Rivista Mat. Univ. Parma 6, 301-317 (1955)

K. K. GOROWARA (*)

General and hyper Darboux lines. (**)

The object of this paper is to define and study some properties of « general DARBOUX lines » and « hyper DARBOUX lines » in a Euclidean space of three dimensions. Some of the known results of DARBOUX lines have been deduced from those general DARBOUX lines as particular cases.

1. – Rectilinear congruence.

Let the coordinates of a point P on the surface of reference of S of rectilinear congruence be given by $x^i = x^i(u^1, u^2)$ $(i = 1, 2, 3)$ and the direction cosines of the ray of the congruence through x^i by $\lambda^i = \lambda^i(u^1, u^2)$ $(i = 1, 2, 3)$. Since in general the rays of the congruence are not normal to S, we have

$$\lambda^i = p^\alpha x^i_{,\alpha} + q x^i \tag{1·1}$$

and

$$\lambda^i \lambda^i = 1, \tag{1·2}$$

where: p^α are the contravariant components of a vector in surface at P; $x^i_{,\alpha}$ $(\alpha = 1, 2)$ (¹) are the direction numbers and denote convariant differentiation of x^i with regard to u^α based on the first fundamental tensor

$$g_{\alpha\beta} = x^i_{,\alpha} \, x^i_{,\beta} \tag{1·3}$$

(*) Address: Lecturer in Mathematics, Hans Raj College (Delhi University), Delhi, India.

(**) Received April 18, 1955.

(¹) In what follows Latin indices take the values 1, 2, 3 and Greek indices the values 1, 2.

of the surface S; and q is a positive scalar function, such that if θ is the angle between the normal to the surface at a point P and the line of the congruence through P, then

$$q = \lambda^i X^i = \cos\theta \tag{1·4}$$

and

$$p^\alpha p^\beta g_{\alpha\beta} = \sin^2\theta\,. \tag{1·5}$$

A DARBOUX line in Euclidean 3 space has been defined as a curve on S which has the property that the osculating sphere at any point P of the curve is tangent to the surface at P. The differential equation of DARBOUX lines is find to be ([**4**], *p.* 356)

$$X^i \frac{\mathrm{d}^3 x^i}{\mathrm{d}s^3} = 0, \tag{1·6}$$

where $\mathrm{d}^3x^i/\mathrm{d}s^3$ is a vector normal to the radius vector of the osculating sphere. A general DARBOUX line (G.D. line) at a point P on the surface in a Euclidean three space is defined as a curve which has the property that the line of the congruence through P passes through the centre of the osculating sphere of the curve at P. Analytically its equation will be

$$\lambda^i \frac{\mathrm{d}^3 x^i}{\mathrm{d}s^3} = 0\,. \tag{1·7}$$

2. – G. D. lines.

We know that

$$\frac{\mathrm{d}x^i}{\mathrm{d}s} = \frac{\partial x^i}{\partial u^\alpha}\frac{\mathrm{d}u^\alpha}{\mathrm{d}s},$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^\beta}{\mathrm{d}s} + \frac{\partial x^i}{\partial u^\alpha}\frac{\mathrm{d}^2u^\alpha}{\mathrm{d}s^2},$$

$$\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3} = \frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma}\frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\frac{\mathrm{d}u^\gamma}{\mathrm{d}s} + 3\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\frac{\mathrm{d}^2u^\alpha}{\mathrm{d}s^2}\frac{\mathrm{d}u^\beta}{\mathrm{d}s} + \frac{\partial x^i}{\partial u^\alpha}\frac{\mathrm{d}^3u^\alpha}{\mathrm{d}s^3}\,.$$

Hence the equation of G.D. lines takes the form

$$(2\cdot1)\qquad (p^{\varphi}x^i_{,\varphi}+qX^i)\left(\frac{\partial^3 x^i}{\partial u^\alpha\partial u^\beta\partial u^\gamma}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}+3\frac{\partial^2 x^i}{\partial u^\alpha\partial u^\beta}\frac{d^2u^\alpha}{ds^2}\frac{du^\beta}{ds}+\frac{\partial x^i}{\partial u^\alpha}\frac{d^3u^\alpha}{ds^3}\right)=0\,.$$

But ([**1**], p. 215)

$$(2\cdot2)\qquad x^i_{,\alpha\beta}=\frac{\partial^2 x^i}{\partial u^\alpha\partial u^\beta}-\frac{\partial x^i}{\partial u^\gamma}\left\{\begin{matrix}\gamma\\ \alpha\ \beta\end{matrix}\right\}$$

With the help of GAUSS's equation

$$x^i_{,\alpha\beta}=d_{\alpha\beta}\,X^i$$

equation (2·2) reduces to

$$(2\cdot3)\qquad \frac{\partial^2 x^i}{\partial u^\alpha\partial u^\beta}=d_{\alpha\beta}\,X^i+\frac{\partial x^i}{\partial u^\delta}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}.$$

Differentiating (2·3) with regard to u^γ we have

$$\frac{\partial^3 x^i}{\partial u^\alpha\partial u^\beta\partial u^\gamma}=\left(\frac{\partial}{\partial u^\gamma}\,d_{\alpha\beta}\right)X^i+d_{\alpha\beta}\frac{\partial X^i}{\partial u^\gamma}+\frac{\partial^2 x^i}{\partial u^\gamma\partial u^\delta}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}+\frac{\partial x^i}{\partial u^\delta}\frac{\partial}{\partial u^\gamma}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}.$$

