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The basic analogue of a class of polynomials. (**)

A set of polynomials $\{P_n(x)\}$ is called an APPELL set, if

$$P_n'(x) = P_{n-1}(x) \qquad (n = 0, 1, 2, \ldots). \tag{1}$$

SHEFFER [1], and later STEFFENSEN [2], have generalised the APPELL set and have shown how all the properties of an APPELL set hold good for their sets. STEFFENSEN calls his set of polynomials, « poweroids ». These generalisations are obtained by replacing the differential operator D in (1) by the operator J, such that

$$J\{P_n(x)\} = P_{n-1}(x) \qquad (n = 0, 1, 2, \ldots), \tag{2}$$

where (in the notation of STEFFENSEN)

$$J = k_1 D + k_2 D^2 + \ldots \qquad (k_1 \neq 0)$$

the expansion being convergent if the symbol of differentiation D is replaced by a sufficiently small number.

NIELSEN [3] considered a remarkable subset of the APPELL set (1) by considering a set of polynomials which satisfy the two functional equations, for $n = 0, 1, 2, \ldots$,

$$P_n'(x) = P_{n-1}(x)\,, \qquad P_n(-x-1) = (-1)^n P_n(x)\,, \tag{3}$$

and has shown their importance in the theory of BERNOULLI and EULER

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(1) NIELSEN [3] calls the set which satisfies (1) to be harmonic.

numbers. Later MORGAN WARD [4] has generalised the set by considering a set of polynomials $\{Y_n(x)\}$, for which

$$(4) \qquad Y_n'(x) = Y_{n-1}(x), \qquad Y_n(ax+b) = \tau_n Y_n(x),$$

for $n = 0, 1, 2, \ldots$, and a, b are any complex numbers.

We propose the following problem:

Does there exist a subset of the class of polynomials which satisfy (2), such that it satisfies a given functional equation as in (4)?

In its general form the question does not seem to present much interest. But, recently CARLITZ [5] has considered polynomials ([2]) in q^x to obtain what he calls q-BERNOULLI and EULER numbers and q-BERNOULLI and EULER polynomials. He considers the difference operator Δ in relation to these polynomials, such that, if

$$f(x) = \sum_{i=0}^{m} a_i q^{ix},$$

then

$$\Delta f(x) = f(x+1) - f(x) = q^x \sum_{i=1}^{m} (q^i - 1)\, a_i\, q^{(i-1)x}.$$

Now, it is easy to see that the operator D_q, defined by

$$\mathrm{D}_q f(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

where q is any real number $\neq 0$, plays the same role for polynomials in x as the difference operator does for polynomials in q^x.

The object of this paper is to obtain some properties of the class of polynomials $\{H_n(x)\}$ in x, such that

$$(5) \qquad \mathrm{D}_q\{H_n(x)\} = H_{n-1}(x) \qquad (n = 0, 1, 2, \ldots);$$

and to examine some subsets of this class of polynomials which have properties analogous to regular and cyclic sets of NIELSEN [3] and WARD [4].

By the way, we may also observe that it seems possible to arrive at the results of CARLITZ (5) on q-BERNOULLI and EULER numbers if we consider a suitable subset of the class of polynomials satisfying (2) with D replaced by D_q in the operator J of STEFFENSEN. This we have, however, not been able to achieve.

([2]) Polinomials in q^x have been treated earlier by TCHEBICHEF and also later by STIELTJÈS and WIGERT (see SZEGÖ [6], p. 32) in different contexts.

