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ROBERT A. GAMBILL (*)

Criteria for parametric instability for linear differential systems with periodic coefficients. (**)

Introduction. In a previous paper [2] ([1]), we have considered linear systems of ordinary differential equations:

$$(1)\qquad \ddot{y}_j + \sigma_j^2 y_j + \lambda \sum_{h}^{n}{}_1 \varphi_{jh}(t) y_h = 0 \qquad (j = 1, 2, \ldots, n),$$

where: (A) $\sigma_1, \ldots, \sigma_n$ are distinct positive numbers; (B) λ is a small real parameter; (C) $\varphi_{jh}(t)$ are real periodic functions of period $T = 2\pi/\omega$, with

$$\int_0^T \varphi_{jh}(t)\,dt = 0, \qquad \varphi_{jh}(t) = \sum_{k=-\infty}^{+\infty} c_{jhk}\, e^{ik\omega t}, \qquad \sum_{k=-\infty}^{+\infty} |c_{jhk}| < C, \quad (j, h = 1, \ldots, n);$$

(D) $m\omega \neq \sigma_j \pm \sigma_h$ $(j, h = 1, \ldots, n;\ m = 1, 2, \ldots)$. As we have pointed out in [2], condition (D) assures that no resonance occurs between the small periodic restoring forces $\lambda \sum_{h=1}^{n} \varphi_{jh}\, y_h$ and the harmonic oscillations of the differential equations $\ddot{y}_j + \sigma_j^2 y_j = 0$.

For $n = 1$, system (1) reduces to the well known HILL equation

$$(2)\qquad \ddot{y} + \sigma^2 y + \lambda \varphi(t) y = 0 .$$

Here, condition (D) reduces to $m\omega \neq 2\sigma$ $(m = 1, 2, \ldots)$ and it is known that under conditions (A), (B), (C), (D), all solutions of (2) are bounded in $(-\infty, +\infty)$ for every λ sufficiently small in absolute value.

(*) Address: U. S. Naval Ordnance Plant, Indianapolis, Indiana, U.S.A..

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([1]) Numbers in brackets refer to the Bibliography at the end of the paper.

However, for systems of the form (1) with $n > 1$, the situation is somewhat different. Conditions (A), (B), (C), (D), are not enough to ensure that every solution of (1) is bounded in $(-\infty, +\infty)$ for $|\lambda|$ sufficiently small, as it has been proved by L. CESARI [1], by an example. Nevertheless, additional general conditions on the matrix $\Phi = \|\varphi_{jh}(t)\|$ assure the boundedness of all solutions of (1) under conditions (A), (B), (C), (D), and for $|\lambda|$ sufficiently small. These additional conditions are essentially conditions of symmetry on the matrix Φ, namely either (α) $\Phi(t)$ is even $[\varphi_{jh}(t) = \varphi_{jh}(-t)$ $(j, h=1, \dots, n)]$, or (β) $\Phi(t)$ is symmetric $[\varphi_{jh}(t) = \varphi_{hj}(t)$ $(j, h = 1, \dots, n)]$ [1], or (γ) $\Phi = \Phi_0 + \Psi$ where $\Phi_0 = \operatorname{diag}(\Phi_1, \dots, \Phi_k)$ is the direct sum of blocks $\Phi_1, \dots, \Phi_k$, each satisfying (α) or (β), and the elements of Ψ on and above [or on and below] the blocks $\Phi_1, \dots, \Phi_k$ of Φ_0 are all zero [2].

We may mention here the concept of parametric stability, in the line of LIAPOUNOFF, used in [2]. We say that the solution $[y_i = 0$ $(i = 1, \dots, n)]$, with $\lambda = 0$, of system (1), is parametrically stable in $(0, +\infty)$ or in $(0, -\infty)$ provided, given $\varepsilon > 0$, there exists a $\delta > 0$, such that for every $|\lambda| < \delta$, $|y_i(0)| < \delta$, $|\dot{y}_i(0)| < \delta$, we have $|y_i(t)| < \varepsilon$, $|\dot{y}_i(t)| < \varepsilon$ for all $0 \leqslant t < +\infty$ $[-\infty < t \leqslant 0]$. If conditions (A), (B), (C), (D) and any one of the conditions (α), (β), (γ), are satisfied, then the solution $[y_i = 0$ $(i = 1, \dots, n)]$ of (1), with $\lambda = 0$, is parametrically stable, as we have seen in [2].

