Rivista Mat. Univ. Parma 6, 65-88 (1955)

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# Hermite polynomials, Hermite functionals and their integrals, in real Hilbert space. (**)

In the quantum theory of fields [1] (1) it appears appropriate to represent the probability amplitudes of certain dynamical variables by points in real HILBERT space and HERMITE functionals defined in the real HILBERT space. The object of this paper is to *develope a mathematical theory of such Hermite functionals and their integrals.*

My method of approach is to express HERMITE functionals and their integrals in $n$-dimensional Euclidean space in a way that does not depend explicitly on the dimension $n$. Such expressions can then be used to define HERMITE functionals and their integrals in a space of aleph-null dimensions, i.e., HILBERT space.

## PART I.

## $n$-dimensional Euclidean space.

We begin with some notations, definitions, and ideas suggested by professor HAROLD GRAD's paper *Note on n-dimensional Hermite polynomials* [2].

Definition I. Given $2n$ dummy indices $i_1, i_2, \ldots, i_{2n}$ the $2n$-th order tensor $\delta^n_{i_1, i_2, \ldots, i_{2n}}$ or $\delta^n$ (where the indices referred to are understood) is the

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(**) Received 15 September, 1954. Condensed from a Thesis (New York University, June 1951).

(1) Numbers in brackets refer to Bibliography at the end of the paper.

sum of all distinct products of KRONECKER $\delta_{i,j}$ of the form

$$\delta_{i_{\lambda_1} i_{\lambda_2}} \delta_{i_{\lambda_3} i_{\lambda_4}} \dots \delta_{i_{\lambda_{2n-1}} i_{\lambda_{2n}}},$$

where $(\lambda_1, \lambda_2, \dots, \lambda_{2n})$ is a permutation of the natural numbers $(1, 2, \dots, 2n)$. Two products are considered identical if one can be obtained from the other by re-arranging the order of the factors and or replacing $\delta_{kl}$ by $\delta_{lk}$, i.e., permuting the subscripts in any one of the factors. Thus there are $(2n)!/(2^n \cdot n!)$ terms in $\delta^n$ and it is symmetric in its $2n$ subscripts:

$$\delta^n_{i_1 \dots i_{2n}} = \frac{1}{2^n \cdot n!} \sum_{(\lambda)} \delta_{i_{\lambda_1} i_{\lambda_2}} \delta_{i_{\lambda_3} i_{\lambda_4}} \dots \delta_{i_{\lambda_{2n-1}} i_{\lambda_{2n}}},$$

where $\sum\limits_{(\lambda)}$ indicates the sum over all permutations $(\lambda_1, \lambda_2, \dots, \lambda_{2n})$ of $(1, 2, \dots, 2n)$.

Let

$$(x, x) = x_1^2 + x_2^2 + \dots + x_n^2, \qquad w = w(x) = e^{-\pi \cdot (x,x)},$$

$$\nabla_i \equiv \frac{\partial}{\partial x_i},$$

then

(1) $$\nabla_i w(x) = -2\pi x_i w(x),$$

(2) $$\Delta_i x_{i_1} x_{i_2} \dots x_{i_n} = \sum_{j=i_1}^{i_n} x_{i_1} x_{i_2} \dots x_{i_n} x_j^{-1} \delta_{ij},$$

(3) $$\int_{E^{(n)}} w(x)\,\mathrm{d}w = 1.$$

Hereafter we shall use the simbol $\int$ to indicate the integral over the $n$-dimensional Euclidean space:

$$\int w(x)\, x_i x_j \,\mathrm{d}x = (2\pi)^{-1} \delta_{ij}$$

and, by mathematical induction using integration by parts and applying (1) and (2), we obtain

(4) $$\int w(x) x_{i_1} x_{i_2} \dots x_{i_{2p}} \,\mathrm{d}x = (2\pi)^{-p} \delta^p_{i_1 i_2 \dots i_{2p}}.$$

If

$$f(x) = \sum_{i_1, \dots, i_p = 1}^{n} a_{i_1 i_2 \dots i_p} x_{i_1} x_{i_2} \dots x_{i_n}$$

is a $p$-ic and

$$g(x) = \sum_{i_1,\dots,i_q=1}^{n} b_{i_1 i_2 \dots i_q} x_{i_1} x_{i_2} \dots x_{i_q}$$

is a $q$-ic, then (4) implies

$$\int \{f(x)\}^2 w(x)\, dx = (2\pi)^{-p} \sum_{\substack{i_1,\dots,i_p=1\\ j_1,\dots,j_p=1}}^{n} a_{i_1 i_2 \dots i_p} a_{j_1 j_2 \dots j_p} \delta^{p}_{i_1 \dots i_p j_1 \dots j_q}, \tag{5}$$

$$\int f(x) g(x) w(x) dx = \begin{cases} (2\pi)^{-(p+q)/2} \sum_{1}^{n} a_{i_1 \dots i_p} b_{j_1 \dots j_q} \delta^{(p+q)/2}_{i_1 \dots i_p j_1 \dots j_q} & \text{if} \quad p+q \text{ is even},\\ 0 \quad \text{if} \quad p+q \text{ is odd}. \end{cases} \tag{6}$$

Notations:

1°)
$$\nabla^{p}_{i_1 \dots i_p} = \frac{\partial^p}{\partial x_{i_p} \partial x_{i_{p-1}} \dots \partial x_{i_1}}.$$

2°) $\sum$ followed by an expression with subscripts means the sum over all permissible values of all the subscripts.

3°) $\sum\limits_{i,j,k,\dots}$ followed by an expression having subscripts $i_1, \dots, i_p$, $j_1, \dots, j_q$, $k_1, \dots, k_r$, etc. indicates that the summation is to be over all permissible values of the subscripts $i_1, \dots, i_p$ and $j_1, \dots, j_q$.

4°) If $(\lambda_1, \lambda_2, \dots, \lambda_p)$ designates a permutation of the integers $1, 2, \dots, p$, then $\sum\limits_{(\lambda)}$ followed by an expression with subscripts $i_1, \dots, i_q, j_{\lambda_1}, j_{\lambda_2}, \dots, j_{\lambda_p}$ indicates the sum over all the permutations $(\lambda_1, \lambda_2, \dots, \lambda_p)$ of $(1, 2, \dots, p)$.

Definition II: *Hermite polinomial in N-dimensional Euclidean space.* The Hermite polynomial of order $p$ denoted by

$$H^{p}_{i_1 \dots i_p} = (-1)^p (2\pi)^{-p} w^{-1} \nabla^{p}_{i_1 \dots i_p} w.$$

$H^{p}_{i_1 \dots i_p}$ is a polynomial of degree $p$ whose highest degree term is $x_{i_1} x_{i_2} \dots x_{i_q}$.

Theorem I. *If $H^{p}_{i_1 \dots i_p}$ is a Hermite polynomial of order $p$ in $n$-dimensional Euclidean space and the set of integers $i_1, \dots, i_p$ is such that $p_1$ of them have the value 1, $p_2$ of them the value 2, ..., $p_n$ the value $n$, then*

$$H^{p}_{i_1 \dots i_p} = H^{p_1}(x_1) H^{p_2}(x_2) \dots H^{p_n}(x_n),$$

*where $H^{p_i}(x_i)$ is the Hermite polynomials of roder $p_i$ in one dimensional space.*

Proof:

$$H^p_{i_1\dots i_p} = (-1)^p(2\pi)^{-p}w^{-1}\nabla^p_{i_1\dots i_p}w = \frac{(-1)^p}{(2\pi)^p w}\frac{\partial^{p_1}e^{-\pi x_1^2}}{\partial x_1^{p_1}}\frac{\partial^{p_2}e^{-\pi x_2^2}}{\partial x_2^{p_2}}\cdots\frac{\partial^{p_n}e^{-\pi x_n^2}}{\partial x_n^{p_n}} =$$

$$= \frac{(\partial p_1/\partial x_1^{p_1})\,e^{-\pi x_1^2}}{(-2\pi)^{p_1}e^{-\pi x_1^2}}\frac{(\partial p_2/\partial x_2^{p_2})\,e^{-\pi x_2^2}}{(-2\pi)^{p_2}e^{-\pi x_2^2}}\cdots\frac{(\partial p_n/\partial x_n^{p_n})\,e^{-\pi x_n^2}}{(-2\pi)^{p_n}e^{-\pi x_n^2}} = H^{p_1}(x_1)\,H^{p_2}(x_2)\dots H^{p_n}(x_n).$$

Theorem II. *Any two Hermite polynomials of different orders, or of the same order referred to different combinations of subscripts, are orthogonal relative to the weight function $w(x)$, i.e.*

$$\int w(x)\,H^p_{i_1\dots i_q}H^q_{j_1\dots j_q}\,\mathrm{d}x = 0$$

*if $p \neq q$ or $(i_1, \dots, i_p)$ is not a permutation of $(j_1, \dots, j_q)$.*