Hence equation (2·1) takes the form

$$(2\cdot4)\qquad (p^{\varphi}x^i_{,\varphi}+qX^i)\left[\left(\frac{\partial}{\partial u^\gamma}\,d_{\alpha\beta}\cdot X^i+d_{\alpha\beta}\frac{\partial X^i}{\partial u^\gamma}+\frac{\partial^2 x^i}{\partial u^\gamma\partial u^\delta}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}+\right.\right.$$

$$\left.+\frac{\partial x^i}{\partial u^\delta}\frac{\partial}{\partial u^\gamma}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}+3\left(d_{\alpha\beta}\,X^i+\frac{\partial x^i}{\partial u^\delta}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}\right)\frac{d^2u^\alpha}{ds^2}\frac{du^\beta}{ds}+\left.\frac{\partial x^i}{\partial u^\alpha}\frac{d^3u^\alpha}{ds^3}\right]=0\,.$$

Equation (2·4) on simplification is

$$(2\cdot5)\qquad p^{\varphi}\left[\left(-d_{\alpha\beta}d_{\gamma\varphi}+[\varphi,\ \gamma\delta]\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}+g_{\varphi\delta}\frac{\partial}{\partial u^\gamma}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}+g_{\varphi\alpha}\frac{d^3u^\alpha}{ds^3}\right]+$$

$$+q\cdot\left(\frac{\partial}{\partial u^\gamma}\,d_{\alpha\beta}+d_{\gamma\delta}\left\{\begin{matrix}\delta\\ \alpha\ \beta\end{matrix}\right\}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}+3(q d_{\alpha\beta}+p^{\varphi}[\varphi,\ \alpha\beta])\frac{d^2u^\alpha}{ds^2}\frac{du^\beta}{ds}=0\,.$$

Thus (2·5) *is the required differential equation of the* G.D. *lines.*

The equation of G.D. lines can also be written as

$$p^{\varphi} x^{i}_{,\varphi} \frac{d^3 x^i}{ds^3} + q\, X^i \frac{d^3 x^i}{ds^3} = 0 . \tag{2·6}$$

From this we get the result that if the vector $p^{\varphi} x^{i}_{,\varphi}$ is orthogonal to d^3x^i/ds^3, then, since q is not in general equal to zero, the equation of G.D. lines reduces to

$$X^i \frac{d^3 x^i}{ds^3} = 0$$

whic is the equation of D. lines. Hence we have the result that:

If the vector $p^{\varphi} x^{i}_{,\varphi}$ i on a surface is orthogonal to d^3x^i/ds^3 the G.D. lines reduced to D. lines.

From (2·6) we also have:

If any two of the following conditions are satisfied, the third is also satisfied:

(i) *The curve be G.D. lines (for which $q \neq 0$).*

(ii) *The curve be D. line.*

(iii) *The vector $p^{\varphi} x^{i}_{,\varphi}$ be orthogonal to the vector d^3x^i/ds^3.*

Particular cases.

(a) If λ^i is normal to the surface, $p^{\varphi} = 0$ and $q = 1$, then the G.D. lines become D. lines (DARBOUX lines) and the equation reduces to

$$\left(\frac{\partial}{\partial u^{\gamma}} d_{\alpha\beta} + d_{\gamma\delta} \left\{ {\delta \atop \alpha\ \beta} \right\}\right) \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} + 3 d_{\alpha\beta} \frac{d^2u^{\alpha}}{ds^2} \frac{du^{\beta}}{ds} = 0 . \tag{2·7}$$

(b) If λ^i is tangent to the surface then p^{φ} is a unit vector on S and $q = 0$, so that the equation of G.D. lines becomes

$$p^{\varphi}\left[\left(- d_{\alpha\beta}\, d_{\gamma\varphi} + [\varphi,\ \gamma\delta] \left\{ {\delta \atop \alpha\ \beta} \right\} + g_{\varphi\delta} \frac{\partial}{\partial u^{\gamma}} \left\{ {\delta \atop \alpha\ \beta} \right\}\right) \frac{\partial u^{\alpha}}{ds} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} + \right. \tag{2·8}$$

$$\left. + 3[\varphi,\ \alpha\beta] \frac{d^2u^{\alpha}}{ds^2} \frac{du^{\beta}}{ds} + g_{\varphi\alpha} \frac{d^3u^{\alpha}}{ds^3}\right] = 0 .$$

We call these curves as « tangent DARDOUX lines » (T.D. lines).

3. – Condition that the G. D. lines be parametric.

We now find the condizions that the G,D. lines be the coordinate curves

$$u^2 = \text{const.}, \qquad u' = \text{const.}.$$

We have, for coordinate curves $u^2 = \text{const.}$,

$$ds^2 = g_{11}(du')^2 .$$

Therefore

$$\frac{d^2u^1}{ds^2}\frac{du^1}{ds} = \frac{d}{ds}\left(\frac{du^1}{ds}\right)\cdot\frac{du^1}{ds} = \frac{\partial}{\partial u'}\frac{1}{\sqrt{g_{11}}}\cdot\left(\frac{du^1}{ds}\right)^2 =$$

$$= -\frac{1}{2}\frac{\partial g_{11}/\partial u^1}{(g_{11})^{3/2}}\left(\frac{du^1}{ds}\right)^2 = -\frac{1}{2}\frac{\partial g_{11}/\partial u^1}{g_{11}}\left(\frac{du^1}{ds}\right)^3 .$$

Also

$$\frac{d^3u^1}{ds^3} = \frac{d^2}{ds^2}\frac{1}{\sqrt{g_{11}}} = \frac{d}{ds}\left[\frac{\partial}{\partial u'}\frac{1}{\sqrt{g_{11}}}\cdot\frac{du^1}{ds}\right] = \frac{d}{ds}\frac{-\partial g_{11}/\partial u^1}{2(g_{11})^2} = -\frac{1}{2}\frac{\partial}{\partial u'}\frac{\partial g_{11}/\partial u^1}{(g_{11})^2}\cdot\frac{du^1}{ds} =$$

$$= \frac{(-\partial^2 g_{11}/\partial(u^1)^2)(g_{11})^2 + 2g_{11}\cdot(\partial g_{11}/\partial u^1)^2}{2(g_{11})^4}\frac{du^1}{ds} = \frac{(-\partial^2 g_{11}/\partial(u^1)^2)g_{11} + 2(\partial g_{11}/\partial u^1)}{2(g_{11})^2}\left(\frac{du^1}{ds}\right)^3 .$$