1. – q-harmonic sequences. If $H_n(x) = \sum_{i=0}^{n} a_i x^{n-1}$, we put

$$H_n[ax+b] = a_0(ax+b)_n + a_1(ax+b)_{n-1} + \dots + a_n,$$

$$(ax+b)_n = (ax+b)(ax+bq)\dots(ax+bq^{n-1}),$$

$$H_n[x] = q^{n(n-1)/2} a_0 x^n + q^{(n-1)(n-2)/2} a_1 x^{n-1} + \dots + a_n.$$

We shall use the following notations:

$$[x] = (q^x - 1)/(q-1), \qquad [x]_s = [x][x-1]\dots[x-s+1],$$

$$[x]! = [x][x-1]\dots[1], \qquad [0]! = 1;$$

$$\left[{x \atop s}\right] = [x]_s/[s]!, \qquad \left[{x \atop 0}\right] = 1.$$

It is easy to observe that $H_n[b+ax]$ is not the same as $H_n[ax+b]$.

In analogy with NIELSEN [**3**] *we shall call the polynomials* $\{H_n(x)\}$ *which satisfy* (5) *to be « q-harmonic »* (3). The following properties of q-harmonic sequences are easy to prove:

(i) If $\{H_n(x)\}$ is a q-harmonic sequence then there exists a sequence of constants $\{h_n\}$ such that

$$H_n(x) = h_0 \frac{x^n}{[n]!} + h_1 \frac{x^{n-1}}{[n-1]!} + \dots + h_n, \qquad H_n(0) = h_n.$$

We shall sometimes indicate the two by $[H_n(x), h_n]$.

(ii) If $\{H_n(x)\}$ is a q-harmonic sequence, then

$$D_q\{H_n[x]\} = H_{n-1}[qx] \qquad (n = 0, 1, 2, \dots).$$

(iii) If $\{H_n(x)\}$ is a q-harmonic sequence, then

$$\sum_{n=0}^{\infty} H_n(x)t^n = e_q(xt)\cdot h(t),$$

where

$$e_q(xt) = \sum_{n=0}^{\infty} \frac{x^n t^n}{[n]!} \qquad \text{and} \qquad h(t) = \sum_{n=0}^{\infty} h_n t^n.$$

(3) The term « q-harmonic » used here has nothing to do with that used by HUMBERT in a different context [*Sur les fonctions dans l'hyperspace*, C. R. Acad. Sci. Paris **162**, p. 1264 (1926)].

(iv) If $\{H_n(x)\}$ is a q-harmonic sequence, then we have the two expansions:

$$H_n[x+b] = x^n \frac{H_0[b]}{[n]!} + x^{n-1}\frac{H_1[b]}{[n-1]!} + \dots + x\frac{H_{n-1}[b]}{[1]!} + H_n[b],$$

$$H_n[b+x] = x^n q^{\frac{n(n-1)}{2}} \frac{H_0(b)}{[n]!} + x^{n-1} q^{\frac{(n-1)(n-2)}{2}} \frac{H_1[b]}{[n-1]!} + \dots + H_n[b].$$

If $[H_n(x), h_n]$ and $[K_n(x), k_n]$ are two q-harmonic sequences, then:

(v) there exists a unique sequence $\{\alpha_n\}$, such that, for all n,

$$K_n(x) = \alpha_0 H_n(x) + \alpha_1 H_{n-1}(x) + \dots + \alpha_n H_0(x);$$

(vi) the expression $A_n(x) = \sum_{s=0}^{n} (-1)^s H_{n-s}(x) K_s[x]$ is a constant, while the polynomials $G_n(x) = (1/2^n)\sum_{s=0}^{n} H_{n-s}(x) K_s[x]$ form a new q-harmonic sequence.

To prove (vi) we observe that ([4]) on using property (ii), we have

$$D_q\{A_n(x)\} = \sum_{s=0}^{n} (-1)^s \{H_{n-s}(x)K_{s-1}[qx] + H_{n-s-1}(x)K_s[qx]\} =$$

$$= \sum_{s=0}^{n-1} (-1)^{s+1} H_{n-s-1}(x)K_s[qx] + \sum_{s=0}^{n-1} (-1)^s H_{n-s-1}(x)K_s[qx] = 0.$$

Hence $A_n(qx) = A_n(x)$ for all values of x and since $A_n(x)$ is a polynomial we conclude that $A_n(x)$ is independent of x. Also

$$A_n(x) \equiv A_n = \sum_{s=0}^{n} (-1)^s h_{n-s} k_s.$$

Similarly $(n = 0, 1, 2, \dots)$

$$D_q\{G_n(x)\} = G_{n-1}(x).$$

(vii) Let $[H_n(x), h_n]$ and $[K_n(x), k_n]$ be two q-harmonic sets related by the equation

$$H_n[x] - H_n[-1+x] = K_{n-1}(x),$$

then if only one of the q-harmonic sets is given, the second is completely determined.