Proof. As in GRAD's paper ([2], page 329, corresponding to (19)), we would have

$$\int w(x)\,H^p_{i_1\dots i_p}H^q_{j_1\dots j_q}\,\mathrm{d}x = (2\pi)^{-p}\sum_{(\lambda)}\delta_{i_1 j_{\lambda_1}}\dots\delta_{i_p j_{\lambda_p}}\,. \tag{7}$$

Theorem III. *Any p-ic $\sum a_{i_1\dots i_p}x_{i_1}x_{i_2}\dots x_{i_p}$ is expressible in the form*

$$\sum_{q=1}^{p}\sum_{i} b_{i_1\dots i_q,q}H^q_{i_1\dots i_q}\,,$$

*and hence any polynomial of degree $p_q$ may be expressed in the form*

$$\sum_{q=1}^{p}\sum_{i} c_{i_1\dots i_q,q}H^q_{i_1\dots i_q}\,.$$

The Theorem follows from $H^q_{i_1\dots i_q}$ being a polynomial in $x_{i_1}, x_{i_2}, \dots, x_{i_q}$ whose highest degree term is $x_{i_1}x_{i_2}\dots x_{i_q}$.

Theorem IV. *If $f(x)$ is a function defined in n-dimensional Euclidean space and*

$$\lim_{\|x\|\to\infty}\left\{\nabla^{q-1-p}_{i_{p+1}\dots i_q}f(x)\,\nabla^p_{i_1\dots i_p}w(x)\right\} = 0 \qquad (p = 0, 1, 2, \dots, q-1),$$

*then*

$$\int f(x)\, w(x)\, H^q_{i_1 i_2 \ldots i_q}\, dx = (2\pi)^{-q} \int w(x)\, \nabla^q_{i_1 \ldots i_q} f(x)\, dx\,.$$

The Theorem is obtained by repeated integration by parts.

Corollary I. If

$$\lim_{\|x\| \to \infty} \left\{ \nabla^{q-1-p}_{i_{k+1} \ldots i_q} f(x)\, \nabla^p_{i_1 \ldots i_p} w(x) \right\} = 0 \qquad (p = 0, 1, 2, \ldots, q-1)$$

and

$$\nabla^q_{i_1 \ldots i_q} f(x) = 0\,,$$

then

$$\int w(x)\, f(x)\, H^q_{i_1 \ldots i_q}\, dx = 0\,.$$

A polynomial of degree less than $q$ is a such function.

Corollary II.

$$\int x_{j_1} \ldots x_{j_q}\, w\, H^p_{i_1 \ldots i_p}\, dx = (2\pi)^{-p} \int w\, \nabla^p_{i_1 \ldots i_p} x_{j_1} \ldots x_{j_q}\, dx =$$

$$= \begin{cases} [(q-p)!\, 2^{(q-p)/2} \cdot \{(q-p)/2\}!\, (2\pi)^{(q+p)/2}]^{-1} \sum\limits_{(\lambda)} \delta_{i_1 j_{\lambda_1}} \ldots \delta_{i_p j_{\lambda_p}} \sum\limits_{(y)} \delta_{j_{y_1} j_{y_2}} \ldots \delta_{j_{y_{q-p-1}} j_{y_{q-p}}} \ (^2) \\ \qquad\qquad \text{if } q > p \text{ and } q-p \text{ is even}\,, \\ 0 \quad \text{if } q < p \text{ or if } q-p \text{ is odd.} \end{cases}$$

If

$$f(x) = \sum_{q=0}^{p} \sum_{i_1, \ldots, i_q = 1}^{n} a_{i_1 \ldots i_q, q} H^q_{i_1 \ldots i_q}\,,$$

where $a_{i_1 \ldots i_q, q}$ is symmetric with respect to $i_1, \ldots, i_q$, by (7)

$$\int w(x)\, f(x)\, H^k_{j_1 \ldots j_k}\, dx = \sum_i a_{i_1 \ldots i_k, k} (2\pi)^{-k} \sum_{(\lambda)} \delta_{i_1 j_{\lambda_1}} \ldots \delta_{i_k j_{\lambda_k}} = k!\, (2\pi)^{-k} a_{j_1 \ldots j_k, k}\,,$$

(8) $$a_{i_1 \ldots i_k, k} = [(2\pi)^k / k!] \int w(x)\, f(x)\, H^k_{i_1 \ldots i_k}\, dx$$

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(2) Where $\sum_{(y)}$ is the sum over all permutations $(y_1, y_2, \ldots, y_{q-p})$ of $(\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_q)$, and $\sum_{(\lambda)}$ is the sum over all permutations $(\lambda_1, \lambda_2, \ldots, \lambda_q)$ of $(1, 2, \ldots, q)$.

and

$$\int w(x)\{f(x)\}^2\,\mathrm{d}x=\sum_{q=0}^{p}\sum_{i_1,\dots,i_q=1}^{n} q!\,(2\pi)^{-q}a^2_{i_1\dots i_q,q}\,. \tag{9}$$

Theorem V. *The operators* $\{x_i-(2\pi)^{-1}\nabla_i\}$ *and* $\{x_j-(2\pi)^{-1}\nabla_j\}$ *are commutative when applied to functions* $f$, *for which* $\nabla_{ij}f$ *is continuous.*

Theorem VI:

$$H^p_{i_1\dots i_p}=\{x_{i_p}-(2\pi)^{-1}\nabla_{i_p}\}\{x_{i_{p-1}}-(2\pi)^{-1}\nabla_{i_{p-1}}\}\dots\{x_{i_1}-(2\pi)^{-1}\nabla_{i_1}\}1\,.$$

The Theorem is established by mathmeatical induction on $p$.

Lemma I:

$$\nabla_i\, x_{i_1}x_{i_2}\dots x_{i_p}=\{(p-1)!\}^{-1}\sum_{(\lambda)}\delta_{ii_{\lambda_1}}x_{i_{\lambda_2}}x_{i_{\lambda_3}}\dots x_{i_{\lambda_p}}\,.$$

Proof. Let

$$u=x_{i_1}\dots x_{i_p}\,,\qquad \frac{1}{u}\frac{\partial u}{\partial x_i}=\sum_{\lambda_1=1}^{p}\delta_{ii_{\lambda_1}}/x_{i_{\lambda_1}}\,,$$

$$\frac{\partial u}{\partial x_i}=\sum_{\lambda_1=1}^{p}\delta_{ii_{\lambda_1}}x_{i_1}x_{i_2}\dots x_{i_p}x^{-1}_{i\lambda_1}=\{(p-1)\}^{-1}\sum_{(\lambda)}\delta_{i_{\lambda_1}i}x_{i_{\lambda_2}}\dots x_{i_{\lambda_p}}\,.$$

Lemma II:

$$\nabla_i\sum_{(\lambda)}x_{i_{\lambda_1}}\dots x_{i_{\lambda_r}}\delta_{i_{\lambda_{r+1}}i_{\lambda_{r+2}}}\dots\delta_{i_{\lambda_{r+2s-1}}i_{\lambda_{r+2s}}}/\{r!\,s!\,2^s\}=$$

$$=\sum_{(\lambda)}\sum_{(\nu)}\delta_{ii_{\nu_1}}x_{i_{\nu_2}}\dots x_{i_{\nu_r}}\delta_{i_{\lambda_{r+1}}i_{\lambda_{r+2}}}\dots\delta_{i_{\lambda_{r+2s-1}}i_{\lambda_{r+2s}}}/\{(r-1)!\,r!\,s!\,2^s\}=\quad(^3)$$

$$=\sum_{(\lambda)}\delta_{ii_{\lambda_1}}x_{i_{\lambda_2}}\dots x_{i_{\lambda_r}}\delta_{i_{\lambda_{r+1}}i_{\lambda_{r+2}}}\delta_{i_{\lambda_{r+3}}i_{\lambda_{r+4}}}\dots\delta_{i_{\lambda_{r+2s-1}}i_{\lambda_{r+2s}}}/\{(r-1)!\,s!\,2^s\}.$$

Theorem VII:

$$H^p_{i_1\dots i_p}=x_{i_1}\dots x_{i_p}-\sum_{(\lambda)}\delta_{i_{\lambda_1}i_{\lambda_2}}x_{i_{\lambda_3}}\dots x_{i_{\lambda_p}}/\{2\cdot(p-2)!\,2\pi\}+$$

$$+\sum_{(\lambda)}\delta_{i_{\lambda_1}i_{\lambda_2}}\delta_{i_{\lambda_3}i_{\lambda_4}}x_{i_{\lambda_5}}\dots x_{i_{\lambda_p}}/\{2^2\cdot 2!\,(p-4)!\,(2\pi)^2\}-$$

$$-\sum_{(\lambda)}\delta_{i_{\lambda_1}i_{\lambda_2}}\delta_{i_{\lambda_3}i_{\lambda_4}}\delta_{i_{\lambda_5}i_{\lambda_6}}x_{i_{\lambda_7}}\dots x_{i_{\lambda_p}}/\{2^3\cdot 3!\,(p-6)!\,(2\pi)^3\}+\dots\,.$$

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(3) Where $\sum_{(\nu)}$ is the sum over permutations $(\nu_1,\dots,\nu_r)$ of $(\lambda_1,\dots,\lambda_r)$, and $\sum_{(\lambda)}$ is the sum over all permutations $(\lambda_1,\dots,\lambda_{r+s})$ of $(1,2,\dots,r+s)$.