Similarly for the coordinate curves $u^1 = \text{const.}$ we have

$$\frac{d^2u^2}{ds^2}\frac{du^2}{ds} = \frac{-\partial g_{22}/\partial u^2}{2g_{22}}\left(\frac{du^2}{ds}\right)^3,$$

$$\frac{d^3u^2}{ds^3} = \frac{(-\partial^2 g_{22}/\partial(u^2)^2)\,g_{22} + 2(\partial g_{22}/\partial u^2)^2}{2(g_{22})^2}\left(\frac{du^2}{ds}\right)^3 .$$

Hence in order that the coordinate curves $u^2 = \text{const.}$ be G.D. lines we must have from (2·5)

$$(3\cdot 1)\qquad p^\varphi\Bigg[-d_{11}d_{1\varphi} + [\varphi,\,1\delta]\begin{Bmatrix}\delta\\11\end{Bmatrix} + g_{\varphi\delta}\frac{\partial}{\partial u'}\begin{Bmatrix}\delta\\11\end{Bmatrix} + g_{\varphi 1}\frac{(-\partial^2 g_{11}/\partial(u^1)^2)g_{11} + 2(\partial g_{11}/\partial u^1)^2}{2(g_{11})^2}\Bigg] +$$

$$+ q\cdot\left(\frac{\partial}{\partial u'}d_{11} + d_{1\delta}\begin{Bmatrix}\delta\\11\end{Bmatrix}\right) + 3\,(qd_{11} + p^\varphi[\varphi,\,11])\,\frac{-\partial g_{11}/\partial u^1}{2g_{11}} = 0 .$$

Similarly in order that the coordinate curves $u' =$ const. be G.D. lines we have

$$(3\cdot 2)\qquad p^{\varphi}\Bigg[- d_{22}d_{2\varphi} + [\varphi,\ 2\delta]\begin{Bmatrix}\delta\\22\end{Bmatrix} + $$
$$+ g_{\varphi\delta}\frac{\partial}{\partial u^2}\begin{Bmatrix}\delta\\22\end{Bmatrix} + g_{\varphi 2}\,\frac{(-\,\partial^2 g_{22}/\partial(u^2)^2)\,g_{22} + 2(\partial g_{22}/\partial u^2)^2}{2(g_{22})^2}\Bigg] +$$
$$+ q\cdot\left(\frac{\partial}{\partial u^2}\,d_{22} + d_{2\delta}\begin{Bmatrix}\delta\\22\end{Bmatrix}\right) + 3(qd_{22} + p^{\varphi}[\varphi,\ 22])\ \frac{-\,\partial g_{22}/\partial u^2}{2g_{22}} = 0\,.$$

In particular, if G.D. lines reduce to D. lines the condition that $u^2 =$ const. is a D. line becomes

$$(3\cdot 3)\qquad \frac{\partial}{\partial u'}\,d_{11} + d_{11}\begin{Bmatrix}1\\11\end{Bmatrix} + d_{12}\begin{Bmatrix}2\\11\end{Bmatrix} - \frac{3d_{11}\cdot(\partial g_{11}/\partial u')}{2g_{11}} = 0$$

which is the same as obtained by F. SEMIN ([**4**], p. 358). Similarly the condition that $u' =$ const. is a D. line is

$$(3\cdot 4)\qquad \frac{\partial}{\partial u^2}\,d_{22} + d_{21}\begin{Bmatrix}1\\22\end{Bmatrix} + d_{22}\begin{Bmatrix}2\\22\end{Bmatrix} - \frac{3d_{22}\cdot(\partial g_{22}/\partial u^2)}{2g_{22}} = 0$$

which is the same as obtained by F. SEMIN ([**4**], p. 358).

4. – Condition that the G. D. lines coincide with lines of curvature.

If a family of lines of curvature are parametric then

$$d_{12} = g_{12} = 0\,.$$

Therefore condition (3·1) that the G.D. lines are the coordinate curves $u^2 =$ = const. reduces to

$$(4\cdot 1)\qquad p^1\cdot\Bigg[-d_{11}^2 + \frac{1}{4}\,g^{11}\left(\frac{\partial}{\partial u'}\,g_{11}\right)^2 - \frac{1}{4}\,g^{22}\left(\frac{\partial}{\partial u^2}\,g_{11}\right)^2 +$$
$$+ \frac{1}{2}\,g_{11}\,\frac{\partial}{\partial u'^2}\log g_{11} - \frac{1}{2}\,\frac{\partial^2}{\partial u'^2}\,g_{11} + g_{11}\left(\frac{\partial}{\partial u'}\log g_{11}\right)^2\Bigg] +$$

$$+ p^2 \cdot \left[-\frac{1}{4} g_{11} \left(\frac{\partial}{\partial u^1} \log g_{11} \right) \left(\frac{\partial}{\partial u^2} \log g_{11} \right) - \right.$$

$$\left. - \frac{1}{4} g_{11} \frac{\partial}{\partial u^2} \log g_{11} \cdot \frac{\partial}{\partial u^1} \log g_{22} - \frac{1}{2} g_{22} \cdot \frac{\partial}{\partial u^1} \left(g^{22} \cdot \frac{\partial}{\partial u^2} g_{11} \right) \right] +$$

$$+ q \cdot \left[\frac{\partial}{\partial u^1} d_{11} + \frac{1}{2} d_{11} \frac{\partial}{\partial u^1} \log g_{11} \right] +$$

$$+ 3 \left[q d_{11} + \frac{1}{2} p^1 g_{11} \cdot \frac{\partial}{\partial u^1} \log g_{11} - \frac{1}{2} p^2 g_{11} \cdot \frac{\partial}{\partial u^2} \log g_{11} \right] \left(-\frac{1}{2} \frac{\partial}{\partial u^1} \log g_{11} \right) = 0 .$$