([4]) For q-differentiation, we easily have

$$D_q\{\varphi(x)\psi(x)\} = \varphi(x)D_q\{\psi(x)\} + \psi(qx)D_q\{\varphi(x)\}.$$

Writing -1 for x and x for b in the first relation in (iv), we have

$$K_{n-1}(x)=\sum_{s=0}^{n-1}(-1)^{n-s-1}=\frac{H_s[x]}{[n-s]!}=\sum_{s=0}^{n-1}\frac{(-1)^{n-s+1}}{[n-s]!}\sum_{j=0}^{s}\frac{h_{s-j}}{[j]!}q^{\frac{j(j-1)}{2}}x^j,$$

which gives identically

$$k_{n-p}=q^{\frac{(p-1)(p-2)}{2}}\sum_{s=0}^{n-p}\frac{(-1)^s}{[s+1]!}h_{n-p-s}.$$

2. – q(I)-and q(II)-regular sequences (when a is not a root of unity).

If the set of polynomials $\{H_n(x)\}$ satisfies the two functional relations $(n=0,1,2,\ldots)$

(6) $$D_q\{H_n(x)\}=H_{n-1}(x),\qquad H_n[ax+b]=\tau_nH_n(x),$$

then we shall call them « q(I)-regular » sequences.

If in (6), $a=0$ or 1, the set $\{H_n(x)\}$ is trivial. Leaving aside such trivial sequences, we have for q(I)-regular sequences

$$\tau_n=a^n,\qquad \text{and}\qquad H_n[b]=a^nH_n(0).$$

Now suppose $\{H_n(x)\}$ to be a q-harmonic sequence for which

$$H_n[b]=a^nH_n(0)\qquad (n=0,1,2,\ldots).$$

Then

$$H_n[ax+b]=\sum_{s=0}^{n}\frac{a^sx^sH_{n-s}[b]}{[s]!}=\sum_{s=0}^{n}\frac{a^sx^sa^{n-s}H_{n-s}(0)}{[s]!}=a^n\sum_{s=0}^{n}\frac{x^sh_{n-s}}{[s]!}=a^nH_n(x).$$

We have thus proved that:

The necessary and sufficient condition for a q-harmonic sequence $\{H_n(x)\}$ to be q(I)-regular is $(n=0,1,2,\ldots)$

(7) $$H_n[b]=a^nH_n(0).$$

If we expand the left side of (7), we obtain

(7′) $$\begin{cases} h_0 & = h_0\\ \frac{bh_0}{[1]!}+h_1 & = ah_1\\ \frac{qb^2h_0}{[2]!}+\frac{bh_1}{[1]!}+h_2 & = a^2h_2\\ \frac{q^3b^3h_0}{[3]!}+\frac{qb^2h_1}{[2]!}+\frac{bh_2}{[1]!}+h_3 & = a^3h_3\\ \cdots\cdots\cdots\cdots & \\ \frac{q^{n(n-1)/2}b^nh_0}{[n]!}+\frac{q^{(n-1)(n-2)}b^{n-1}h_1}{[n-1]!}+\cdots\cdots+h_n & = a^nh_n. \end{cases}$$