Proof. The Theorem is true for $p=1$. Assuming the Theorem for $p=k$, and by Theorem VI,

$$H^{k+1}_{i_1\dots i_{k+1}} = \{x_{i_{k+1}} - (2\pi)^{-1}\nabla_{i_{k+1}}\}H^k_{i_1\dots i_k} =$$

$$= \{x_{i_{k+1}} - (2\pi)^{-1}\nabla_{i_{k+1}}\}\Big[x_{i_1}\dots x_{i_k} - \sum_{(\lambda)} \delta_{i_{\lambda_1} i_{\lambda_2}} x_{i_{\lambda_3}} \dots x_{i_{\lambda_k}} / \{2\cdot(k-2)!\, 2\pi\}^{-1} +$$

$$+ \sum_{(\lambda)} \delta_{i_{\lambda_1} i_{\lambda_2}} \delta_{i_{\lambda_3} i_{\lambda_4}} x_{i_{\lambda_5}} \dots x_{i_{\lambda_k}} / \{2^2\cdot 2!\,(k-4)!\,(2\pi)^2\}^{-1} -$$

$$- \sum_{(\lambda)} \delta_{i_{\lambda_1} i_{\lambda_2}} \delta_{i_{\lambda_3} i_{\lambda_4}} \delta_{i_{\lambda_5} i_{\lambda_6}} x_{i_{\lambda_7}} \dots x_{i_{\lambda_k}} / \{2^3\cdot 3!\,(k-6)!\,(2\pi)^3\}^{-1} + \dots\Big].$$

Applying Lemmas I and II, we obtain

$$H^{k+1}_{i_1\dots i_{k+1}} = x_{i_1}\dots x_{i_{k+1}} - (2\pi)^{-1} \sum_{(\lambda)} \delta_{i_{\lambda_1} i_{\lambda_2}} x_{i_{\lambda_3}} \dots x_{i_{\lambda_{k+1}}} / \{2\cdot(k-1)!\}^{-1} +$$

$$+ (2\pi)^{-2} \sum_{(\lambda)} \delta_{i_{\lambda_1} i_{\lambda_2}} \delta_{i_{\lambda_3} i_{\lambda_4}} x_{i_{\lambda_5}} \dots x_{i_{\lambda_{k+1}}} / \{2^2\cdot 2!\,(k-3)!\}^{-1} -$$

$$- (2\pi)^{-3} \sum_{(\lambda)} \delta_{i_{\lambda_1} i_{\lambda_2}} \delta_{i_{\lambda_3} i_{\lambda_4}} \delta_{i_{\lambda_5} i_{\lambda_6}} x_{i_{\lambda_7}} \dots x_{i_{\lambda_{k+1}}} / \{2^3\cdot 3!\,(k-5)!\}^{-1} + \dots,$$

which is the Theorem for $p=k+1$.

## Part II.

## Real Hilbert space.

Definition III: *Multilinear form.* $A$ is a multilinear form of order $n$ on the real Hilbert space $H$, if:

1. For every $n$ elements $x_1, x_2, \dots, x_n$ of $H$, $A(x_1, \dots, x_n)$ is a complex number and is not identically 0.

2. $A(x_1, \dots, x_{i-1}, x_i+y_i, x_{i+1}, \dots, x_n) =$

$$= A(x_1, \dots, x_n) + A(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$$

for $i = 1, 2, \dots, n$.

3. $$A(x_1, \dots, x_{i-1}, \lambda x_i, x_{i+1}, \dots, x_n) = \lambda A(x_1, \dots, x_n)$$

for any real number $\lambda$ and $i = 1, 2, \dots, n$.

Theorem VIII. *If $A(x_1, \ldots, x_n)$ is a multilinear form, then*

$$A(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) = \lambda_1 \lambda_2 \ldots \lambda_n A(x_1, \ldots, x_n)$$

*and*

$$A(x_1, \ldots, x_{i-1}, \sum \lambda_j x_j, \quad x_{i+1}, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j A(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n).$$

Definition IV: *Boundedness.* A multilinear form $A(x_1, \ldots, x_n)$ is bounded if there exists a constant $C$ such that

$$|A(x_1, \ldots, x_n)| \leqslant C\|x_1\| \|x_2\| \ldots \|x_n\|$$

for every $n$-tuple $(x_1, \ldots, x_n)$ of elements of $H$.

Definition V: *Continuity.* A multilinear form $A(x_1, \ldots, x_n)$ is continuous at $(x_1^0, \ldots, x_n^0)$ if given any $\varepsilon > 0$, there is a $\delta(\varepsilon)$ such that

$$|A(x_1^0 + x_1, \ldots, x_n^0 + x_n) - A(x_1^0, \ldots, x_n^0)| < \varepsilon$$

whenever

$$\|x_i - x_i^0\| = \sqrt{(x_i - x_i^0, x_i - x_i^0)} < \delta(\varepsilon) \qquad \text{for } i = 1, 2, \ldots, n.$$

Theorem IX. *A bounded multilinear form is continuous.*

Theorem X. *If $A(x_1, \ldots, x_n)$ is a continuous multilinear form and $\{\varphi_i\}$ is a complete orthonormal sequence in $H$, then*

$$A(x_1, x_2, \ldots, x_n) = \sum (x_1, \varphi_{i_1})(x_2, \varphi_{i_2}) \ldots (x_n, \varphi_{i_n}) A(\varphi_{i_1}, \ldots, \varphi_{i_n}).$$

Definition VI: *Symmetry.* A multilinear form $A(x_1, \ldots, x_n)$ is symmetric if

$$A(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n) = A(x_1, x_2, \ldots, x_n)$$
$$(i, j = 1, 2, \ldots, n).$$

Definition VII. $A(x, x, \ldots, x)$ is called an $n$-ic if $A$ is a multilinear form of the $n$-th order. A 1-ic is called a *linear* form, a 2-ic a *quadratic* form, etc..

Definition VIII. If $A_0$ is a complex number, $A_1$ is a linear form, ..., and $A_p$ is a $p$-ic, then $\sum_{i=0}^{n} A_i$ is a polynomial of the $n$-th degree.

Definition IX. If $A(x)$ is a $n$-ic, $A(x)$ is continuous if the multilinear form $A(x_1, ..., x_n)$ defining it is continuous, $A(x)$ is bounded if $A(x_1, ..., x_n)$ is bounded and $A(x)$ is symmetric if $A(x_1, ..., x_n)$ is symmetric.

Definition X: *Inner product of two $n$-ics.* If $A_p(x)$ and $B_q(x)$ are $n$-ics of order $p$ and $q$ respectively, we define the inner product with respect to the complete orthonormal system $\{\varphi_i\}$ by

$$(A, B) = \begin{cases} 0 & \text{if} \quad p \neq q\,, \\ \sum A(\varphi_{i_1}, ..., \varphi_{i_q})\overline{B(\varphi_{i_1}, ..., \varphi_{i_p})} & \text{if} \quad p = q\,. \end{cases}$$

$(A, B)$ is also called the inner product of the multilinear forms $A(x_1, ..., x_p)$ and $B(x_1, ..., x_q)$.

Definition XI: *Norm of an $n$-ic or multilinear form.* If $A(x)$ is a $n$-ic defined by the multilinear form $A(x_1, ..., x_n)$, the norm of $A$ denoted $\|A\| = \sqrt{(A, A)}$.

Theorem XI. *If $A(x_1, ..., x_n)$ and $B(x_1, ..., x_n)$ are multilinear forms with finite norms referred to the c. o. s. (complete orthonormal system) $\{\varphi_i\}$, then $(A, B)$ with respect to $\{\varphi_i\}$ is absolutely convergent.*

Proof: $(A, B) \leqslant \|A\| \, \|B\|$ by the Schwarz inequality.

