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Infinitesimal Deformations of a Riemannian Space. (**)

The object of this paper is to study the infinitesimal deformation of Riemannian spaces under the transformations defined by

$$\bar{x}^i = x^i + \varepsilon\ \lambda^i(x).$$

1. - Deformations of V_n.

Let V_n be n-dimensional Riemannian space with fundamental metric tensor $g_{ij}(x)$, whose differentiability class is at least two. We consider an infinitesimal deformation defined by

$$\bar{x}^i = x^i + \varepsilon\ \lambda^i(x), \tag{1·1}$$

where ε in an infinitesimal constant and $\lambda^i(x)$ is a vector field, whose differentiability class is at least two.

The deformed tensor field of any tensor field, say, T^l_{ij} is defined by (K. YANO and S. SASAKI [4] (1))

$$\bar{T}^l_{ij} = T^l_{ij} + LT^l_{ij} = T^l_{ij} + \varepsilon\ (T^l_{ij,k}\lambda^k - T^k_{ij}\lambda^l_{,k} + T^l_{kj}\lambda^k_{,i} + T^l_{ik}\lambda^k_{,j}),$$

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where L before a tensor field stands for the LIE derivative of that tensor field with respect to the infinitesimal deformation (1·1) and comma (,) followed by an index denotes covariant derivative with respect to the CHRISTOFFEL symbols $\left\{ {h \atop i\,j} \right\}$ calculated with respect to g_{ij}. Thus the fundamental metric tensors g_{ij} and g^{ij} deform into $\bar{g}_{ij}$ and $\bar{g}^{ij}$ respectively, such that

$$(1\cdot2)\qquad \begin{cases} \bar{g}_{ij} = g_{ij} + \varepsilon\,(\lambda_{i,j} + \lambda_{j,i}) \\ \bar{g}^{ij} = g^{ij} - \varepsilon\,(g^{ih}\lambda^{j}_{,h} + g^{hj}\lambda^{i}_{,h}). \end{cases}$$

Also the CHRISTOFFEL symbols $\left\{ {h \atop i\,j} \right\}$ deform into $\overline{\left\{ {h \atop i\,j} \right\}}$, such that (T. SUGURI [2])

$$(1\cdot3)\qquad \overline{\left\{ {h \atop i\,j} \right\}} = \left\{ {h \atop i\,j} \right\} + \varepsilon\,(\lambda^{h}_{,ij} + R^{h}_{ijk}\,\lambda^{k}).$$

The Riemannian space $\bar{V}_n$ with fundamental tensors $\bar{g}_{ij}$, and $\bar{g}^{ij}$ and the coefficients of connection $\overline{\left\{ {h \atop i\,j} \right\}}$ is called the deformed space of the Riemannian space V_n.

If u^i is any vector in V_n, then its deformed vector $\bar{u}^i$ is given by

$$\bar{u}^i = u^i + \varepsilon\,(u^{i}_{,k}\lambda^k - u^l\lambda^{i}_{,l}).$$

Therefore, using (1·2),

$$\bar{g}_{ij}\bar{u}^i\bar{u}^j = [g_{ij} + \varepsilon\,(\lambda_{i,j} + \lambda_{j,i})][u^i + \varepsilon\,(u^{i}_{,k}\lambda^k - u^l\lambda^{i}_{,l})][u^j + \varepsilon\,(u^{j}_{,p}\lambda^p - u^p\lambda^{j}_{,p})] =$$

$$= g_{ij}u^iu^j + \varepsilon\,[g_{ij}(u^{i}_{,k}\lambda^k u^j + u^iu^{j}_{,p}\lambda^p) - g_{ij}(u^iu^l\lambda^{j}_{,l} + u^iu^p\lambda^{j}_{,p}) + u^iu^j(\lambda_{i,j} + \lambda_{j,i})] =$$

$$= g_{ij}u^iu^j + \varepsilon\,[(g_{ij}u^iu^j)_{,k}\,\lambda^k].$$

If the vector u is of constant magnitude, then

$$u^2 = g_{ij}u^iu^j = \text{constant}$$

and hence

$$(g_{ij}u^iu^j)_{,k}{}^{k} = 0.$$

Therefore

$$\bar{g}_{ij}\,\bar{u}^i\,\bar{u}^j = g_{ij}\,u^i\,u^j\,.$$

Hence:

When the space V_n admits a one-parameter group of motions generated by an infinitesimal transformation (1·1), *a vector of constant magnitude is deformed into a vector of the same magnitude.*

If the vector λ^i is deformed into a vector $\bar{\lambda}^i$, then

$$\bar{\lambda}^i = \lambda^i + \varepsilon\,(\lambda^i_{,k}\lambda^k - \lambda^k\lambda^i_{,k}) = \lambda^i,$$

hence:

If the space V_n admits a one-parameter group of motions generated by an infinitesimal transformation $\bar{x}^i = x^i + \varepsilon\lambda^i$, the vector λ^i deforms into itself.

Let $e_{h/}$ $(h = 1, 2, \ldots, n)$ be an othogonal ennuple of unit vectors in V_n. The deformed vector of $e^i_{k/}$ is $\bar{e}^i_{k/}$, such that

$$\bar{e}^i_{k/} = e^i_{k/} + \varepsilon\,(e^i_{k/,p}\lambda^p - e^p_{k/}\lambda^i_{,p})\,. \tag{1·4}$$

Therefore

$$\bar{g}_{ij}\,\bar{e}^i_{k/}\,\bar{e}^j_{k/} = [g_{ij} + \varepsilon\,(\lambda_{i,j} + \lambda_{j,i})]\,[e^i_{k/} + \varepsilon\,(e^i_{k/,p}\lambda^p - e^p_{k/}\lambda^i_{,p})]\,[e^j_{h/} + \varepsilon\,(e^j_{h/,l}\lambda^l - e^l_{h/}\lambda^j_{,l})] =$$

$$= \delta^h_k + \varepsilon\,[(g_{ij}\,e^i_{h/}\,e^j_{k/})_{,p}\lambda^p - g_{ij}\,e^i_{k/}\,e^l_{h/}\,\lambda^j_{,l} - g_{ij}\,e^p_{k/}\,e^j_{h/}\,\lambda^i_{,p} + e^i_{k/}e^j_{h/}(\lambda_{i,j} + \lambda_{j;i})] = \delta^h_k\,,$$

where δ^h_k are KRONECKER deltas. Hence:

When the space V_n admits a one parameter group of motions generated by an infinitesimal transformation (1·1), *an orthogonal ennuple in V_n is deformed into an orthogonal ennuple.*

Let x^i and $x^i + \mathrm{d}x^i$ be two consecutive points in V_n, the distance between them being $\mathrm{d}s$, so that

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j.$$

If $\bar{x}^i$ and $\bar{x}^i + \mathrm{d}\bar{x}^i$ are the two points in $\bar{V}_n$, corresponding to x^i and $x^i + \mathrm{d}x^i$ in V_n, the distance $\mathrm{d}\bar{s}$ between them is given by the relation

$$\mathrm{d}\bar{s}^2 = \bar{g}_{ij}\,\mathrm{d}\bar{x}^i\,\mathrm{d}\bar{x}^j.$$

Therefore

$$d\bar{s}^2 =$$

$$= [g_{ij} + \varepsilon\,(\lambda_{i,j} + \lambda_{j;i})][dx^i + \varepsilon\{(dx^i)_{,k}\lambda^k - dx^k\lambda^i_{,k}\}][dx^j + \varepsilon\{(dx^j)_{,q}\lambda^q - dx^q\lambda^j_{,k}\}] =$$

$$= ds^2 + \varepsilon\,[(g_{ij}dx^i dx^j)_{,k}\lambda^k - dx^k dx^j\lambda^i_{,k}g_{ij} - g_{ij}dx^i dx^l\lambda^j_{,l} + dx^i dx^j(\lambda_{i,j} + \lambda_{j,i})] =$$

$$= ds^2 + \varepsilon\,\frac{\partial}{\partial x^k}\,(ds)^2\lambda^k = ds^2 + 2\varepsilon\,ds\;\psi,$$

where ψ is the derivative of ds along the vector λ^k.