Similarly the condition (3·2) that the G.D. lines are the coordinate curves $u' =$ = const. reduces to

$$(4 \cdot 2) \qquad p^1 \cdot \left[-\frac{1}{4} g_{11} \cdot \frac{\partial}{\partial u^1} \log g_{11} \cdot \frac{\partial}{\partial u^2} \log g_{11} - \right.$$

$$\left. - \frac{1}{4} g_{11} \cdot \frac{\partial}{\partial u^1} \log g_{11} \cdot \frac{\partial}{\partial u^2} \log g_{22} - \frac{1}{2} g_{11} \cdot \frac{\partial}{\partial u^2} \left(\frac{\partial}{\partial u^1} \log g_{11} \right) \right] +$$

$$+ p^2 \cdot \left[- d_{22}^2 - \frac{1}{4} g^{11} \cdot \left(\frac{\partial}{\partial u^1} g_{22} \right)^2 + \frac{1}{4} g_{22} \cdot \left(\frac{\partial}{\partial u^2} \log g_{22} \right)^2 + \frac{1}{2} g_{22} \frac{\partial^2}{\partial (u^2)^2} \log g_{22} - \right.$$

$$\left. - \frac{1}{2} \frac{\partial^2}{(\partial u^2)^2} g_{22} + g_{22} \cdot \left(\frac{\partial}{\partial u^2} \log g_{22} \right)^2 \right] + q \cdot \left[\frac{\partial}{\partial u^2} d_{22} + \frac{1}{2} d_{22} \frac{\partial}{\partial u^2} \log g_{22} \right] +$$

$$+ 3 \left[q d_{22} - \frac{1}{2} p^1 g_{11} \frac{\partial}{\partial u^1} \log g_{11} + \frac{1}{2} p^2 g_{22} \frac{\partial}{\partial u^2} \log g_{22} \right] \left(-\frac{1}{2} \frac{\partial}{\partial u^2} \log g_{22} \right) = 0 .$$

If the G.D. lines become D. lines the condition (4·1) reduces to

$$\frac{\partial}{\partial u'} d_{11} + \frac{1}{2} d_{11} \frac{\partial}{\partial u^1} (g_{21} \log g_{11}) - \frac{3}{2} d_{11} \frac{\partial}{\partial u^1} \log g_{11} = 0$$

or

$$(4 \cdot 3) \qquad \frac{\partial}{\partial u^1} d_{11} - d_{11} \frac{\partial}{\partial u^1} \log g_{11} = 0 .$$

Similarly if the G. D. lines become D. lines the condition (4·2) reduces to

$$\frac{\partial}{\partial u^2}\, d_{22} - d_{22}\, \frac{\partial}{\partial u^2} \log g_{22} = 0\,. \tag{4·4}$$

From (4·3) we get the known result ([4], p. 361) that: *For a developable surface the curves $u^2 =$ const. are the lines curvature and Darboux lines.*

Also from (4·4) we have: *If the surface is a doubly developable surface then the coordinate curves $u' =$ const. are also the lines of curvature and D. lines.*

5. – Another form for G. D. lines.

We know that

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} = \varrho^\alpha x^i_{,\alpha} + k_n X^i,$$

where k_n is the normal curvature of the surface in the directions of the curve C: $x^i = x^i(s)$ and ϱ^α is the curvature vector of the curve C. Therefore

$$\frac{\mathrm{d}^3 x^i}{\mathrm{d}s^3} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\, \varrho^\alpha \frac{\mathrm{d}u^\beta}{\mathrm{d}s} + x^i_{,\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\,\varrho^\alpha + \frac{\mathrm{d}}{\mathrm{d}s}\, k_n\cdot X^i + k_n \frac{\partial X^i}{\partial u^\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}s} =$$

$$= \left[d_{\alpha\beta} X^i + x^i_{,\gamma}\left\{ {\gamma \atop \alpha\ \beta} \right\}\right] \varrho^\alpha \frac{\mathrm{d}u^\beta}{\mathrm{d}s} + x^i_{,\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\,\varrho^\alpha + \frac{\mathrm{d}}{\mathrm{d}s}\, k_n\cdot X^i + k_n \frac{\partial X^i}{\partial u^\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\,.$$

Hence

$$X^i\cdot\frac{\mathrm{d}^3 x^i}{\mathrm{d}s^3} = d_{\alpha\beta}\, \varrho^\alpha \frac{\mathrm{d}u^\beta}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\, k_n\,,$$

$$x^i_{,\varphi}\, \frac{\mathrm{d}^3 x^i}{\mathrm{d}s^3} = [\varphi,\ \alpha\beta]\varrho^\alpha \frac{\mathrm{d}u_\beta}{\mathrm{d}s} + g_{\varphi\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\,\varrho^\alpha - k_n d_{\beta\varphi}\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\,.$$

The equation of the G.D. lines can then be written as

$$p^\varphi\cdot\left\{[\varphi,\ \alpha\beta]\varrho^\alpha \frac{\mathrm{d}u_\beta}{\mathrm{d}s} + g_{\varphi\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\,\varrho^\alpha - k_n d_{\beta\varphi}\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\right\} + q\cdot\left\{d_{\alpha\beta}\, \varrho^\alpha \frac{\mathrm{d}u^\beta}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\, k_n\right\} = 0. \tag{5·1}$$

Particular cases.