By solving the above equations, we have ($n = 0, 1, 2, \ldots$)

$$h_n = \frac{h_0 b^n \Delta_n(a, q)}{(a-1)(a^2-1)\ldots(a^n-1)}, \tag{8}$$

where (if a is not a root of unity)

$$\Delta_n(a, q) \equiv \begin{vmatrix} \frac{1}{[1]!} & (1-a) & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{q}{[2]!} & \frac{1}{[1]!} & (1-a^2) & 0 & & & & \cdot & \cdot & \cdot \\ \frac{q^3}{[3]!} & \frac{q}{[2]!} & \frac{1}{[1]!} & (1-a^3) & & 0 & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{q^{n(n-1)/2}}{[n]!} & \frac{q^{(n-1)(n-2)/2}}{[n-1]!} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{[1]!} \end{vmatrix}.$$

We find from (8) that (if a is not a root of unity)

$$h_1 = \frac{bh_0}{a-1}, \qquad h_2 = \frac{b^2(1+aq)h_0}{[2]!\,(a-1)(a^2-1)},$$

$$h_3 = \frac{b^3\{(1+aq)(1+aq+a^2q^2) - aq(1-a)(1-q)\}h_0}{[3]!\,(a-1)(a^2-1)(a^3-1)}.$$

We can also define another set of polynomials $\{H_n(x)\}$ *satisfying the two functional relations* ($n = 0, 1, 2, \ldots$)

$$\mathrm{D}_q\{H_n(x)\} = H_{n-1}(x), \qquad H_n[b+ax] = \tau_n H_n[x], \tag{9}$$

and call them « q(II)-regular » sequences.

It is easy to prove that *the necessary and sufficient condition for a q-harmonic sequence* $\{H_n(x)\}$ *to be « q(II)-regular » is that* ($n = 0, 1, 2, \ldots$).

$$H_n(b) = a^n H_n(0). \tag{10}$$

We also similarly have (if a is not a root of unity)

$$h_n = \frac{h_0 b^n \Delta_n(a, q)}{(a-1)(a^2-1)\ldots(a^n-1)}, \tag{11}$$

where

$$\Delta_n(a, q) \equiv \begin{vmatrix} \frac{1}{[1]!} & (1-a) & 0 & \cdots & \cdots & 0 \\ \frac{1}{[2]!} & \frac{1}{[1]!} & (1-a^2) & 0 & \cdots & 0 \\ \frac{1}{[3]!} & \frac{1}{[2]!} & \frac{1}{[1]!} & (1-a^3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{[n-1]!} & \frac{1}{[n-2]!} & \cdots & \cdots & \cdots & (1-a^{n-1}) \\ \frac{1}{[n]!} & \frac{1}{[n-1]!} & \cdots & \cdots & \cdots & \frac{1}{[1]!} \end{vmatrix}.$$

For both q(I)-regular and q(II)-regular sequences we see that $h_1, h_2, \ldots$ are finite only if a is not a root of unity. If, however, $a^r = 1$ [$r \equiv 0 \pmod p$], then it is easy to see from (7′) that an infinite sequence of polynomials $\{H_n(x)\}$ does not exist which satisfies (6) or (9).

3. – $q(T)$-regular sequences and the case when a is a root of unity.

In order to be able to discuss the case when a is a root of unity, we replace the second condition in (6) and (9) by conditions (12) and introduce a special matrix in the following way:

Consider a triangular matrix T of non zero numbers $\{c_{i,n}\}_0^n$ given by [where $c_{n,0} = 1$ $(n = 0, 1, 2, 3, \ldots)$]

$$T \equiv \left\{ \begin{array}{lllll} c_{0,0} & & & & \\ c_{1,0} & c_{0,1} & & & \\ c_{2,0} & c_{1,1} & c_{0,2} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n,0} & c_{n-1,1} & c_{n-2,2} & \cdots & c_{0,n} \end{array} \right.$$

and let us use the following notations:

$$H_n^T(x) = \frac{h_0 c_{n,0}}{[n]!} x^n + \frac{h_1 c_{n-1,1}\, x^{n-1}}{[n-1]!} + \ldots + h_n c_{0,n},$$

$$H_n^T(ax+b) = \frac{h_0}{[n]!} (ax+b)_{T_n} + \frac{h_1}{[n-1]!} (ax+b)_{T_{n-1}} + \ldots + h_n (ax+b)_{T_0},$$

where

$$(ax+b)_{T_n} = a^n x^n c_{n,0} + a^{n-1}x^{n-1}bc_{n-1,1} + \dots + c_{0,n}b^n .$$