Theorem XII. *If $A$ and $B$ are continuous multilinear forms for which $(A, B)$ is absolutely convergent, $(A, B)$ is independent of the choice of the c.o.s. $\{\varphi_i\}$ with respect to which it is defined.*

The theorem follows from Theorem X and

$$\sum_j |(\varphi_i, \psi_j)|^2 = 1\,, \qquad \text{where} \quad \{\psi_j\} \text{ is a c.o.s.}\,.$$

Not every continuous $n$-ic has a finite norm, i.e., $(x, x)$ is evidently continuous and bounded but has an infinite norm.

Definition XII. A continuous $n$-ic or multilinear form with a finite norm is said to be « strictly bounded ».

Notation. If $A(x_1, ..., x_n)$ is a multilinear form and $\{\varphi_i\}$ is a c.o.s., then $A_{i_1...i_n}$ denotes $A(\varphi_{i_1}, ..., \varphi_{i_n})$ and $x_i$ denotes $(x, \varphi_i)$.

Theorem XIII. *If $A(x, y, ..., w)$ is a strictly bounded multilinear form, $A(x, y, ..., w)$ is bounded and*

$$|A(x, y, ..., w)| \leqslant \|A\| \, \|x\| \, \|y\| ... \|w\| .$$

Theorem XIV. *For any given $n$ the set $C_n$ of strictly bounded $n$-ics and the constant 0 form a Hilbert space.*

Proof. A. $C_n$ is a linear space by its definition and the SCHWARZ inequality.

B. $(A, B)$ has the properties of a scalar product.

C. The set multilinear forms $(x_1, \varphi_{i_1})$, $(x_2, \varphi_{i_2})$, ..., $(x_n, \varphi_{i_n})$ form a sequence of strictly bounded multilinear forms, each form corresponding to a set $(i_1, ..., i_n)$ of $n$ natural numbers, such that any $k$ of the forms are linearly independent if $\{\varphi_i\}$ is an c.o.s..

D. The set of $n$-ics of the form

$$\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_n=1}^{n_n} a_{i_1...i_n} x_{i_1} x_{i_2} ... x_{i_n} ,$$

where $a_{i_1...i_n}$ is a complex number with rational real and imaginary parts, form a denumerable everywhere dense set of elements of $C_n$.

E. $C_n$ is complete.

MURRAY and VON NEUMANN [3] have given a different proof of Theorem XIV.

Theorem XV. *If $A(x_1, ..., x_n) = \sum A_{i_1...i_n}(x_1, \varphi_{i_1}) ... (x_n, \varphi_{i_n})$ is a continuous multilinear form and $\{\varphi_i\}$ a c.o.s., then*

$$\sum |A_{i_1...i_n}(x_1, \varphi_{i_1}) ... (x_n, \varphi_{i_n})| \leqslant \|A\| \, \|x_1\| ... \|x_n\| .$$

Corollary. If $A(x) = \sum A_{i_1...i_n} x_{i_1} ... x_{i_n}$ is a strictly bounded $n$-ic, $\sum |A_{i_1...i_n} x_{i_1} ... x_{i_n}|$ converges for every $x$ in $h$.

Definition XIII. If $\sum\limits_{i=0}^{n} A_i$ and $\sum\limits_{i=0}^{m} B_i$ are polynomials of degree $m$ and $n$, where $A_i$ and $B_i$ are $i$-ics, their inner product

$$(\sum_{i=0}^{n} A_i, \sum_{i=0}^{m} B_i) = \sum_{i=0}^{l} (A_i, B_i)$$

where $l = \min(n, m)$, and

$$(A_0, B_i) = (A_i, B_0) = \begin{cases} 0 & \text{if } i \neq 0, \\ A_0 B_0 & \text{if } i = 0. \end{cases}$$

Definition XIV: *Power series and their inner products.* If $A_0$ is a complex number and $A_i$ is an $i$-ic for $i \geqslant 1$, we shall call the formal sum $\sum_{i=0}^{\infty} A_i$ a pover series. If $\sum A_i$ and $\sum B_i$ are two power series or a power series and a polynomial, then the inner product

$$(\sum A_i, \sum B_i) = \sum (A_i, B_i).$$

Theorem XVI. *The space C consisting of all polynomials of finite norms and power series of finite norms is a Hilbert space.*

This follows from Theorem 1.27 in [4].

Definition XV. If $A(x)$ is a symmetric continuous $n$-ic, the partial derivative of $A$ with respect to $x_j$, referred to $\{\varphi_i\}$,

$$\frac{\delta A}{\delta x_j} = nA(x, \ldots, x, \varphi_j).$$

If $A$ is a complex number, then $\delta A / \delta x_j = 0$. If $P = \sum_r A_r$ is a polynomial or a power series, then

$$\frac{\delta P}{\delta x_j} = \sum_r \frac{\delta A_r}{\delta x_j}.$$

Definition XVI: *Trace or Laplacian.* If $A$ is a continuous symmetric $n$-ic, then the trace or Laplacian of $A$ is defined as a generalization of von Neymann's [8] trace of a quadratic form by

$$\mathrm{T}A = \frac{1}{4\pi} \sum_j \frac{\delta A}{\delta x_j} = \frac{n(n-1)}{4\pi} \sum A(x, x, \ldots, x, \varphi_j, \varphi_j).$$

We define

$$\frac{\delta(\mathrm{T}A)}{\delta x_i} = \frac{n(n-1)}{4\pi} \sum_j \frac{\delta}{\delta x_i} A(x, \ldots, x, \varphi_j, \varphi_j) .$$

We define $\mathrm{T}^0 A$ as $A$ and $\mathrm{T}^k A$ is defined inductively for $k > 1$ by

$$\mathrm{T}^k A = \mathrm{T}\mathrm{T}^{k-1} A = \frac{n!}{(n-2k)!\,(4\pi)^k} \sum A(x, x, \ldots, x, \varphi_{i_1}, \varphi_{i_1}, \varphi_{i_2}, \varphi_{i_2}, \ldots, \varphi_{i_k}, \varphi_{i_k}) .$$

If $\sum A_r$ is a polynomial or a power series, $\mathrm{T} \sum A_r = \sum \mathrm{T} A_r$.

Definition XVII: *Gradient.* If $A$ is a continuous symmetric $n$-ic for which $\sum |\delta A / \delta x_j|^2 < \infty$, the gradient of $A$,

$$\nabla A = \sum \frac{\delta A}{\delta x_j} \varphi_j = n \sum A(x, \ldots, x, \varphi_j) \varphi_j .$$

If $A$ is a complex number, $\nabla A = 0$. If $\sum_r A_r$ is a polynomial or power a series, $\nabla \sum_r A_r = \sum_r \nabla A_r$. Moreover

$$\nabla \mathrm{T}^k A = \frac{n!}{(n-2k-1)!\,(4\pi)^k} \sum A(x, x, \ldots, x, \varphi_j, \varphi_{i_1}, \varphi_{i_1}, \ldots, \varphi_{i_k}, \varphi_{i_k}) \varphi_j .$$

A $n$-ic may be strictly bounded and not have a finite trace.

Theorem XVII. *If* $\mathrm{T}A$ *is absolutely convergent, it is independent of the c.o.s. used to define it.*

Theorem XVIII. $\nabla A(x)$ *maps every element* $x$ *of* $H$ *into an element of* $H$ *and this mapping is independent of the c.o.s. used to define* $\nabla A$.

Theorem XIX. *A symmetric strictly bounded* $n$-*ic has a gradient.*

Proof:

$$\sum |\delta A / \delta x_j|^2 \leq \sum_j n^2 \{ \sum_i |A_{i_1 \ldots i_{n-1}, j}\, x_{i_1} \ldots x_{i_{n-1}}| \}^2 \leq n^2 \|x\|^{2(n-1)} \|A\|^2 .$$

Corollary. If $L(x)$ is a bounded linear form on $H$, $\nabla L$ exists and is equal to $\sum L_i \varphi_i$.

Theorem XX. *If* $A$ *is a continuous symmetric* $n$-*ic and* $\nabla A$ *exists,* $A(x) - A(y) = (\nabla A(y), x - y) + r(y, x - y)$ *where* $r(y, x-y)$ *is a numerically* *valued function for which*

$$\lim_{x\to y} r(y,\, x-y)/\|x-y\| = 0\,.$$

This is the GOLOMB [6] and ROTHE [7] definition of gradient.