(a) If λ^i is normal to the surface; then $p^\varphi = 0$, $q = 1$ and the equation (5·1) reduces to

$$d_{\alpha\beta}\, \varrho^\alpha \frac{du^\beta}{ds} + \frac{d}{ds} k_n = 0,$$

or

$$d_{\alpha\beta}\, k_g\, \mu^\alpha \frac{du^\beta}{ds} + \frac{d}{ds} k_n = 0,$$

or

$$(5\cdot2) \qquad \tau_g\, k_g + \frac{d}{ds} k_n = 0.$$

Since

$$\tau_g = d_{\alpha\beta}\, \mu^\alpha \frac{du^\beta}{ds},$$

where τ_g is the geodesic torsion. Equation (5·2) is well known form of the differential equation of D. lines in Euclidean 3 space ([**4**], p. 359).

(b) If λ^i is tangent to the surface, then $q = 0$ and p^φ is a unit vector. The equation of T. D. lines is then

$$(5\cdot3) \qquad p^\varphi \cdot \left\{ [\varphi,\ \alpha\beta]\varrho^\alpha \frac{du^\beta}{ds} + g_{\varphi\alpha} \cdot \frac{d}{ds} \varrho^\alpha - k_n\, d_{\beta\varphi} \frac{du^\beta}{ds} \right\} = 0 .$$

Since for a straight line on a surface both geodesic curvature and normal curvature vanish, therefore, for a straight line the differential equation of the G. D. lines are satisfied. Hence:

Any straight line on a surface is a G. D. *line.*

We have obtained that the equation of T. D. lines is

$$p^\varphi \cdot \left\{ [\varphi,\ \alpha\beta]\varrho^\alpha \frac{du^\beta}{ds} + g_{\varphi\alpha} \cdot \frac{d}{ds} \varrho^\alpha - k_n\, d_{\beta\varphi} \frac{du^\beta}{ds} \right\} = 0 .$$

We suppose that these are not straight lines. If the T.D. lines are geodesic on the surface this becomes

$$p^{\varphi} k_n d_{\alpha\varphi} \frac{du^{\beta}}{ds} = 0,$$

or

$$p^{\varphi} d_{\alpha\varphi} \frac{du^{\beta}}{ds} = 0,$$

or

$$p^{\varphi} d_{1\varphi} \frac{du'}{ds} + p^{\varphi} d_{2\beta} \frac{du^2}{ds} = 0\,.$$

If these are the parametric curves $u^2 =$ const. and $u^1 =$ const. we have

$$p^1 d_{11} + p^2 d_{12} = 0,$$

$$p^1 d_{21} + p^2 d_{22} = 0.$$

Elimination of p^1 and p^2 gives

$$d_{11} d_{22} - d_{12}^2 = 0.$$

Hence we have the result:

If the T.D. *lines are geodesic and parametric curves on a surface, the surface is of zero Gaussian curvature i. e. it must be a developable surface.*

6. – Hyper Darboux lines (H. D. lines).

We now find the equation of H. D. lines. H. D. lines in Euclidean space of 3 dimensions (PRAVANOVITCH [3]) are defined as the curves on the surface which have the property that λ^i lies in the plane of dx^i/ds and

$$\varrho \beta^i - \frac{1}{\tau} \frac{d\varrho}{ds} \gamma^i$$

where: ϱ is the radius of curvature, τ the torsion of the curve, β^i and γ^i being the direction cosines of the unit vector along the principal normal and bynormal at point $P(x^i)$ of the curve. Hence λ^i must satisfy a relationshship of the form

$$(6\cdot1) \qquad \lambda^i = a\,\frac{\mathrm{d}x^i}{\mathrm{d}s} + b\cdot\left(\varrho\beta^i - \frac{1}{\tau}\,\frac{\mathrm{d}\varrho}{\mathrm{d}s}\,\gamma^i\right).$$

Now we have

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} = \frac{1}{\varrho}\,\beta^i,$$

$$\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3} = -\frac{1}{\varrho^2}\,\frac{\mathrm{d}\varrho}{\mathrm{d}s}\,\beta^i - \frac{1}{\varrho^2}\,\frac{\mathrm{d}x^i}{\mathrm{d}s} - \frac{\tau}{\varrho}\,\gamma^i\,.$$

Multiplying both sides of (6·1) by $\mathrm{d}^3x^i/\mathrm{d}s^3$ we obtain

$$(p^a x^i_{,a} + qX^i)\,\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3} = a\,\frac{\mathrm{d}x^i}{\mathrm{d}s}\,\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3} = a x^i_{,a}\,\frac{\mathrm{d}u^a}{\mathrm{d}s}\,\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3}\,,$$

or

$$x^i_{,a}\,\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3}\left(a\,\frac{\mathrm{d}u^a}{\mathrm{d}s} - p^a\right) = qX^i\,\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3}\,.$$

Multiplying both sides of (6·1) by $\mathrm{d}x^i/\mathrm{d}s$ we obtain

$$p^a\,x^i_{,a}\,x^i_{,\beta}\,\frac{\mathrm{d}u^\beta}{\mathrm{d}s} = a,$$

or

$$a = p^a g_{a\beta}\,\frac{\mathrm{d}u^\beta}{\mathrm{d}s} = p^\beta\,\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\,.$$

Hence the equation of H. D. lines assumes the form

$$(6\cdot2) \qquad x^i_{,a}\,\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3}\left(p_\beta\,\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\,\frac{\mathrm{d}u^a}{\mathrm{d}s} - p^a\right) = qX^i\,\frac{\mathrm{d}^3x^i}{\mathrm{d}s^3}\,.$$

In particular if the congruence is formed by the normals to the surface the H. D. lines reduces to D. lines. The H. D. lines also reduce to D. lines if the vector $x^i_{,\alpha}$ is orthogonal to the vector d^3x^i/ds^3. The H. D. lines also reduce to D. lines if the vector

$$x^i_{,\alpha}\cdot\left(p_\beta \frac{du^\beta}{ds}\frac{du^\alpha}{ds} - p^\alpha\right)$$

is orthogonal to d^3x^i/ds^3.