Also, if T' is the matrix

$$(T') \equiv \begin{cases} c_{n,0} \\ c_{n-1,0} \quad c_{n-1,1} \\ c_{n-2,0} \quad c_{n-2,1} \quad c_{n-2,2} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_{0,0} \quad c_{0,1} \quad c_{0,2} \quad \dots \quad c_{0,n} , \end{cases}$$

we put

$$H_n^{T'_k}(b) = \frac{h_0 b^k c_{n-k,k}}{[n]_k} + \frac{h_1 b^{k-1} c_{n-k,k-1}}{[n-1]_{k-1}} + \dots + \frac{h_{k-1} b c_{n-k,1}}{[n-k+1]} + h_k c_{n-k,0}$$

and, in particular,

$$H_n^{T'_n}(b) = \frac{h_0 b^n c_{0,n}}{[n]!} + \frac{h_1 b^{n-1} c_{0,n-1}}{[n-1]!} + \dots + h_n c_{0,0} ,$$

$$H_n^{T'_0}(b) = H_n^{T'}(b) = h_0\, c_{n,0} .$$

From these, it is easy to see that

$$H_n^T(ax+b) = \frac{a^n x^n}{[n]!} H_n^{T'}(b) + \frac{a^{n-1}x^{n-1}}{[n-1]!} H_n^{T'_1}(b) + \dots + H_n^{T'_n}(b) .$$

Now, if we consider the class of polynomials which satisfy

(12) $\qquad \mathrm{D}_q\{H_n(x)\} = H_{n-1}(x) \qquad$ and $\qquad H_n^T(ax+b) = \tau_n H_n(x) ,$

we at once have (since $c_{n,0}=1$)

(13) $\qquad \tau_n = a^n \qquad$ and $\qquad H_n^{T'_i}(b) = a^i H_i(0) \qquad (i = 0, 1, \dots, n).$

In this case the equations got from relation (13) easily give $\{h_n\}$ (when a is not a root of unity) in the form

$$h_n = \frac{h_0 b^n \Delta_n(a, T)}{(a-1)(a^2-1)\ldots(a^n-1)}, \tag{14}$$

where

$$\Delta_n(a, T) \equiv \begin{vmatrix} \frac{c_{n-1,1}}{[n]} & (1-a) & 0 & & \ldots & 0 \\ \frac{c_{n-2,2}}{[n]_2} & \frac{c_{n-2,1}}{[n-1]} & (1-a^2) & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{c_{0,n}}{[n]!} & \frac{c_{0,n-1}}{[n-1]!} & \cdot & \cdot & \cdot & c_{0,1} \end{vmatrix}.$$

When $a^r = 1$ $[r \equiv 0 \pmod p]$, *for* $p < n$, *we see that the necessary and sufficient condition in order that the equations* (13) *be consistent is that the following determinant of order* p *vanishes identically for* $\exp(2\pi ik/p)$ $(k = 1, 2, \ldots, p-1)$:

$$\begin{vmatrix} \frac{c_{n-1,1}}{[n]} & (1-a) & 0 & 0 & \ldots & 0 \\ \frac{c_{n-2,2}}{[n]_2} & \frac{c_{n-2,1}}{[n-1]} & (1-a^2) & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{c_{n-p,p}}{[n]_p} & \frac{c_{n-p,p-1}}{[n-1]_{p-1}} & \cdot & \cdot & \cdot & \frac{c_{n-p,1}}{[n-p+1]} \end{vmatrix}. \tag{14'}$$

If all determinants of order p, $2p$, $3p$, ... vanish identically when $a^r = 1$ $[r \equiv 0 \pmod p]$, then there exists solutions giving $\{h_n\}$ for all n.