Theorem XXI. *If $A$ is a continuous symmetric $n$-ic,*

$$\frac{\delta^2 A}{\delta x_k\,\delta x_j} = \frac{\delta^2 A}{\delta x_j\,\delta x_k}\,,$$

$$\frac{\delta}{\delta x_k}(\mathrm{T}A) = \mathrm{T}\left(\frac{\delta A}{\delta x_k}\right) \quad\text{and}\quad \frac{\delta}{\delta x_j}(\mathrm{T}^k A) = \mathrm{T}^k\left(\frac{\delta A}{\delta x_j}\right).$$

Definition XVIII. The HERMITE polynomial $H^n(A)$ associated with the continuous symmetric $n$-ic $A(x)$ is defined by the formal series

$$H^n(A) = A(x) - \mathrm{T}A + \frac{\mathrm{T}^2 A}{2!} - \frac{T^3 A}{3!} + \dots + (-1)^{[n/2]}\frac{T^{[n/2]}A}{[n/2]!}\,.$$

We shall consider the HERMITE polynomial associated witha $n$-ic regardless of whether the traces in its definition converge or not. The HERMITE polynomial need not be a functional on $H$, but is a symbol for an indicated finite sum, some of whose terms are formal series which may or may not converge. If $\mathrm{T}^k A$ converges absolutely for $k = 0, 1, 2, \dots, [n/2]$, then $H^n(A)$ is absolutely convergent and is a functional on our real HILBERT space.

For applications to the quantum theory of fields, it is important to define the following operators on $H^n(A)$, where $L(x)$ is a linear form and $a$ a complex number:

$$L(x)\,H^n(A) = L(x)\,A(x) - L(x)\,\mathrm{T}A + L(x)\,\frac{\mathrm{T}^2 A}{2!} + \dots + (-1)^{[n/2]}L(x)\,\frac{\mathrm{T}^{[n/2]}A}{[n/2]!}\,,$$

$$aH^n(A) = H^n(aA)\,,$$

$$\frac{\delta}{\delta x_i}H^n(A) = \frac{\delta A}{\delta x_i} - \frac{\delta \mathrm{T}A}{\delta x_i} + \frac{1}{2!}\,\frac{\delta \mathrm{T}^2 A}{\delta x_i} - \dots\,,$$

$$\mathrm{T}^k H^n(A) = \mathrm{T}^k A - \mathrm{T}^{k+1}A + \frac{\mathrm{T}^{k+2}A}{2!} - \frac{\mathrm{T}^{k+3}A}{3!} + \dots\,,$$

$$\nabla H^n(A) = \nabla A - \nabla \mathrm{T}A + \frac{\nabla \mathrm{T}^2 A}{2!} - \dots\,,$$

$$(\nabla L,\, \nabla H^n(A)) = (\nabla L,\, \nabla A) - (\nabla L,\, \nabla \mathrm{T}A) + \frac{(\nabla L,\, \nabla \mathrm{T}^2 A)}{2!} - \dots\,.$$

Theorem XXII. *If $A(x)$ is a continuous symmetric n-ic, then*

$$\frac{\delta}{\delta x_j} H^n(A) = H^{n-1}\left(\frac{\delta A}{\delta x_j}\right).$$

Corollary. If $A(x)$ is a continuous symmetric $n$-ic and $T^k A$ is a continuous symmetric $(n-2k)$-ic, then

$$T^k H^n(A) = H^{n-2k}(T^k A).$$

Theorem XXIII. *If $L$ is a continuous linear form, and $A$ is a continuous $n$-ic for which $T^k A$ is a $(n-2)$-ic whose gradient is defined for $k=1, 2, \dots, [n/2]$, then $(\nabla L, \nabla HA) = H(\nabla L, \nabla A)$.*

Definition XIX. If $L(x)$ is a bounded linear form and $A(x)$ is a continuous symmetric $n$-ic, the symmetric part of the multilinear form $L(x_{n+1})A(x_1, \dots, x_n)$ denoted by

$$\mathrm{Sy}\, LA(x_1, \dots, x_{n+1}) = \frac{1}{n+1}\sum_{\lambda=1}^{n+1} L(x_\lambda)A(x_1, x_2, \dots, x_{\lambda-1}, x_{\lambda+1}, x_{\lambda+2}, \dots, x_{n+1}).$$

The symmetric continuous $n$-ic $\mathrm{Sy} LA(x)$ is called the symmetric form of $LA(x)$.

Theorem XXIV. *If $L(x)$ is a b.l.f. and $A(x)$ a continuous symmetric n-ic for which $T^k(\mathrm{Sy}\, LA)$ is absolutely convergent, then*

$$\frac{T^k(\mathrm{Sy}\, LA)}{k!} = \frac{1}{2\pi}\left(\nabla L, \frac{\nabla T^{k-1}A}{(k-1)!}\right) + L(x)\frac{T^k A}{k!}.$$

Proof:

$$\frac{T^k(\mathrm{Sy}\, LA)}{k!} =$$

$$= \frac{(n+1)!\,2k}{(n+1-2k)!\,k!\,(n+1)(4\pi)^k}\sum L(\varphi_{i_1})A(\varphi_{i_1}, \varphi_{i_2}, \varphi_{i_2}, \varphi_{i_3}, \varphi_{i_3}, \dots, \varphi_{i_k}, \varphi_{i_k}, x, x, \dots, x) +$$

$$+ \frac{(n+1)!\,(n+1-2k)}{(n+1-2k)!\,k!\,(n+1)(4\pi)^k}\sum L(x)A(\varphi_{i_1}, \varphi_{i_1}, \varphi_{i_2}, \varphi_{i_2}, \dots, \varphi_{i_k}, \varphi_{i_k}, x, x, \dots, x).$$

Theorem XXV. *If $L(x)$ is a b.l. form and $A(x)$ is a continuous symmetryc n-ic for which $T^k(\mathrm{Sy}\, LA)$ is absolutely convergent for $k = 1, 2, 3, \dots$, then*

$$L(x)H^n(A) = H^{n+1}(A)(\mathrm{Sy}\, LA) + (\nabla L, \nabla H(A))/(2\pi).$$

Definition XX. The integral of $H^p(A)\cdot\overline{H^q(B)}$ denoted

$$\mathrm{I}[H^p(A)\cdot\overline{H^q(B)}] = \{p!/(2\pi)^p\}(A, B).$$

In particular if $A = B$:

$$\mathrm{I}[|H^p(A)|^2] = \{p!/(2\pi)^p\}\|A\|^2.$$

$\mathrm{I}[|H^p(A)|^2]$ may exist even though $H^p(A)$ is not a functional, i.e., if some of the traces do not converge. If $A_i$ and $B_i$ are continuous symmetric $n$-ics of order $p_i$ and $q_i$ respectively for $i = 1, 2, 3, \ldots, n$ or $i = 1, 2, 3, \ldots,$ we define

$$\mathrm{I}[(\sum H^{p_i}(A_i))\,(\overline{\sum H^{q_i}(B_i)})] = \mathrm{I}[\sum H^{p_i}(A_i)\cdot\overline{H^{q_j}(B_j)}] = \sum \mathrm{I}[H^{p_i}(A_i)\cdot\overline{H^{q_i}(B_j)}].$$

In particular, if $f(x) = \sum_p H^p(A_p)$, where $A_p$ is a continuous symmetric $p$-ic,

$$\mathrm{I}[|f(x)|^2] = \sum_p \mathrm{I}[|H^p(A_p)|^2] = \sum_p \{p!/(2\pi)^p\}\|A_p\|^2.$$

If $a$ and $b$ are complex numbers, we define $\mathrm{I}[|a|^2] = |a|^2$ and $\mathrm{I}[a\bar{b}] = a\bar{b}$. By Theorem XIV the strictly bounded $n$-ics form a HILBERT space $D_n$ with the inner product $\mathrm{I}[H^n(A)\cdot\overline{H^n(B)}]$. Corresponding to Theorem XVI, the space $D$ of all functions $f(x) = \sum_p H^p(A_p)$ with inner product $\mathrm{I}[f(x)\cdot\overline{g(x)}]$ is a HILBERT space, where $A_p$ is a strictly bounded $p$-ic.

Theorem XXVI. *If $L$ is a bounded linear form and $A$ is a strictly bounded $n$-ic, then*

(1) $$\mathrm{I}[|H^{n+1}(\mathrm{Sy}\,LA)|^2] \leqslant \{(n+1)/(2\pi)\}\|L\|^2\,\mathrm{I}[|H^n(A)|^2],$$

(2) $$\mathrm{I}[|H^{n-1}(\nabla L, \nabla A)|^2] \leqslant 2\pi n\|L\|^2\,\mathrm{I}[|H^n(A)|^2].$$

Proof:

(1) $$\|\mathrm{Sy}\,LA\| \leqslant \|LA\| = \|L\|\,\|A\|,$$

(2) $$\|(\nabla L, \nabla A)\|^2 \leqslant n^2\|L\|^2\|A\|^2.$$

Theorem XXVI. *If $L(x)$ is a b.l. form, $A_q(x)$ is a continuous symmetric $q$-ic for which $\mathrm{T}^k(\mathrm{Sy}\,LA_q)$ is absolutely convergent and $\mathrm{T}^kA_q$ has a gradient for*

*$q, k = 1, 2, 3, \ldots$ and*

$$\sum_{q=1}^{\infty} q\mathrm{I}[|H(A_q(x))|^2] < \infty, \qquad f(x) = \sum_{q=1}^{\infty} H(A_q),$$

*then*

$$L(x)f(x) = a_0 + \sum_{i=1}^{\infty} a_i H^i(B_i),$$

*where $B_i$ is an i-ic and $a_i$ a complex number and* $\mathrm{I}[|L(x)f(x)|^2] < \infty$.

