j = 1$. But, it is clear that in the beginning of our procedure, we could have replaced either of the σ_j $(j = 1, 2, \ldots, n)$ by τ and carried out the above process. Therefore, the Theorem is proved.

3.4. – A third general statement. Let us assume in (3.2.4) that

$$\text{(3.4.1)} \qquad \begin{cases} q_j(\vec{x}, -\dot{\vec{x}}; \varepsilon) = -q_j(\vec{x}, \dot{\vec{x}}; \varepsilon)\,, \\ q_j(-\vec{x}, \dot{\vec{x}}; \varepsilon) = -q_j(\vec{x}, \dot{\vec{x}}; \varepsilon)\,, \end{cases} \qquad (j = 1, 2, 3, \ldots, n).$$

Then each of the numbers $\nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_n$ in (3.2.4) is odd. Moreover, it is easy to show by a simple induction proof that we must have

$$x_{mj} = \sum_{m_1, m_2 \geq 0} \sigma_{mj}^{(m_1, m_2)} x_{01}^{m_1} x_{02}^{m_2}\,, \qquad s_{mj} = \sum_{m_1, m_2 \geq 0} \varrho_{mj}^{(m_1, m_2)} x_{01}^{m_1} x_{02}^{m_2}\,,$$

where $m_1 + m_2$ is odd if m is even and $m_1 + m_2$ is even if m is odd. Furthermore, since there will be no secular terms in the odd approximations, we will have

$$S_{2r+1,1} = S_{2r+1,2} = 0 \qquad (r = 0, 1, 2, \ldots). \tag{3.4.2}$$

We shall now prove the following Lemma.

Lemma 3.4.i. If the preceding algorithm is applied to system (3.2.4), satisfying (3.4.1) and with zeroth approximation given by (3.2.5), we have

$$\sigma^{(m_1,m_2)}_{m,2j-1} = (-1)^m \sigma^{(m_2,m_1)}_{m,2j}, \quad \varrho^{(m_1,m_2)}_{mk} = (-1)^m \varrho^{(m_2,m_1)}_{mk}, \quad S_{m1} = -S_{m2},$$

for every m_1, $m_2 \geq 0$; $m = 0, 1, 2, \ldots$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, 2n$.

Proof. (By induction.) From (3.2.5), we see that $\sigma^{(1,0)}_{0,2j-1} = \sigma^{(0,1)}_{0,2j}$ $(j = 1, 2, \ldots, n)$. Assume that $\sigma^{(m_1,m_2)}_{p,2j-1} = (-1)^p \sigma^{(m_2,m_1)}_{p,2j}$ for all m_1, $m_2 \geq 0$, and $p = 0, 1, \ldots, m-1$. Then

$$x_{p,2j} = \sum_{m_1,m_2 \geq 0} \sigma^{(m_1,m_2)}_{p,2j} x_{01}^{m_1} x_{02}^{m_2} = \sum_{m_1,m_2 \geq 0} (-1)^p \sigma^{(m_1,m_2)}_{p,2j-1} x_{01}^{m_2} x_{02}^{m_1},$$

for $j = 1, 2, \ldots, n$; $p = 0, 1, \ldots, m-1$, and, thus,

$$\begin{cases} x_{p,2j-1} + x_{p,2j} = \sum\limits_{m_1,m_2 \geq 0} \sigma^{(m_1,m_2)}_{p,2j-1} (x_{01}^{m_1} x_{02}^{m_2} + (-1)^p x_{01}^{m_2} x_{02}^{m_1}), \\ x_{p,2j-1} - x_{p,2j} = \sum\limits_{m_1,m_2 \geq 0} \sigma^{(m_1,m_2)}_{p,2j-1} (x_{01}^{m_1} x_{02}^{m_2} + (-1)^{p+1} x_{01}^{m_2} x_{02}^{m_1}). \end{cases} \tag{3.4.3}$$