In the present paper, we shall see that if such additional conditions on the matrix $\Phi(t)$ are not satisfied, it is likely that system (1) with $n > 1$ has unbounded solutions in $(0, +\infty)$ no matter how small $|\lambda|$ is $(\lambda \neq 0)$, and that the solution $[y_i = 0$ $(i = 1, \dots, n)]$ of (1), with $\lambda = 0$, is parametrically unstable. (This is a rather unexpected situation since by (D), no particular relation exists between the period of the small periodic restoring forces and the periods of the harmonic oscillations, that is, no resonance is expected.) We shall give a general criterion for this situation, namely, we shall define certain functions P_{1j}, P_{2j}, ... $(j = 1, \dots, n)$ and we shall see that the condition for the mentioned parametric instability is that any one of the expressions P_{1j}, P_{2j}, ... $(j=1, \dots, n)$ is not zero.

We shall use consistently in the next sections the concepts and formulas given in [2]. For the sake of brevity, we shall refer to them by quoting section and formula number.

§ 1. - Explicit expression for the imaginary part of $d_{1,j}$.

We have seen in [2, § 1] that the characteristic exponents $\tau_1, \tau_2, \dots, \tau_{2n}$ of system (1) with conditions (A), (B), (C), (D), are given by the system of equations

$$
(1.1)\qquad \begin{cases} i\sigma_1 = \tau_1 - \lambda d_1(\tau_1, \ldots, \tau_{2n}, \lambda) \\ -i\sigma_1 = \tau_2 - \lambda d_2(\tau_1, \ldots, \tau_{2n}, \lambda) \\ \cdots\cdots\cdots \\ i\sigma_n = \tau_{2n-1} - \lambda d_{2n-1}(\tau_1, \ldots, \tau_{2n}, \lambda) \\ -i\sigma_n = \tau_{2n} - \lambda d_{2n}(\tau_1, \ldots, \tau_{2n}, \lambda), \end{cases}
$$

where each $d_j(\tau_1, \ldots, \tau_{2n}, \lambda)$ is a holomorphic function of λ, and of the $2n$ variables $\tau_1, \ldots, \tau_{2n}$, for $|\lambda|$ sufficiently small, and $\tau_1, \ldots, \tau_{2n}$ belonging to convenient small circles $C_1, \ldots, C_{2n}$ with centers $i\sigma_1, -i\sigma_1, \ldots, i\sigma_n, -i\sigma_n$ respectively. We have already seen in [**2**, § **2**] that if we replace $\tau_1, \ldots, \tau_{2n}$ by $i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n$; $\tau_1, \ldots, \tau_n$ real, then the equations (1.1) are replaced by the $2n$ equations

$$
(1.2)\qquad \begin{cases} i\sigma_1 = i\tau_1 - \dfrac{\lambda i}{2\sigma_1} d_{1,1}(i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n, \lambda) \\ -i\sigma_1 = -i\tau_1 + \dfrac{\lambda i}{2\sigma_1} d_{2,1}(i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n, \lambda) \\ \cdots\cdots\cdots \\ i\sigma_n = i\tau_n - \dfrac{\lambda i}{2\sigma_n} d_{1,n}(i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n, \lambda) \\ -i\sigma_n = -i\tau_n + \dfrac{\lambda i}{2\sigma_n} d_{2,n}(i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n, \lambda). \end{cases}
$$

For the new functions $d_{1,j}(i\tau_1, -i\tau_1, \ldots, i\tau_n, -i\tau_n, \lambda)$ we have seen in [**2**, formula (2.9)], the following expression:

$$
(1.3)\qquad d_{1,j} = \frac{\lambda}{2} \sum_{1}^{n}{}_t \sum_{0}{}_{k_1+k_2} \sum_{1}^{2}{}_u (-1)^{u-1} c_{jtk_1} c_{tjk_2} \{\sigma_t [(-1)^u \tau_t + k_2\omega + \tau_j]\}^{-1} +
$$
$$
+ \frac{\lambda^2}{4} \sum_{1}^{n}{}_{t_1, t_2} \sum_{0}{}_{k_1+k_2+k_3} \sum_{1}^{2}{}_{u_1, u_2} (-1)^{u_1+u_2} c_{jt_1k_1} c_{t_1t_2k_2} c_{t_2jk_3} \{\sigma_{t_1}\sigma_{t_2} [(-1)^{u_1}\tau_{t_1} + (k_2 + k_3)\omega + \tau_j] \cdot
$$
$$
\cdot [(-1)^{u_2} \tau_{t_2} + k_3\omega + \tau_j]\}^{-1} + O(\lambda^3) \qquad (j = 1, \ldots, n),
$$