We therefore have the following results:

If any two of the following conditions are satisfied the third is also satisfied:

(i) *The curve be a* H. D. *line* $(q \neq 0)$.

(ii) *The curve be* D. *line.*

(iii) *The vector* $x^i_{,\alpha}\cdot\left(p^\beta \frac{du^\beta}{ds}\frac{du^\alpha}{ds} - p^\alpha\right)$ *is orthogonal to* $\frac{d^3x^i}{ds^3}$.

From equation (6·2) we have that the H. D. lines coincide with D. lines along the directions given by

$$x^i_{,\alpha}\frac{d^3x^i}{ds^3} = 0, \tag{6·3}$$

or

$$p_\beta \frac{du^\beta}{ds}\frac{du^\alpha}{ds} - p^\alpha = 0\,. \tag{6·4}$$

7. – Curvature of a H. D. line.

By definition of H. D. line we have

$$\lambda^i = a\frac{dx^i}{ds} + b\cdot\left[\varrho\beta^i - \frac{1}{\tau}\frac{d\varrho}{ds}\gamma^i\right]. \tag{7·1}$$

Multiplying both sides of (7·1) by dx^i/ds we have

$$a = \lambda^i\frac{dx^i}{ds} = (p^\alpha x^i_{,\alpha} + qX^i)x^i_{,\beta}\frac{du^\beta}{ds} = p^\alpha g_{\alpha\beta}\frac{du^\beta}{ds} = p^\beta\frac{du^\beta}{ds}\,. \tag{7·2}$$

Let ψ be the angle between the line of the congruence and the vector

$$\varrho\,\beta^i - \frac{1}{\tau}\,\frac{d\varrho}{ds}\,\gamma^i .$$

Multiplying both sides of (7·1) by

$$\varrho\,\beta^i - \frac{1}{\tau}\,\frac{d\varrho}{ds}\,\gamma^i$$

we have

$$(7\cdot3)\qquad \lambda^i\cdot\left(\varrho\,\beta^i - \frac{1}{\tau}\,\frac{d\varrho}{ds}\,\gamma^i\right) = bR^2,$$

where R is the magnitude of the vector $\varrho\,\beta^i - \frac{1}{\tau}\,\frac{d\varrho}{ds}\,\gamma^i$.

Equation (7·3) gives us $R\cdot\cos\psi = bR^2$ or

$$b = \frac{\cos\psi}{R}\,.$$

Hence (7·1) takes the form

$$(7\cdot4)\qquad \lambda^i = p^\beta\,\frac{du^\beta}{ds}\,\frac{dx^i}{ds} + \frac{\cos\psi}{R}\left(\varrho\beta^i - \frac{1}{\tau}\,\frac{d\varrho}{ds}\,\gamma^i\right).$$

If k is the curvature of the H. D. lines we have form FRENET's formulae

$$k = \beta^i\,\frac{d\alpha^i}{ds} = \gamma^i\times\alpha^i\,\frac{d\alpha^i}{ds} = \begin{bmatrix}\gamma^i & \alpha^i & \dfrac{d\alpha^i}{ds}\end{bmatrix} =$$

$$= \begin{bmatrix}\dfrac{\tau R\cdot\sec\psi}{d\varrho/ds}\left(p^\beta\,\dfrac{du^\beta}{ds}\,\dfrac{dx^i}{ds} + \varrho\,\dfrac{\cos\psi}{R}\,\beta^i - \lambda^i\right) & \dfrac{dx^i}{ds} & \dfrac{d^2x^i}{ds^2}\end{bmatrix}.$$

Dropping out vanishing determinants we have

$$(7\cdot5)\qquad k = -\frac{\tau R\cdot\sec\psi}{d\varrho/ds}\left(\lambda^i \quad \frac{dx^i}{ds} \quad \frac{d^2x^i}{ds^2}\right) =$$

$$= -\frac{\tau R\cdot\sec\psi}{d\varrho/ds}\,(p^\eta x^i_{,\eta} + qX^i \qquad x^i_{,\mu}u'^\mu \qquad \varrho^\delta x^i_{,\delta} + k_n\,X^i) =$$

$$= -\frac{\tau R\cdot\sec\psi}{d\varrho/ds}\,[q\,e_{\mu\delta}\,\varrho^\delta u'^\mu + k_n\,p^\eta\,e_{\eta\mu}\,u'^\mu] = -\frac{\tau R\cdot\sec\psi}{d\varrho/ds}\,e_{\mu\delta}[q\varrho^\delta - k_n\,p^\delta]u'^\mu .$$

Also the union curvature of the curve is given by (C. E. SPRINGER [5])

$$k_u = \frac{1}{q} \cdot e_{\mu\delta}\, u'^{\mu} \cdot [q\varrho^{\delta} - k_n\, p^{\delta}] .$$

Hence

$$(7\cdot6) \qquad k = -\frac{\tau R \cdot \sec\psi}{\mathrm{d}\varrho/\mathrm{d}s}\, q k_u ,$$

or

$$(7\cdot7) \qquad k_u = -\frac{1}{q}\, \frac{\mathrm{d}\log\varrho}{\mathrm{d}s}\, \frac{\cos\psi}{\tau R} .$$

If H. D. curve is also a union curve, then either

(i) $\log\varrho$ that is ϱ is constant, i. e. it is a curve of MONGE; or

(ii) $\cos\psi = 0$ which shows that $b = 0$, therefore $\lambda^i = a \cdot \mathrm{d}x^i/\mathrm{d}s$, i. e. λ^i are functions of a single parameter which is impossible; or

(iii) R is infinite, i. e. it is a plane curve; or

(iv) τ is ∞ which is not true.