If we substitute this expression (3.4.3) in expression (3.2.8) for $s_{r,2j-1,k}$, we see that the last sum in (3.2.8) consists of a sum of terms of the form

$$\prod_{\delta=1}^{\nu_l} (x_{01}^{m_{1\delta}} x_{02}^{m_{2\delta}} + (-1)^{\alpha_\delta} x_{01}^{m_{2\delta}} x_{02}^{m_{1\delta}}) \prod_{\lambda=1}^{\mu_l} (x_{01}^{n_{1\lambda}} x_{02}^{n_{2\lambda}} + (-1)^{\beta_\lambda + 1} x_{01}^{n_{2\lambda}} x_{02}^{n_{1\lambda}}), \tag{3.4.4}$$

where $x_{01}^{m_{1\delta}} x_{02}^{m_{2\delta}} + (-1)^{\alpha_\delta} x_{01}^{m_{2\delta}} x_{02}^{m_{1\delta}}$, $x_{01}^{n_{1\lambda}} x_{02}^{n_{2\lambda}} + (-1)^{\beta_\lambda+1} x_{01}^{n_{2\lambda}} x_{02}^{n_{1\lambda}}$ are terms in $x_{\alpha_\delta,2l-1} + x_{\alpha_\delta,2l}$, $x_{\beta_\lambda,2l-1} - x_{\beta_\lambda,2l}$, respectively.

Let us consider any term in the development of (3.4.4), say $x_{01}^{\theta_1} x_{02}^{\theta_2}$. Then, the corresponding term $x_{01}^{\theta_2} x_{02}^{\theta_1}$ will also be in the development of (3.4.4) and the coefficients of these two terms will differ by a factor

$$(-1)^{\alpha_1+\alpha_2+\ldots+\alpha_{\nu_l}} \cdot (-1)^{\beta_1+\beta_2+\ldots+\beta_{\mu_l}} (-1)^{\mu_l} =$$

$$= (-1)^{\alpha_1+\ldots+\alpha_{\nu_l}+\beta_1+\ldots+\beta_{\mu_l}} (-1)^{\mu_l} = (-1)^{r-1} \cdot (-1) = (-1)^r,$$

from (3.2.9) and the fact that μ_l is odd. Therefore, using the same reasoning as for the case of even functions, we see this implies that

$$\varrho_{r,2j-1}^{(m_1,m_2)} = (-1)^r \varrho_{r,2j-1}^{(m_2,m_1)} \qquad (r = 1, 2, ..., m;\ j = 1, 2, ..., n;\ m_1, m_2 \geqslant 0). \tag{3.4.5}$$

In the opening remarks of this section, we have observed that in order to have a term $\varrho_{r1}^{(0+1,0)} x_{01}^{0+1} x_{02}^{0}$, we must have r even. From (3.4.5) this implies $\varrho_{2s,1}^{(0+1,0)} = \varrho_{2s,1}^{(0,0+1)}$. But, from the definition (1.1.9), we see as in the preceding section that $S_{2s,1} = -S_{2s,2}$. Moreover, from (3.4.2), we have

$$S_{r1} = -S_{r2} \qquad (r = 1, 2, ..., m). \tag{3.4.6}$$

If we substitute (3.4.6) in (1.1.17), we have

$$x_{m1} = \sum_{m_1,m_2\geq 0} [\varrho_{m1}^{(m_1,m_2)} - S_{11}\sigma_{m-1,1}^{(m_1,m_2)} - ... - S_{m-1,1}\sigma_{11}^{(m_1,m_2)}][(m_1 - m_2 - 1)i\tau]^{-1} x_{01}^{m_1} x_{02}^{m_2},$$

$$x_{m2} = \sum_{m_1,m_2\geq 0} [\varrho_{m2}^{(m_1,m_2)} + S_{11}\sigma_{m-1,2}^{(m_1,m_2)} + ... + S_{m-1,1}\sigma_{12}^{(m_1,m_2)}][(m_1 - m_2 + 1)i\tau]^{-1} x_{01}^{m_1} x_{02}^{m_2},$$

$$x_{m,2j-1} = \sum_{m_1,m_2\geq 0} \varrho_{m,2j-1}^{(m_1,m_2)}[(m_1 - m_2)i\tau - i\sigma_j]^{-1} x_{01}^{m_1} x_{02}^{m_2},$$

$$x_{m,2j} = \sum_{m_1,m_2\geq 0} \varrho_{m,2j}^{(m_1,m_2)}[(m_1 - m_2)i\tau + i\sigma_j]^{-1} x_{01}^{m_1} x_{02}^{m_2}, \qquad (j = 2, 3, ..., n).$$