Theorem XXVII. *If $A(x)$ is a continuous symmetric n-ic on a real Hilbert space $D$ and $h$ is an element of $D$,*

(1) $$A(x+h) = A(x) + nA(x, \ldots, x, h) + {}^nC_2 A(x, \ldots, x, h, h) + \\ + \ldots + nA(x, h, \ldots, h) + A(h).$$

(2) *The functional obtained by replacing $x$ by $x+h$ in $\delta A(x)/\delta x_j$ is equal to the functional obtained by applying the operator $\delta/\delta x_j$ to the polynomial (1) by expanding $A(x+h)$.* Hence this functional may be denoted unambisguously by $\delta A(x+h)/\delta x_j$.

Corollary I. If $A(x)$ is a continuous symmetric *n-ic* whose trace $\mathrm{T}A(x)$ is a strictly bounded $(n-2)$-ic, then the functional obtained by replacing $x$ by $x+h$ in $\mathrm{T}A(x)$ is equal to the trace of (1) in the Theorem. Hence $\mathrm{T}A(x+h)$ is unambiguous and equal to

$$\mathrm{T}A(x) + n\mathrm{T}A(x, \ldots, x, h) + \ldots + \mathrm{T}A(x, x, h, \ldots, h).$$

Corollary II. If $A(x)$ is a continuous symmetric $n$-ic all of whose traces are strictly bounded and $H^n A(x+h)$ denotes the value of the functional $H^n A(x)$ when $x$ is replaced by $x+h$, then

$$H^n A(x+h) = H^n A(x) + nH^{n-1}A(x, \ldots, x, h) + {}^nC_2 H^{n-2}A(x, \ldots, x, h, h) + \ldots = \\ = A(x+h) - \mathrm{T}A(x+h) + \frac{\mathrm{T}^2 A(x+h)}{2!} - \frac{\mathrm{T}^3 A(x+h)}{3!} + \ldots.$$

Definition XXI. If $A(x)$ is a continuous symmetric $n$-ic, we define

$$H^n A(x+h) = H^n A(x) + nH^{n-1}A(x, \ldots, x, h) + {}^nC_2 H^{n-2}A(x, \ldots, x, h, h) + \ldots.$$

Corollary III:

$$\mathrm{I}[|H^nA(x+h)|^2] = \mathrm{I}[|H^nA(x)|^2] + n^2\mathrm{I}[|H^{n-1}A(x, ..., x, h)|^2] + \\ + {}^nC_2^2\,\mathrm{I}[|H^{n-2}A(x, ..., x, h, h)|^2] + \ldots .$$

Corollary IV:

$$\mathrm{I}[|H^nA(x+h)|^2] \leqslant \mathrm{I}[|H^nA(x)|^2]\left(1 + \frac{\|h\|^2}{2\pi}\right)^n.$$

Corollary V:

$$\mathrm{I}[|H^nA(x+h)|^2] \to \mathrm{I}[|H^nA(x)|^2] \qquad \text{as} \qquad \|h\| \to 0\,.$$

Corollary VI (TAYLOR's Theorem). If $A(x)$ is a continuous symmetric $n$-ic, then

$$A(x+\beta\varphi_i) = A(x) + \beta\frac{\delta A}{\delta x_i} + \frac{\beta^2}{2!}\frac{\delta^2 A}{\delta x_i^2} + \ldots + \frac{\beta^n}{n!}\frac{\delta^n A}{\delta x_i^n}\,.$$

Corollary VII. If $A(x)$ is a continuous symmetric $n$-ic, then

$$A(x_1 + x_2 + \ldots + x_k) = \sum_{i_1+i_2+\ldots+i_k=n} \frac{n!}{i_1!\,i_2!\ldots i_k!} A(\underbrace{x_1, \ldots, x_1}_{i_1\,\text{arguments}}, \underbrace{x_2, \ldots, x_2}_{i_2\,\text{arguments}}, \ldots, \underbrace{x_k, \ldots, x_k}_{i_k\,\text{arguments}})\,.$$

Definition XXII. If $A(x)$ is a continuous symmetric $n$-ic, the HERMITE function associated with $A(x)$, denoted by $J^nA(x)$ is defined by

$$J^nA(x) = H^nA(x)\,e^{-\pi\cdot(x,x)/2}\,,$$

where the symbol $=$ is interpreted as numerical equality when $H^nA(x)$ is a functional and as formally identical when $H^nA(x)$ is a formal sum.

If $A(x)$ is a $p$-ic and $B(x)$ is a $q$-ic, we define the integral

$$\int J^pA(x)\overline{J^qB(x)}\,\mathrm{d}x = \mathrm{I}[H^pA(x)\overline{H^qA(x)}].$$

If

$$f(x) = \sum_p J^pA_p(x) \qquad \text{and} \qquad g(x) = \sum_q J^qB_q(x)\,,$$

where the sums may be numerical or merely formal sums of a finite or infinite series of HERMITE functions, then

$$\int f(x)\overline{g(x)}\,\mathrm{d}x = \sum_{p,q}\int J^pA_p(x)\overline{J^qB_q(x)}\,\mathrm{d}x\,, \qquad \int|f(x)|^2\,\mathrm{d}x = \sum_p\int|J^pA_p(x)|^2\,\mathrm{d}x\,,$$

$$J^nA(x+y) = H^nA(x+y)e^{-(\pi/2)(x+y,\,x+y)}\,.$$

Theorem XXVIII. *If $\{\varphi_i\}$ is a c.o.s. in the real Hilbert space $H$, $E_n$ is the linear manifold spanned by $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $A$ and $B$ are continuous symmetric n-ics of order $p$ and $q$, respectively, and*

$$\int_H J^p(A)\overline{J^q(B)}\,\mathrm{d}x = \lim_{n\to\infty}\int_{E_n} H^p(A)\overline{H^q(B)}\,e^{-\pi\cdot(x,x)}\,\mathrm{d}x\,,$$

*where in the left member of the equation $(x, x) = x_1^2 + \ldots + x_n^2$,*

$$H^p(A) = \sum_{i_1\ldots i_p=1}^{n} A_{i_1\ldots i_p} H^p_{i_1\ldots i_p}(x)\,,$$

*$H^p_{i_1\ldots i_p}$ being the basic Hermite polynomial of order $p$ in $E_n$.*

The Theorem follows from formula (7) in the proof of Theorem II.

Corollary. If $f(x) = \sum_n J^n A_n$, then

$$\int_H |f(x)|^2\,\mathrm{d}x = \lim_{n\to\infty}\int_{E_n} |f(x)|^2\,\mathrm{d}x\,.$$

Theorem XXIX. *If $f(x) = \sum_{n=0}^{N} J^n A_n(x)$, where $A_n(x)$ is a continuous symmetric n-ic on the real Hilbert space $H$, for which $\int_H |J^n A_n(x)|^2\,\mathrm{d}x$ converges for $n = 1, 2, 3, \ldots, N$; $h$ is any element of $H$, and $H_{\perp_h}$ the Hilbert space which is the orthogonal complement of $h$, then $\int_{H_{\perp_h}} |f(x+h)|^2\,\mathrm{d}x$ converges to a positive real number.*

Proof. Since $(x, h) = 0$ on $H_{\perp_h}$ for $x$ in $H_{\perp_h}$,

$$f(x+h) = e^{-(\pi/2)(h,h)}e^{-(\pi/2)(x,x)}\sum_{n=0}^{N} HA_n(x+h) =$$

$$= e^{-(\pi/2)(h,h)}\sum_{i=0}^{N}\sum_{j=1}^{N} j_{c_{j-1}} J^i A_j(\underbrace{x, \ldots, x}_{i\text{ arguments}}, h, \ldots, h)\,,$$

$$\int_{H_{\perp_h}} |f(x+h)|^2\,\mathrm{d}x \leqslant e^{-(\pi/2)(h,h)}\sum_{i=0}^{N}\sum_{j=1}^{N} j_{c_{j-1}}\|h\|^{2(j-1)}\int_H |J^i A_i(x)|^2\,\mathrm{d}x\,.$$

Hence $\int_{H_{\perp_h}} |f(x+h)|^2\mathrm{d}x$ converges. It converges to a positive number by Corollary of Theorem XXVIII.