We call a set of polynomials $\{H_n(x)\}$ *to be «* $q(T)$*-regular » if it satisfies the functional relations* $(n = 0, 1, 2, \ldots)$

$$\mathrm{D}_q\{H_n(x)\} = H_{n-1}(x), \qquad H_n^T(ax+b) = \tau_n H_n(x), \tag{15}$$

and the elements of the matrix T *satisfy the conditions*

$$\frac{c_{n-1,1}}{[n]} = c_{0,1}, \qquad \frac{c_{n-2,2}}{[n]_2} = \frac{c_{0,2}}{[2]!}, \ \ldots, \qquad \frac{c_{1,n-1}}{[n]} = c_{0,n-1}. \tag{16}$$

In this case the matrix T is given by

$$T = \begin{cases} c_{0,0} \\ c_{1,0} \quad c_{0,1} \\ c_{2,0} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} c_{0,1} \quad c_{0,2} \\ c_{3,0} \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} c_{0,1} \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix} c_{0,2} \quad c_{0,3} \\ c_{4,0} \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix} c_{0,1} \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix} c_{0,2} \quad \begin{bmatrix} 4 \\ 3 \end{bmatrix} c_{0,3} \quad c_{0,4} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_{n,0} \quad \begin{bmatrix} n \\ 1 \end{bmatrix} c_{0,1} \quad \begin{bmatrix} n \\ 2 \end{bmatrix} c_{0,2} \quad \begin{bmatrix} n \\ 3 \end{bmatrix} c_{0,3} \quad \cdot \quad \cdot \quad \cdot \quad c_{0,n} \, . \end{cases}$$

Then the conditions (13) can be replaced by

$$H_n^{T'^n}(b) = a^n H_n(0) \qquad (n = 0, 1, 2, \ldots), \tag{17}$$

which is a necessary and sufficient condition for a sequence to be $q(T)$-regular.

The above remarks can be summed up in the following way:

If a is not a root of unity then there is a unique $q(T)$-regular sequence. If a is a primitive p^{th} root of unity ($\neq 1$) then there are infinitely many $q(T)$-regular sequences, only if the matrix T is such that the determinants (14′) of order p, $2p$, $3p$, ... vanish when $a^r = 1$ $[r \equiv 0 \pmod p]$.

The regular sequences for which a is a root of unity are of much greater interest. *We define a $q(T)$-cyclic sequence of order p to mean any solution of* (15) *satisfying the conditions* (16) *for which a is a primitive p^{th} root of unity, i.e. when $a^r = 1$ $[r \equiv 0 \pmod p]$.*

We will now give two examples:

Example 1. Consider the matrix

$$T = \begin{cases} 1 \\ 1 \quad [1] \\ 1 \quad [2] \quad q \\ 1 \quad [3] \quad [3]\, q \quad q^3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 1 \quad [n] \quad \begin{bmatrix} n \\ 2 \end{bmatrix} q \quad \begin{bmatrix} n \\ 3 \end{bmatrix} q^3 \quad \cdot \quad \cdot \quad \cdot \quad q^{n(n-1)/2} \, . \end{cases}$$

In this case

$$H_n^T(ax + b) = H_n[ax + b]$$

and the q-harmonic sequence of polynomials $\{H_n(x)\}$ satisfies the relation

$$H_n[ax + b] = a^n H_n(x) ,$$

if and only if

$$H_n[b] = a^n H_n(0) \qquad (n = 0, 1, 2, \ldots).$$

We have thus arrived again at $q(\mathrm{I})$-regular sequences treated earlier (§ **2**).