Furthermore, since $s_{r,2j-1} = -s_{r,2j}$, we have $\varrho_{m,2j-1}^{(m_1,m_2)} = -\varrho_{m,2j}^{(m_1,m_2)}$, and if we make use of (3.4.5) and our assumption on the $\sigma_{sj}^{(m_1,m_2)}$, we have

$$x_{m2} = \sum_{m_1,m_2\geq 0} [(-1)^m \varrho_{m1}^{(m_1,m_2)} - S_{11}(-1)^{m-1}\sigma_{m-1,1}^{(m_1,m_2)} - ... - S_{m-1,1}^{(m_1,m_2)}(-1)\sigma_{11}^{(m_1,m_2)}] \frac{x_{01}^{m_2} x_{02}^{m_1}}{(m_1 - m_2 - 1)i\tau},$$

$$x_{m,2j} = \sum_{m_1,m_2\geq 0} \frac{(-1)^m \varrho_{m,2j-1}^{(m_1,m_2)} x_{01}^{m_2} x_{02}^{m_1}}{(m_1 - m_2)i\tau - i\sigma_j} \qquad (j = 1, 2, ..., n).$$

As a consequence, since $S_{2r-1,1} = S_{2r-1,2} = 0$, we have $\sigma_{m,2j-1}^{(m_1,m_2)} = (-1)^m \sigma_{m,2j}^{(m_2,m_1)}$, and the induction on the x_{mj} is completed. Finally, (3.4.5) and (3.4.6) hold for all r and the Lemma is proved.

Theorem 3.4.i. *Consider the system of equations*

$$\ddot{x}_j + \sigma_j^2 x_j = \varepsilon q_j(\vec{x}, \dot{\vec{x}}; \varepsilon) \qquad (j = 1, 2, ..., n), \tag{3.4.7}$$

where $\sigma_1, ..., \sigma_n$ are distinct positive numbers such that $m\sigma_j + \sigma_k \neq 0$; $j \neq k$; $j, k = 1, 2, ..., n$; $m = 0, \pm 1, \pm 2, ...,$ and each $q_j(\vec{x}, \dot{\vec{x}}; \varepsilon)$ is analytic for

$|x_j|$, $|\dot{x}_j| < A$, $\varepsilon < \varepsilon_0$, *and, also,*

$$q_j(\vec{x}, -\dot{\vec{x}}; \varepsilon) = -q_j(\vec{x}, \dot{\vec{x}}; \varepsilon), \quad q_j(-\vec{x}, \dot{\vec{x}}; \varepsilon) = -q_j(\vec{x}, \dot{\vec{x}}; \varepsilon), \quad (j = 1, 2, \ldots, n).$$

Then, for ε sufficiently small and every complex number a, $|a| < A$, there exists a periodic solution to the equations (3.4.7) of the form (3.3.7) as in Theorem (3.3.i). As j takes on the values $1, 2, \ldots, n$, there exist n such systems (3.3.7), each depending upon the two parameters $|a| < A$, φ arbitrary. (Thus, n two-manifolds of periodic solutions.)

Proof. The proof is exactly the same as the proof of Theorem (3.3.i).

Example 1. By Theorem (3.3.i), there is a two-manifold of periodic solutions to the second order equation

$$\ddot{x} + \sigma^2 x = \varepsilon f(x, \dot{x}), \qquad f(x, -\dot{x}) = f(x, \dot{x}).$$

In particular, if $x = 0$ is a solution of this equation, then the solution $x = 0$ is stable in the sense of LIAPOUNOFF. The function f may be the function $f(x, \dot{x}) = x + x^2 + \dot{x}^2$.