where the indices k_i are never zero and in the sums, all terms are excluded whose denominators are zero. In (1.3), put $k_1 = -k_2$ in the terms of order λ, and $k_1 = -k_2 - k_3$ in the terms of order λ^2, then using the property that $c_{jhk} = \bar{c}_{jh-k}$, express all sums $\sum_{-\infty}^{+\infty}{}_k$ as sums of the form $\sum_{1}^{\infty}{}_k$. In the so obtained expression, put $c_{jhk} = (a_{jhk} - ib_{jhk})/2$, $k > 0$, where a_{jhk}, b_{jhk} are the coefficients of the real FOURIER series of $\varphi_{jh}(t) = \sum_1^\infty{}_k \, (a_{jhk} \cos k\omega t + b_{jhk} \sin k\omega t)$. We now extract the imaginary part from this expression, obtaining

$$
\begin{aligned}
(1.4)\qquad \operatorname{Im}(d_{1,j}) = {} & \frac{\lambda}{8} \sum_1^n{}_t \sum_1^\infty{}_{k_2} \frac{1}{\sigma_t} (a_{tjk_2} b_{jtk_2} - a_{jtk_2} b_{tjk_2})(-\delta_{k_2}^{tj22} - \delta_{k_2}^{tj11} + \delta_{k_2}^{tj21} + \delta_{k_2}^{tj12}) + \\
& + \frac{\lambda^2}{32} \sum_1^n{}_{t_1,t_2} \sum_1^\infty{}_{k_2,k_3} \Big\{ \frac{1}{\sigma_{t_1}\sigma_{t_2}} (-a_{jt_1,k_2+k_3} a_{t_1t_2k_2} b_{t_2jk_3} - a_{jt_1k_2+k_3} a_{t_2jk_3} b_{t_1,t_2k_2} + \\
& \qquad + a_{t_1t_2k_2} a_{t_2jk_3} b_{jt_1,k_2+k_3} - b_{jt_1,k_2+k_3} b_{t_1t_2k_2} b_{t_2jk_3}) \cdot \\
& \cdot ([\delta_{k_2+k_3}^{t_1j22} + \delta_{k_2+k_3}^{t_1j11}][\delta_{k_3}^{t_2j22} + \delta_{k_3}^{t_2j11}] - [\delta_{k_2+k_3}^{t_1j12} + \delta_{k_2+k_3}^{t_1j21}][\delta_{k_3}^{t_2j12} + \delta_{k_3}^{t_2j21}]) + \\
& + (-a_{jt_1,k_2-k_3} a_{t_1t_2k_2} b_{t_2jk_3} + a_{jt_1,k_2-k_3} a_{t_2jk_3} b_{t_1t_2k_2} - \\
& \qquad - a_{t_1t_2k_2} a_{t_2jk_3} b_{jt_1,k_2-k_3} - b_{jt_1,k_2-k_3} b_{t_1t_2k_2} b_{t_2jk_3}) \cdot \\
& \cdot ([\delta_{-k_2+k_3}^{t_1j22} + \delta_{-k_2+k_3}^{t_1j11}][\delta_{k_3}^{t_2j22} + \delta_{k_3}^{t_2j11}] - [\delta_{k_2-k_3}^{t_1j22} + \delta_{k_2-k_3}^{t_1j11}][\delta_{k_3}^{t_2j12} + \delta_{k_3}^{t_2j21}] \Big\} + O(\lambda^3),
\end{aligned}
$$

where

$$
(1.5)\qquad \begin{cases} \delta_k^{jh11} = (\tau_j + k\omega + \tau_h)^{-1}, & \delta_k^{jh12} = (\tau_j + k\omega - \tau_h)^{-1}, \\ \delta_k^{jh21} = (\tau_j - k\omega + \tau_h)^{-1}, & \delta_k^{jh22} = (\tau_j - k\omega - \tau_h)^{-1}. \end{cases}
$$

We have therefore

$$
(1.6)\qquad \operatorname{Im}(d_{1,j}) = P_{1,j}(\tau_1, \dots, \tau_n)\lambda + P_{2,j}(\tau_1, \dots, \tau_n)\lambda^2 + \dots .
$$

We shall consider now the numbers

$$
(1.7)\qquad P_{k,j} = P_{k,j}(\sigma_1, \dots, \sigma_n) \qquad (j = 1, 2, \dots, n;\; k = 1, 2, \dots)
$$

obtained by replacing the variables $\tau_1, \dots, \tau_n$ by the numbers $\sigma_1, \dots, \sigma_n$ in the coefficients $P_{k,j}$ of λ^k in (1.6).