Corollary. If $f(x) = \sum_{n=0}^{N} J^n A_n(x)$, where $A_n(x)$ is a continuous symmetric $n$-ic on the real Hilbert space $H$, for which $\int_H |J^n A_n(x)|^2\,\mathrm{d}x$ converges, the sequence $h_1, h_2, h_3, \ldots$ is a complete orthonormal system on $H$, $H^{(k)}$ denotes the HILBERT space spanned by $h_{k+1}, h_{k+2}, h_{k+3}, \ldots$, then

$$\int_{H^{(k)}} |f(x + \beta_1 h_1 + \beta_2 h_2 + \ldots + \beta_k h_k)|^2 dx = e^{-\pi\cdot(\beta_1^2+\beta_2^2+\ldots+\beta_k^2)} \sum_{i_1,\ldots,i_k=0}^{N} a_{i_1\ldots i_k} \beta_1^{i_1} \ldots \beta_k^{i_k},$$

where

$$\sum_{i_1,\ldots,i_k=0}^{N} a_{i_1\ldots i_k} \beta_1^{i_1} \ldots \beta_k^{i_k}$$

is a symmetric positive function of $\beta_1, \ldots, \beta_k$.

The proof of the Corollary is similar to that of the Theorem after applying Corollary VII of Theorem XXVII.

## PART III.

## Measure theorie in Hilbert space, or fields of probability in Hilbert space.

Definition XXIII: *Cylinder sets.* Let $h_1, h_2, \ldots$ be a complete orthonormal system in $H$, and $A$ a sub-set of the linear manifold $[h_1, h_2, \ldots, h_N]$ spanned by $h_1, h_2, \ldots, h_N$. The $N$ dimensional cylinder set $C(A)$ denotes the set of elements of $H$ of the form $y = x + h$ where $h \in A$ and $x \in H^{(N)}$, the linear manifold spanned by $h_{N+1}, h_{N+2}, \ldots$. If $S$ is a system of sub-sets of $[h_1, \ldots, h_N]$, $C(S)$ denotes the corresponding system of sests $C(A)$ of $H$ where $A \in S$. We denote $[h_1, h_2, \ldots, h_n]$ by $E^n$. If $A$, $B$ and $A_i$ $(i = 1, 2, \ldots)$ are sets in $E^n$, we note that

$$C(A + B) = C(A) + C(B), \qquad C(AB) = C(A)\,C(B),$$
$$C(A - B) = C(A) - C(B), \qquad C(\sum A_i) = \sum C(A_i).$$

The system of sets $C(S)$ in $H$ is a field (or BOREL field) whenever the system of sets $S$ in $E^n$ is a field (or BOREL field).

A system of sets is called a field if the sum, product and difference of two sets of the system also belong to the same system. A BOREL field is a field such that if $A_i$ $(i = 1, 2, \ldots)$ is a sequence of sets in the field, $\sum A_i$ belongs to the field.

By the Corollary to Theorem XXIX, if $f(x)$ satisfies the hypothesis of Theorem XXIX, then

$$f(\beta_1, \ldots, \beta_k) = \{\int_{H^{(k)}} |f(x + \beta_1 h_1 + \ldots + \beta_k h_k)|^2 dx\} / \int |f(x)|^2 dx > 0$$

for every point $\sum_{i=1}^{k} \beta_i h_i$ in $E_k$. It is also a consequence of the following theorem that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\beta_1, \beta_2, \ldots, \beta_k)\, d\beta_1\, d\beta_2 \ldots d\beta_k = 1\,.$$

Therefore $f(\beta_1, \ldots, \beta_k)$ may be regarded as the probability density at the point $\sum_{i=1}^{k} \beta_i h_i$ in $E_k$ and it defines the completely additive set function $P(A)$ on the BOREL field consisting of the BOREL's sets in $E_k$:

$$P(A) = \iint \cdots \int_A f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n\,.$$

Theorem XXX. *The hypothesis of the Corollary to Theorem XXIX implies*

$$\int_{-\infty}^{\infty} \Big\{ \int_{H^{(k)}} |f(x+\beta_1 h_1 + \ldots + \beta_k h_k)|^2\, dx \Big\} d\beta_k = \int_{H^{(k-1)}} |f(x+\beta_1 h_1 + \beta_2 h_2 + \ldots + \beta_{k-1} h_{k-1})|^2\, dx\,.$$

Proof. By the Corollary to Theorem XXVIII and formula (12) preceding Theorem V.

$$(1) \quad \int_{E_n - E_k} |f(x+\beta_1 h_1 + \beta_2 h_2 + \ldots + \beta_k h_k)|^2\, dx \leqslant \int_{H^{(k)}} |f(x+\beta_1 h_1 + \beta_2 h_2 + \ldots + \beta_k h_k)|^2\, dx.$$

By the Corollary to Theorem XXIX,

$$\int_{H^{(k)}} |f(x + \beta_1 h_1 + \beta_2 h_2 + \ldots + \beta_k h_k)|^2\, dx$$

is summable in $E_k$. The left member of (1) is a summable function of $\beta_k$ for every set of real values of $\beta_1, \beta_2, \ldots, \beta_{k-1}$ and

$$\int_{-\infty}^{\infty} \int_{E_n - E_k} |f(x + \beta_1 h_1 + \ldots + \beta_k h_k)|^2\, dx\, d\beta_k = \int_{E_n - E_{k-1}} |f(x + \beta_1 h_1 + \ldots + \beta_k h_k)|^2\, dx\,.$$

Taking the limits of both sides of this equation as $n \to \infty$, we obtain the Theorem.

For any $E_k$ having $h_1, \ldots, h_k$ as a basis, let $A$ be a BOREL set in $E_k$ then the cylinder sets $C(A)$ form a BOREL field $F^N$. Let $P(C(A)) = P(A)$. $P(C(A))$ and the BOREL field $F^N$ determine a field of probability in the sense of KOLMOGOROV [**9**].

Theorem XXXI. *The system of cylinder sets which belong to $F^1 + F^2 + F^3 + \ldots$ form a field of cylinder sets $F^H$ and the set function $P(C(A))$ defined* *above is additive on $F^H$, i.e., it determines a generalized field of probability in the sense of Kolmogorov on the system of sub-sets of $H$, $F^H$.*

The Theorem follows from the observation that if $C(A) \in F^n$, $C(A) \in F^m$ for $m > n$.

Theorem XXXII. *$P$ does not determine a field of probability on $F^H$, i.e., $P$ is not completely additive on $F^H$.*

Proof. If $A = C(B)$ where $B$ is a BOREL set in $E_n$, then

$$P(A) = \int_B f(\beta_1, \ldots, \beta_n)\, e^{-\pi\cdot(\beta_1^2+\ldots+\beta_n^2)}\, d\beta_1 \ldots d\beta_n\,.$$

There is a real $m$ independent of $n$ such that $f(\beta_1, \ldots, \beta_n) > m > 0$. Let

$$P^1(A) = \int_B e^{-\pi\cdot(\beta_1^2+\ldots+\beta_n^2)}\, d\beta_1 \ldots d\beta_n\,, \qquad \text{then} \qquad P(A) \geqslant mP^1(A) \geqslant 0\,.$$

Hence if $A_1 \supset A_2 \supset \ldots$ is a decreasing sequence of cylinder sets for which $P(A_i) \to 0$, $P^1(A_i) \to 0$, i.e., if $P$ determines a field of probability on $F^H$, $P^1$ determines a field of probability on $F^H$. But KOLMOGOROV ([**9**], Chapter II) showed that if $P^1$ determines a field of probability on $F^H$, it satisfies the covering Theorem: « If $A$, $A_1$, $A_2$, $A_3$, ... belong to $F^H$ and $A \subset A_1 + A_2 + \ldots$ then $P^1(A) \leqslant \sum_i P^1(A_i)$. » The following example shows that the covering Theorem is not satisfies by $P^1$. Let $C_{k_j}$ be the points $(x_1, x_2, \ldots)$ of $H$ whose first $k_j$ coordinates $x_i$ are such that $|x_i| < j$. Choose $k_j$ sufficiently large so that

$$P^1(C_{k_j}) = \Big\{\int_{-j}^{j} e^{-\pi\beta^2}\, d\beta\Big\}^{k_j} < \varepsilon/2^j \qquad \text{for} \qquad j = 1, 2, \ldots\,.$$

The sequence of sets $C_{k_j}$ are BOREL cylinder sets belonging to $F^H$, $H \subset C_{k_1} + C_{k_2} + \ldots$, because if $(x_1, x_2, \ldots) \in H$, $\sum x_i^2$ converges, but $P^1(H) = 1$ and $\sum P^1(C_{k_j}) < \varepsilon$ contrary to the covering Theorem.