Example 2. Consider the matrix

$$T = \left| \begin{array}{llllll} 1 & & & & & \\ 1 & [1] & & & & \\ 1 & [2] & \frac{[2]!}{2!} & & & \\ 1 & [3] & \frac{[3]!}{2!} & \frac{[3]!}{3!} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & [n] & \frac{[n][n-1]}{2!} & \frac{[n][n-1][n-2]}{3!} & \ldots & \frac{[n]!}{n!} \end{array} \right. .$$

In this case, we get (5)

$$h_r = \frac{h_0}{r!}\left(\frac{b}{a-1}\right)^r \qquad (r = 0, 1, 2, \ldots),$$

for all values of a $(\neq 1)$, and $\left(\text{if } \lambda = \dfrac{b}{a-1}\right)$

$$(18) \qquad H_n(x) = h_0\left(\frac{x^n}{[n]!} + \frac{x^{n-1}\lambda}{[n-1]!} + \frac{x^{n-2}\lambda^2}{[n-2]!\,2!} + \ldots + \frac{\lambda^n}{n!}\right).$$

In this case, this is the *simplest* $q(T)$*-regular sequence.*

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(5) It is interesting to observe that this is also the value for h_r obtained by WARD [4]. The polynomials $H_n(x)$ in WARD's case reduce to $(h_0/n!)(x + \lambda)^n$. On taking $q = 1$, we get the result of WARD, as a particular case of our result.

We can prove the following result for $q(T)$-cyclic sequences:

If $[K_n(x), k_n]$ *is a* $q(T)$ *cyclic sequence of order* p, *then* $K_n(x)$ *is uniquely represented by*

$$K_n(x) = \alpha_0 H_n(x) + \alpha_p H_{n-p}(x) + \dots + \alpha_{rp} H_{n-rp}(x) ,$$

where $\{H_n(x)\}$ *is given by* (18) *and* $\alpha_0, \alpha_p, \dots, \alpha_{rp}$ *are constants* $\left[rp \leqslant n < (r+1)p\right]$.

In order to prove this we observe that since $\{H_n(x)\}$ and $\{K_n(x)\}$ are q-harmonic sequences, there exists [by (v)] a unique set $\{\alpha_n\}$ such that

$$K_n(x) = \sum_{r=0}^{n} \alpha_r H_{n-r}(x) .$$

Then

$$K_n^T(ax+b) = \sum_{r=0}^{n} \alpha_r H_{n-r}^T(x) = \sum_{r=0}^{n} \alpha_r a^{n-r} H_{n-r}(x) .$$

Also

$$K_n^T(ax+b) = a^n K_n(x) = a^n \sum_{r=0}^{n} \alpha_r H_{n-r}(x) .$$

Since, $H_{n-r}(x) = \mathrm{D}_q^r\{H_n(x)\}$, we have identically

$$\sum_{r=0}^{n} \alpha_r a^{n-r} \mathrm{D}_q^r \equiv a^n \sum_{r=0}^{n} \alpha_r \mathrm{D}_q^r ,$$

so that

$$\alpha_r(1 - a^r) = 0.$$

We then have $\alpha_r = 0$, for values of r for which $a^r \neq 1$. *Conversely, every sequence* $\{K_n(x)\}$ *of the form*

$$K_n(x) = \alpha_0 H_n(x) + \alpha_p H_{n-p}(x) + \dots + \alpha_{rp} H_{n-rp}(x) ,$$

determines a $q(T)$ *cyclic sequence of order* p.

Putting $x = 0$ in this, we get $\left[rp \leqslant n < (r+1)p\right]$

$$a^{-n} K_n^T(b) = K_n(0) \equiv k_n = \frac{\alpha_0 \lambda^n}{n!} + \alpha_p \frac{\lambda^{n-p}}{(n-p)!} + \dots + \alpha_{rp} \frac{\lambda^{n-rp}}{(n-rp)!} .$$

These equations show that $\alpha_0, \alpha_p, \alpha_{2p}, \ldots$ are determined uniquely, when we know $k_0, k_p, k_{2p}, \ldots$. Also, this relation gives values of k_n for all n, so that knowing $k_0, k_p, \ldots$ all the k_n are uniquely determined. We notice that the set $\{k_n\}$ is the same as given by MORGAN WARD [4].

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