§ 2. – A criterion for parametric instability.

Theorem I. *Consider the differential system*

$$
(2.1)\qquad \ddot{y}_j + \sigma_j^2 y_j + \lambda \sum_1^n{}_h \varphi_{jh}(t) y_h = 0 \qquad (j = 1, \dots, n),
$$

where the conditions (A), (B), (C), (D) *of the introduction are satisfied. If any of the numbers* $P_{1,j}(\sigma_1, \dots, \sigma_n)$, $P_{2,j}(\sigma_1, \dots, \sigma_n)$, ..., *are not zero, then system* (2.1) *has unbounded solutions in* $(0, +\infty)$, *no matter how small* $|\lambda|$ *is* $(\lambda \neq 0)$, *and the solution* $[y_j = 0 \ (j = 1, ..., n)]$ *with* $\lambda = 0$, *is parametrically unstable.*

Proof. As we have seen in [**2**], the characteristic exponents $\tau_1, ..., \tau_{2n}$ of system (2.1) are given by (1.1). Let us now assume, as we did in [**2**, § **2**] that (τ_1, τ_2), (τ_3, τ_4), ..., (τ_{2n-1}, τ_{2n}), are pairs of complex conjugate, purely imaginary numbers. Then the system of equations (1.1) is replaced by the system (1.2). Let us write (1.2) in the form

$$
(2.2) \qquad \left\{
\begin{aligned}
i\sigma_1 &= \quad i\tau_1 - \frac{\lambda i}{2\sigma_1} \mathrm{Rl}\,(d_{1,1}) + \frac{\lambda}{2\sigma_1} \mathrm{Im}(d_{1,1}) \\
-i\sigma_1 &= -i\tau_1 + \frac{\lambda i}{2\sigma_1} \mathrm{Rl}(d_{2,1}) - \frac{\lambda}{2\sigma_1} \mathrm{Im}(d_{2,1}) \\
& \dots\dots\dots\dots\dots\dots\dots\dots \\
i\sigma_n &= \quad i\tau_n - \frac{\lambda i}{2\sigma_n} \mathrm{Rl}(d_{1,n}) + \frac{\lambda}{2\sigma_n} \mathrm{Im}\,(d_{1,n}) \\
-i\sigma_n &= -i\tau_n + \frac{\lambda i}{2\sigma_n} \mathrm{Rl}(d_{2,n}) - \frac{\lambda}{2\sigma_n} \mathrm{Im}(d_{2,n}).
\end{aligned}
\right.
$$

Now the condition that one of the numbers $P_{kj} = P_{kj}(\sigma_1, ..., \sigma_n)$ is not zero, implies that $\mathrm{Im}\,(d_{1,j}) \neq 0$ for all real τ_j in a convenient small neighborhood of σ_j $(j = 1, ..., n)$ and for all $|\lambda|$ sufficiently small, and thus equations (1.2) cannot have a real solution $\tau_1, ..., \tau_n$. All this can be restated by saying that the hypothesis that the characteristic exponents $\tau_1, ..., \tau_{2n}$ are two by two complex conjugate and purely imaginary has led to a contradiction. Hence, at least one characteristic exponent must have a non-zero real part. Since for systems (2.1), the sum of the characteristic exponents is zero [**2**, § **1**], we have finally that at least one characteristic exponent has a positive real part. Thus, at least one solution of system (2.1) is unbounded in $(0, +\infty)$. Thereby Theorem I is proved.

§ 3. - Examples.