Example 2. By Theorem (3.4.i), there is a two-manifold of periodic solutions to the equation

$$\ddot{x} + \sigma^2 x + \varepsilon f(x)\dot{x} = 0, \qquad f(-x) = -f(x).$$

As before, the solution $x = 0$ is stable in the sense of LIAPOUNOFF. In particular, we might have $f(x) = x + x^3$.

3.5. – Theorem. *Consider the system of equations*

$$(3.5.1) \qquad \begin{cases} \ddot{x}_1 + \sigma_1^2 x_1 = \varepsilon q_1(x_1, \dot{x}_1), \\ \ddot{x}_j + \alpha_j \dot{x}_j + \sigma_j^2 x_j = \varepsilon q_j(\vec{x}, \dot{\vec{x}}; \varepsilon) & (j = 2, 3, \ldots, n), \\ \dot{x}_\mu + \beta_\mu x_\mu = \varepsilon q_\mu(\vec{x}, \dot{\vec{x}}; \varepsilon) & (\mu = n+1, \ldots, N), \end{cases}$$

where $\sigma_1 > 0$, $\sigma_j \geqslant 0$, $\alpha_j \geqslant 0$, $\beta_\mu > 0$, $\vec{x} = (x_1, \ldots, x_N)$, $\dot{\vec{x}} = (\dot{x}_1, \ldots, \dot{x}_n)$, and $m\sigma_1 + \sigma_j \neq 0$; $j = 2, 3, \ldots, n$; $m = 0, \pm 1, \pm 2, \ldots$, and each q_1, q_j, q_μ is analytic for $|x_k|$, $|\dot{x}_j| < A$, $(k = 1, \ldots, N;\ j = 1, \ldots, n)$. Moreover, we assume that either $q_1(x_1, -\dot{x}_1) = q_1(x_1, \dot{x}_1)$, or $q_1(-x_1, \dot{x}_1) = -q_1(x_1, \dot{x}_1)$, $q_1(x_1, -\dot{x}_1) = -q_1(x_1, \dot{x}_1)$. Then there is a periodic solution of (3.5.1) of the form (3.3.7) as in Theorem (3.3.i) for every $|a| < A$ (a two-manifold of periodic solutions).

Proof. From Theorem (3.3.i), we see that the first equation has a two-manifold of periodic solutions of the form (3.3.7) if $q_1(x_1, -\dot{x}_1) = q_1(x_1, \dot{x}_1)$. If $q_1(-x_1, \dot{x}_1) = -q_1(x_1, \dot{x}_1)$, $q_1(x_1, -\dot{x}_1) = -q_1(x_1, \dot{x}_1)$, then we see from Theorem (3.4.i) that the first equation has a two-manifold of periodic solutions of the form (3.3.7). Since the first equation depends only on x_1, $\dot{x}_1$, the Theorem follows immediately from Theorem (2.2.i).

Example. Consider the system

$$\begin{cases} \ddot{x} + \omega^2 x = 0\,, \\ \ddot{y} - \varepsilon(1 - y^2)\dot{y} + y = \varepsilon x\,, \end{cases}$$

where $m\omega \neq 1$; $m = 0, \pm 1, \ldots$. This system satisfies the conditions of the above Theorem, and, thus, there is a two-manifold of periodic solutions of period $2\pi/\omega$. Furthermore, we see that we have obtained a periodic solution to the equation

$$\ddot{y} - \varepsilon(1 - y^2)\dot{y} + y = \varepsilon a \cos(\omega t + \varphi)\,,$$

for every a, φ.

§ 4. – Appendix.

As remarked after Theorem (2.2.i), the condition (β) of Theorem (2.2.i) is precisely the condition obtained by Coddington and Levinson [**4**] for the existence of a periodic solution of systems of the type (2.1.1). We wish to prove this statement in the present section.