Hence *if any of the above results hold then* H. D. *lines are union curves.*

8. – Torsion of H. D. lines.

From (7·1) we have

$$\gamma^i = \frac{\tau R \cdot \sec\psi}{\mathrm{d}\varrho/\mathrm{d}s} \left[p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} + \varrho \frac{\cos\psi}{R}\, \beta^i - \lambda^i \right] .$$

Therefore

$$(8\cdot1) \qquad \frac{\mathrm{d}\gamma^i}{\mathrm{d}s} = \frac{\mathrm{d}L}{\mathrm{d}s} \left[p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} + \varrho \frac{\cos\psi}{R}\, \beta^i - \lambda^i \right] +$$

$$+ L \cdot \Bigg[p_{\beta,\delta} \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}u^\delta}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} + p_\beta \frac{\mathrm{d}^2 u^\beta}{\mathrm{d}s^2} \frac{\mathrm{d}x^i}{\mathrm{d}s} + p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} +$$

$$+ \frac{\mathrm{d}}{\mathrm{d}s} \frac{\varrho \cdot \cos\psi}{R} \cdot \beta^i + \frac{\varrho \cdot \cos\psi}{R} \frac{\mathrm{d}\beta^i}{\mathrm{d}s} - \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \Bigg] ,$$

where:

$$L = \frac{\tau R \cdot \sec \psi}{\mathrm{d}\varrho/\mathrm{d}s},$$

τ the torsion of the H. D. lines is therefore given by

$$\tau = \beta^i \frac{\mathrm{d}\gamma^i}{\mathrm{d}s} = \beta^i \cdot \left\{ \frac{\mathrm{d}L}{\mathrm{d}s} \left[p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} + \varrho \frac{\cos\psi}{R} \beta^i - \lambda^i \right] + \right.$$

$$+ L \cdot \left[p_{\beta,\delta} \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}u^\delta}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} + p_\beta \frac{\mathrm{d}^2 u^\beta}{\mathrm{d}s^2} \frac{\mathrm{d}x^i}{\mathrm{d}s} + \right.$$

$$\left. + p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \frac{\mathrm{d}}{\mathrm{d}s} \frac{\varrho \cdot \cos\psi}{R} \cdot \beta^i + \varrho \frac{\cos\psi}{R} \frac{\mathrm{d}\beta^i}{\mathrm{d}s} - \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right] =$$

$$= \beta^i \cdot \left\{ \frac{\mathrm{d}L}{\mathrm{d}s} \left(\varrho \frac{\cos\psi}{R} \beta^i - \lambda^i \right) + L \cdot \left(p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \frac{\mathrm{d}}{\mathrm{d}s} \frac{\varrho \cdot \cos\psi}{R} \cdot \beta^i - \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right) \right\} =$$

$$= \gamma^i \times \alpha^i \cdot \left\{ \frac{\mathrm{d}L}{\mathrm{d}s} \left(\varrho \frac{\cos\psi}{R} \beta^i - \lambda^i \right) + L \cdot \left(p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \frac{\mathrm{d}}{\mathrm{d}s} \frac{\varrho \cdot \cos\psi}{R} \cdot \beta^i - \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right) \right\} =$$

$$= \frac{\tau R \cdot \sec\psi}{\mathrm{d}\varrho/\mathrm{d}s} \left| p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} + \varrho \frac{\cos\psi}{R} \beta^i - \lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}L}{\mathrm{d}s} \left(\varrho \frac{\cos\psi}{R} \beta^i - \lambda^i \right) + \right.$$

$$\left. + L \cdot \left(p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \frac{\mathrm{d}}{\mathrm{d}s} \frac{\varrho \cdot \cos\psi}{R} \cdot \beta^i - \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right) \right| .$$

Dropping out vanishing determinants,

$$(8 \cdot 2) \qquad \tau = \frac{\tau R \cdot \sec\psi}{\mathrm{d}\varrho/\mathrm{d}s} \left| - \varrho \frac{\cos\psi}{R} \left(\beta^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right) - L p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \left(\lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} \right) - \right.$$

$$\left. - L \frac{\mathrm{d}}{\mathrm{d}s} \frac{\varrho \cdot \cos\psi}{R} \cdot \left(\lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \beta^i \right) + L \left(\lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right) \right| =$$

$$= \frac{\tau R \cdot \sec\psi}{\mathrm{d}\varrho/\mathrm{d}s} \left[- \varrho^2 \frac{\cos\psi}{R} \left(\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right) - L p_\beta \frac{\mathrm{d}u^\beta}{\mathrm{d}s} \left(\lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} \right) - \right.$$

$$\left. - L \varrho \frac{\mathrm{d}}{\mathrm{d}s} \frac{\varrho \cdot \cos\psi}{R} \cdot \left(\lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} \right) + L \left(\lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}\lambda^i}{\mathrm{d}s} \right) \right] .$$

Now

$$\left(\lambda_i \quad \frac{dx^i}{ds} \quad \frac{d^2x^i}{ds^2}\right) = -\frac{k}{\tau R\cdot \sec\psi}\,\frac{d\varrho}{ds},$$

$$\left(\lambda_i \quad \frac{dx^i}{ds} \quad \frac{d\lambda^i}{ds}\right) = [p^\delta x^i_{,\delta} + qX^i \quad x^i_{,\nu} \quad \mu^\eta_\varphi x^i_{,\eta} + v_\varphi X^i]\, u'^\nu u'^\varphi =$$

$$= e_{\delta\nu}\cdot(p^\delta v_\varphi - q\,\mu^\delta_\varphi)\, u'^\nu u'^\varphi \qquad \text{(Mishra [2])},$$

$$\left(\frac{d^2x^i}{ds^2} \quad \frac{dx^i}{ds} \quad \frac{d\lambda^i}{ds}\right) = [\varrho^\delta x^i_{,\delta} + k_n X^i \quad x^i_\nu \quad \mu^\eta_\varphi x^i_{,\eta} + v_\varphi X^i]\, u'^\nu u'^\varphi =$$

$$= e_{\delta\nu}\cdot(\varrho^\delta v_\varphi - k_n\,\mu^\delta_\varphi)\, u'^\nu u'^\varphi .$$