Example I. Consider the system

$$
(3.1) \qquad \left\{
\begin{aligned}
& \ddot{y}_1 + \sigma_1^2 y_1 + \lambda(\sin \omega t + \sin 2\omega t) y_2 = 0 \\
& \ddot{y}_2 + \sigma_2^2 y_2 + \lambda\,(\sin \omega t + 2 \sin 2\omega t) y_1 + \lambda \sin \omega t \cdot y_2 = 0\,.
\end{aligned}
\right.
$$

We see here that $b_{121} = b_{122} = b_{211} = b_{221} = 1$, $b_{212} = 2$, $b_{jhk} = a_{jhk} = 0$ otherwise, and

$$
P_{2,1}(\sigma_1, \sigma_2) = \frac{3}{2}\,\sigma_1 \omega(-2\omega^2 - \sigma_1^2 + \sigma_2^2)\,\delta_2^{2122}\,\delta_2^{2111}\,\delta_2^{2112}\,\delta_2^{2121}\,\delta_1^{2122}\,\delta_1^{2111}\,\delta_1^{2112}\,\delta_1^{2121},
$$

where, in δ_k^{jkuv}, replace everywhere τ_t by σ_t. For a fixed pair (σ_1, σ_2), $P_{2,1}(\sigma_1, \sigma_2)$ is different from zero for all ω with the exception of the value ω for which $-2\omega^2 - \sigma_1^2 + \sigma_2^2 = 0$. Thus, with the possible exception of this particular value of ω, system (3.1) presents parametric instability according to Theorem I. Even at that excepted value of ω, we may say that we have parametric instability provided we consider both λ, ω as parameters. This example shows that condition (α) of the introduction cannot be replaced by the condition that the matrix Φ be odd.

Example II. Consider the system

$$(3.2)\qquad \begin{cases} \ddot{y}_1 + \sigma_1^2 y_1 = \lambda \sin \omega t \cdot y_2 \\ \ddot{y}_2 + \sigma_2^2 y_2 = \lambda \cos \omega t \cdot y_1 . \end{cases}$$

We see here that $b_{121} = a_{211} = -1$, $a_{jhk} = b_{jhk} = 0$ otherwise, and

$$P_{1,1} = \omega\sigma_1 \, \delta_1^{2122} \, \delta_1^{2111} \, \delta_1^{2112} \, \delta_1^{2121} \neq 0 .$$

Thus, system (3.2) presents parametric instability for every ω. This example was studied by L. CESARI [1], for particular values of σ_1, σ_2 and $\omega = 1$. The numerical analysis used there has been replaced here by the general discussion above.

Example III. Consider the system

$$(3.3)\qquad \begin{cases} \ddot{y}_1 + \sigma_1^2 y_1 + \lambda \sin \omega t \cdot y_2 = 0 \\ \ddot{y}_2 + \sigma_2^2 y_2 + \lambda (\sin \omega t - \cos 2\omega t) y_1 + \lambda \cos \omega t \cdot y_2 = 0 . \end{cases}$$

We see here that $a_{221} = b_{121} = b_{211} = 1$, $a_{212} = -1$, $a_{jhk} = b_{jhk} = 0$ otherwise, and the number $P_{2,1}(\sigma_1, \sigma_2)$ relative to (3.3) is the negative of $P_{2,1}(\sigma_1, \sigma_2)$ relative to system (3.1). Hence the discussion of Example I applies here. We remark that the unstable system (3.3) can be obtained from the stable system

$$(3.4)\qquad \begin{cases} \ddot{y}_1 + \sigma_1^2 y_1 + \lambda \cos \omega t \cdot y_2 = 0 \\ \ddot{y}_2 + \sigma_2^2 y_2 + \lambda (\cos \omega t + \cos 2\omega t) y_1 + \lambda \cos \omega t \cdot y_2 = 0 \end{cases}$$

by changing symmetric elements of the matrix

$$\Phi(t) = \begin{pmatrix} 0 & \cos \omega t \\ \cos \omega t + \cos 2\omega t & \cos \omega t \end{pmatrix},$$

relative to (3.4), by a given phase; i.e. let $\varphi_{12}(t) \to \tilde{\varphi}_{12}(t - \pi/(2\omega))$, $\varphi_{21}(t) \to \tilde{\varphi}_{21}(t - \pi/(2\omega))$, $\varphi_{22}(t) \to \tilde{\varphi}_{22}(t)$. We then have the matrix

$$\tilde{\Phi}(t) = \begin{pmatrix} 0 & \sin \omega t \\ \sin \omega t - \cos 2\omega t & \cos \omega t \end{pmatrix}$$

relative to (3.3).

Remark. We have seen in [**2**, § **2**], that the reality of the functions $d_{1,j}$, for all real $\tau_j \in c_j$ $(j = 1, 2, ..., n)$, and $|\lambda|$ sufficiently small, is a sufficient condition for the boundedness of all solutions of (2.1). We remark here, that it follows from the proof of Theorem I, that the reality of the functions $d_{1,j}$ is also necessary for the boundedness of all solutions of (2.1).

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