In equations (2.1.1), let $y_1 = -\sigma_1 x_1$, $y_2 = \dot{x}_1$, $y_{2\mu-1} = x_\mu$, $y_{2\mu} = \dot{x}_\mu$, ($\mu = 2, 3, \ldots, n$), $y_{n+j} = x_j$ ($j = n+1, \ldots, N$), and we obtain the system

$$\begin{cases} \dot{y}_1 = -\sigma_1 y_2\,, \\ \dot{y}_2 = \sigma_1 y_1 + \varepsilon q_1(-\sigma_1^{-1} y_1, y_3, \ldots, y_{2n-1}, y_2, y_4, \ldots, y_{2n}; \varepsilon)\,, \\ \dot{y}_{2\mu-1} = y_{2\mu}\,, \\ \dot{y}_{2\mu} = -\sigma_\mu^2 y_{2\mu-1} - \alpha_\mu y_{2\mu} + \varepsilon q_\mu(\ldots)\,, \qquad (\mu = 2, 3, \ldots, n), \\ \dot{y}_{n+j} = -\beta_j y_{n+j} + \varepsilon q_j(\ldots) \qquad (j = n+1, \ldots, N), \end{cases}$$

or,

$$\dot{\vec{y}} = A\vec{y} + \varepsilon\vec{q}(\vec{y}; \varepsilon)\,,$$

and the matrix A is in a form which satisfies the conditions required of the matrix A in Coddington and Levinson [4, p. 30, formula (2.1)]. Moreover, it should be noted that the first component in the vector $\vec{q}$ is equal to zero.

Using the notation of Coddington and Levinson, A is of the form $A = \operatorname{diag}(S_1, C)$ where $S_1 = \begin{Vmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{Vmatrix}$, and C is defined as in [4, p. 22]. Therefore, according to the terminology in [4], the first and second components of the vectors under discussion shall be called the exceptional components. As a consequence, as is pointed out in [4, p. 30, formula (2.5)], the only undetermined constants are c_{01}, c_{02}, and since the above system is independent of t, we may choose $c_{02} = 0$. Thus, equations (2.8) of [4, p. 31] become

$$\begin{cases} \displaystyle\int_0^{2\pi} q_1(\sigma_1^{-1} c_{01} \cos \sigma_1 s, -c_{01} \sin \sigma_1 s, 0, \ldots, 0; 0) \sin \sigma_1 s \, ds = 0 , \\ \displaystyle -c_{01}\nu + \int_0^{2\pi} q^1(\sigma_1^{-1} c_{01} \cos \sigma_1 s, -c_{01} \sin \sigma_1 s, 0, \ldots, 0; 0) \cos \sigma_1 s \, ds = 0 . \end{cases}$$

If we replace $\sigma_1 s$ by $s + (\pi/2)$, we see that these equations become

$$\begin{cases} \displaystyle\int_0^{2\pi} q_1(c_{01}\sigma_1^{-1} \sin s, \ c_{01} \cos s, 0, \ldots, 0; 0) \cos s \, ds = 0 , \\ \displaystyle -c_{01}\nu + \sigma_1^{-1} \int_0^{2\pi} q_1(c_{01}\sigma_1^{-1} \sin s, \ c_{01} \cos s, 0, \ldots, 0; 0) \sin s \, ds = 0 . \end{cases}$$

Condition A, (v) of [4, p. 32] requires that these equations have a solution for some $c_{01} = b$ and $\nu = \nu_0$, and that the Jacobian of these equations with respect to c_{01}, ν be different from zero for these values $c_{01} = b$, $\nu = \nu_0$. But, this Jacobian is precisely

$$[\partial(\int_0^{2\pi} q_1 \cos s \, ds)/\partial c_{01}] \cdot (-c_{01}) ,$$

where $c_{01} = b$. Since the second factor of this product if $\neq 0$, we have

$$| \partial(\int_0^{2\pi} q_1 \cos s \, ds)/\partial c_{01} |_{c_{01}=b} \neq 0$$

which is condition (β).

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