Hence we have

$$(8\cdot 3)\qquad \tau = \frac{\tau R\cdot\sec\psi}{d\varrho/ds}\left[-\varrho^2\frac{\cos\psi}{R}\, e_{\delta\nu}\cdot(\varrho^\delta v_\varphi - k_n\,\mu^\delta_\varphi)\, u'^\nu u'^\varphi + \right.$$

$$\left. + \frac{k}{\tau R\cdot\sec\psi}\,\frac{d\varrho}{ds}\left(Lp_\beta\frac{du^\beta}{ds} + L\varrho\cdot\frac{d}{ds}\,\frac{\varrho\cdot\cos\psi}{R}\right) + Le_{\delta\nu}\cdot(p^\delta v_\varphi - q\mu^\delta_\varphi)\, u'^\nu u'^\varphi\right] =$$

$$= -\frac{\varrho^2\tau}{d\varrho/ds}\, e_{\delta\nu}\cdot(\varrho^\delta v_\varphi - k_n\,\mu^\delta_\varphi)\, u'^\nu u'^\varphi + \frac{\tau R\cdot\sec\psi}{\varrho\cdot d\varrho/ds}\, p^\beta\,\frac{du^\beta}{ds} + \frac{\tau R\cdot\sec\psi}{d\varrho/ds}\cdot\frac{d}{ds}\,\frac{\varrho\cdot\cos\psi}{R} +$$

$$+ \left(\frac{\tau R\cdot\sec\psi}{d\varrho/ds}\right)^2 (p^\delta v_\varphi - q\mu^\delta_\varphi)\, u'^\nu u'^\varphi .$$

9. – Another form of H. D. lines.

We now givenanother form of the differential equation of H. D. lines. As before we have

$$\frac{d^3x^i}{ds^3} = \left[d_{\alpha\beta}X^i + x^i_{,\gamma}\left\{ {\gamma \atop \alpha\,\beta} \right\}\right]\varrho^\alpha\frac{du^\beta}{ds} + x^i_{,\alpha}\cdot\frac{d}{ds}\,\varrho^\alpha + \frac{d}{ds}\,k_n\cdot X^i + k_n\,\frac{\partial X^i}{\partial u^\beta}\,\frac{du^\beta}{ds} .$$

Hence the differential equation of the H. D. lines can also be written as

$$x^i_{,\alpha}\cdot\left[\left(d_{\delta\beta}\,X^i+x^i_{,\gamma}\begin{Bmatrix}\gamma\\ \beta\ \delta\end{Bmatrix}\right)\varrho^\delta\frac{\mathrm{d}u^\beta}{\mathrm{d}s}+x^i_{,\delta}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\varrho^\delta+\right.$$

$$\left.+\frac{\mathrm{d}}{\mathrm{d}s}k_n\cdot X^i+k_n\frac{\partial X^i}{\mathrm{d}u^\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\right]\left(p_\beta\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\frac{\mathrm{d}u^\delta}{\mathrm{d}s}-p^\delta\right)=$$

$$=qX^i\cdot\left[\left(d_{\delta\beta}\,X^i+x^i_{,\gamma}\begin{Bmatrix}\gamma\\ \delta\ \beta\end{Bmatrix}\right)\varrho^\delta\frac{\mathrm{d}u^\beta}{\mathrm{d}s}+x^i_{,\delta}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\varrho^\delta+\frac{\mathrm{d}}{\mathrm{d}s}k_n\cdot X^i+k_n\frac{\mathrm{d}X^i}{\mathrm{d}u^\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\right],$$

or

$$(9\cdot1)\qquad \left\{g_{\alpha\gamma}\begin{Bmatrix}\gamma\\ \delta\ \beta\end{Bmatrix}\varrho^\delta\frac{\mathrm{d}u^\beta}{\mathrm{d}s}+g_{\alpha\delta}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\varrho^\delta-k_n d_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\right\}\left(p_\beta\frac{\mathrm{d}u^\beta}{\mathrm{d}s}\frac{\mathrm{d}u^\delta}{\mathrm{d}s}-p^\delta\right)=$$

$$=q\cdot\left[d_{\alpha\beta}\varrho^\delta\frac{\mathrm{d}u^\beta}{\mathrm{d}s}+\frac{\mathrm{d}}{\mathrm{d}s}k_n\right].$$

From equation (9·1) we conclude:

Any straight line on a surface is a H. D. *line.*

We therefore have the result:

A straight line on a given surface is a G. D. *line or a* H. D. *line or a* D. *line.*

I am thankful to my teacher Dr. R. S. MISHRA for his guidance in the preparation of this paper.

References.

[1] L. P. EISENHART, *An introduction to differential geometry,* Princeton University Press, Princeton 1947.

[2] R. S. MISHRA, *A problem in rectilinear congruences using tensor calculus,* Bull. Calcutta Math. Soc. **42,** 118-122 (1950).

[3] M. PRAVANOVITCH, *Hyperlignes de Darboux appartenant à l'espace riemannien,* Bull. Sci. Math. (2) **78,** 89-96 (1954).

[4] F. SEMIN, *On Darboux lines,* Rev. Fac. Sci. Univ. Istanbul (A) **17,** 351-383 (1952).

[5] C. E. SPRINGER, *Union curves and union curvature,* Bull. Amer. Math. Soc. **51,** 686-691 (1945).