Definition XXIV. A $n$-dimensional rectangle is a set of points in $E_n$ obtained from $n$ sets of real numbers $A_1$, $A_2, \ldots, A_n$, by forming all sets of $n$-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in A_i$.

Theorem XXXIII. *If $S_1 \supset S_2 \supset S_3 \supset \ldots$ is a decreasing sequence of rectangular cylinder sets, i.e., $S_i = C(R_i)$, where $R_i$ is a finite dimensional measurable rectangle, such that $S_1 S_2 S_3 \ldots = 0$ and $S_i \in F^H$. Let*

$$P(S_i) = \int_{R_i} e^{-\pi\cdot(\beta_1^2+\ldots+\beta_n^2)}\, d\beta_1\, d\beta_2 \ldots d\beta_n$$

*when $R_i$ is an $n$-dimensional rectangle, then* $\lim_{n\to\infty} P(S_n) = 0$.

Proof. Let $A_1 \supset A_2 \supset A_3 \supset \ldots$ be a decreasing sequence of rectangular cylinder sets of $F^H$ for which $\lim_{n\to\infty} P(A_n) = L > 0$. We will show that the product $A_1A_2A_3\ldots$ is not empty.

We may assume without essentially restricting the problem that in the definition of the first $n$ sets $A_k$, $A_k = C(B_k)$ where $B_k$ is a $k$-dimensional measurable rectangle in the closed linear manifold $[h_{\nu_1}, h_{\nu_2}, \ldots, h_{\nu_k}]$ spanned by the $k$ orthonormal vectors $h_{\nu_1}, h_{\nu_2}, \ldots, h_{\nu_k}$. In each set $B_n$ there is a closed bounded rectangle $U_n$ such that $P(B_n - U_n) \leqslant \varepsilon/2^n$. Let $V_n = C(U_n)$, then

$$P(A_n - V_n) = P(B_n - U_n) \leqslant \varepsilon/2^n .$$

Let $W_n = V_1V_2\ldots V_n$, then $A_n - W_n \subset (A_1 - V_1) + (A_2 - V_2) + \ldots + (A_n - V_n)$, $P(A_n - W_n) < \varepsilon$. But $W_n \subset V_n \subset A_n$, hence $P(W_n) \geqslant P(A_n) - \varepsilon \geqslant L - \varepsilon$ or we can choose $\varepsilon$ so that $P(W_n) > M > 0$ and $W_n$ is not empty: $W_1 \supset W_2 \supset W_3 \supset \ldots$. Let $W_i = C(X_i)$, then $X_n = U_1U_2\ldots U_n$ is a closed bounded non-null rectangle in $[h_{\nu_1}, h_{\nu_2}, \ldots, h_{\nu_n}]$. Let $S_i^j$ for every $j < i$ be the projection of $W_i$ on $[h_{\nu_j}]$, then $S_i^j$ is a closed bounded non-null set: $S_n^j \supset S_{n+1}^j \supset S_{n+2}^j \supset \ldots$ for every $j < n$. Hence the infinite product $\prod_j S_i^j$ is a closed, bounded, non-empty set. Let $\beta_j h_{\nu_j}$ be a point in $\prod_i S_i^j$ nearest the origin, i.e., for which $|\beta_i^j|$ is a minimum for a fixed $j$. The point $\sum_{i=1}^{\infty} \beta_i h_{\nu_i}$ belongs to $W_n$ for $n = 1, 2, 3, \ldots$, if it belongs to $H$.

$$P(W_n) = P(X_n) = P(S_n^1)P(S_n^2)\ldots P(S_n^n) > M > 0 .$$

Since $0 \leqslant P(S_n^i) \leqslant 1$, $P(S_n^i) < \sqrt[k]{M}$ for at most $k$ values of $i$. Let $\min|\beta_n^i|$ denote the minimum value of $|\beta_n^i|$ for all $\beta_n^i h_{\nu_i} \in S_n^i$, then

$$\min|\beta_k^i| \leqslant \min|\beta_{k+1}^i| \leqslant \min|\beta_{k+2}^i| \leqslant \ldots \leqslant \lim_{n\to\infty} \min|\beta_n^i| = |\beta_i| .$$

There is an $N$ such that for any $k > N$, there are at most $k$ values of $i$ (these at most $k$ values of $i$ being the same for all $n$), such that $P(S_n^i) < \sqrt[k]{M}$ ($\sqrt[k]{M} > 199/200$). Hence there are at most $k$ values of $i$ for which $\min|\beta_n^i| \geqslant 1/10$ for any $n$, since for such $i$ and $nP(S_n^i) < 199/200$. For all values of $i$ except at most $k > N$, $\min|\beta_n^i| < 1/10$ for all $n$ and

$$\sqrt[k]{M} \leqslant P(S_n^i) < 1 - \min|\beta_n^i| < 1 - |\beta_i| .$$

Therefore there are at most $k$ values of $i$ such that for $k \geqslant N$, $|\beta_i| > 1 - \sqrt[k]{M}$.

Let $\beta_{\lambda_1}, \beta_{\lambda_2}, \ldots, \beta_{\lambda_a}$ be all the $\beta_i$ for which $|\beta_i| > 1 - \sqrt[N]{M}$.

Let

$$x_1 = \beta_{\lambda_1}, \quad x_2 = \beta_{\lambda_2}, \ \ldots, \quad x_a = \beta_{\lambda_a},$$
$$x_{a+1} = 1 - \sqrt[N]{M}, \ x_{a+2} = 1 - \sqrt[N]{M}, \ \ldots, \ x_N = 1 - \sqrt[N]{M} .$$

If there are $\beta_i$ for which $1-\sqrt[N]{M} \geqslant |\beta_i| > 1-\sqrt[n+1]{M}$ call them $\beta_{\lambda_{a+1}}, \beta_{\lambda_{a+2}}, \ldots, \beta_{\lambda_{a_1}}$ and let $x_{N+1} = 1-\sqrt[N]{M} \cdot a_1 \leqslant N+1$. For every $k>0$, if there are $\beta_i$ for which

$$1-\sqrt[N+k]{M} \geqslant |\beta_i| > 1-\sqrt[n+k+1]{M},$$

call them $\beta_{\lambda_{a_{k+1}}}, \beta_{\lambda_{a_{k+2}}}, \ldots, \beta_{\lambda_{a_{k+1}}}$ and let

$$x_{N+k+1} = 1-\sqrt[N+k]{M},$$

$a_k < N+k$.

Since $1-\sqrt[k]{M} \to 0$ as $k \to \infty$, $\{\beta_{\lambda_i}\}$ is a re-arrangement of the sequence $\{\beta_i\}$, $|\beta_i| \leqslant |x_i|$. For $i>N$, $x_{i+1} = 1-\sqrt[i]{M}$, $0<M<1$,

$$\sum_i (1-\sqrt[i]{M})^2 < \sum (1-\sqrt[i]{M})^2 (1+\sqrt[i]{M}+\ldots+\sqrt[i]{M^{i-1}})^2/\{i^2(\sqrt[i]{M^{i-1}})^2\} =$$

$$= \sum \frac{(1-M)^2}{i^2 M^{(2i-2)/i}} < \sum \frac{(1-M)^2}{i^2 M^2}$$

which is convergent. Therefore $\sum \beta_i^2$ converges and $\sum \beta_i h_{v_i} \in H$ and hence to $W_n$. Q.E.D..

We can without altering the proof replace $e^{-\pi \cdot (x_1^2+x_2^2+\ldots+x_n^2)}$ in the above Theorem by the product $f_1(x_1)f_2(x_2)\ldots f_n(x_n)$, where each $f_i(x_i)$ is a non-negative LEBESGUE integrable function with the properties:

1. $\int_{-\infty}^{\infty} f_i(x)\,\mathrm{d}x = 1.$
2. There is an $m>0$ such that $|x|<m$ implies $\int_{-x}^{x} f_i(t)\,\mathrm{d}t > x.$

F. H. BROWNELL [11] has recently shown how to construct BOREL measures on some locally compact subspaces of HILBERT space. That his results do not include the above Theorem is shown by the following example:

Let $R_n$ be the BOREL rectangular cylinder set whose base is the rectangle in $E_n$ containing the points $(x_1, x_2, \ldots, x_n)$ such that $|x_i| > n$ for $i = 1, 2, 3, \ldots, n$. The sets $R_n$ form a monotone decreasing sequence whose intersection is null, but $R_n$ is not contained in any locally compact subspace of $H$.

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