Hence to the first order of smallness

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1 + \frac{2\varepsilon}{ds}\,\psi,$$

i. e.

$$\frac{d\bar{s}}{ds} = 1 + \frac{\varepsilon\psi}{ds},$$

and

$$\frac{ds}{d\bar{s}} = 1 - \frac{\varepsilon}{ds}\,\psi. \tag{1·5}$$

2. - Ricci's coefficients of rotation.

Let $e_{h|}$ $(h = 1, 2, \ldots, n)$ be the unit tangents to the n congruences of an orthogonal ennuple in a Riemannian V_n, then from § **1** the deformed vector $\bar{e}_{h|}$ $(h = 1, 2, \ldots, n)$ will also be the unit tangents to the n congruences of an orthogonal ennuple in $\bar{V}_n$. Hence RICCI's coefficients of rotation $\bar{\gamma}_{hkl}$ in $\bar{V}_n$ are given by

$$\bar{\gamma}_{hkl} = \bar{e}_{h|;j}\,\bar{e}_{k|i}\,\bar{e}^j_{l|}, \tag{2·1}$$

where semicolon (;) followed by an index denotes covariant derivative with respect to $\bar{V}_n$, and $\bar{e}^i_{h|;j}$ is given by

$$\bar{e}^i_{h|;j} = e^i_{h|,j} + \varepsilon\,(e^i_{h|,jk}\lambda^k - e^l_{h|,j}\lambda^i_{,l} + e^i_{h|,l}\lambda^l_{,j}). \tag{2·2}$$

Also $\bar{e}^i_{k/}$ and $\bar{e}^j_{l/}$ are given by the relations

$$\bar{e}_{k/i} = e_{k/i} + \varepsilon\,(e_{k/i,j}\lambda^j + e_{k/l}\lambda^l_{,i})\,, \tag{2·3}$$

$$\bar{e}^j_{l/} = e^j_{l/} + \varepsilon\,(e^j_{l/,k}\lambda^k - e^k_{l/}\lambda^j_{,k})\,. \tag{2·4}$$

Now using (2·2), (2·3) and (2·4), the equation (2·1) assumes the form

$$\bar{y}_{hkl} = y_{hkl} + \varepsilon\,\frac{\partial}{\partial x^p}(y_{hkl})\lambda^p, \tag{2·5}$$

where y_{hkl} are the RICCI's coefficients of rotation in V_n.

From (2·5) it follows that $\bar{y}_{hkl}$ is the deformed scalar of y_{hkl}. Hence:

The deformed scalar of the Ricci's coefficients of rotations in V_n are Ricci's coefficients of rotation in $\bar{V}_n$.

Also the following results follow from the equation (2·5):

When the space V_n admits a one-parameter group of motions generated by an infinitesimal transformation (1·1), *then if the curves of the congruence whose unit tangent is $e_{h/}$ be geodesics in V_n, the curves of the congruence in $\bar{V}_n$ whose unit tangent is $\bar{e}_{h/}$ will also be geodesics.*

When the space V_n admits a one-parameter group of motions generated by an infinitesimal transformation (1·1), *then if all the congruences of an orthogonal ennuple are normal in V_n, so are all the congruences of an orthogonal ennuple in $\bar{V}_n$.*

When the space V_n admits a one-parameter group of motions generated by an infinitesimal transformation (1·1), *then if the congruence $e_{h/}$ is irrotational in V_n so is the congruence $\bar{e}_{h/}$ in $\bar{V}_n$.*

3. – Condition that a curve in V_n may deform into a curve.

Any vector $\bar{u}^l$ of the deformed Riemannian space is of the form

$$\bar{u}^l = u^l + \varepsilon\,(u^l_{,k}\lambda^k - u^p\lambda^l_{,p}). \tag{3·1}$$

Let C be a curve in V_n and $\xi^i_{(1)}$ its unit tangent vector at the point P. When the space V_n is deformed into the space $\bar{V}_n$, under the transformation (1·1), let the curve C deform into a curve C'. Then the unit tangent to C', $d\bar{x}^i/d\bar{s}$ is given by

$$(3\cdot 2)\qquad \bar{\xi}^i_{(1)} \equiv \frac{d\bar{x}^i}{d\bar{s}} = \frac{d\bar{x}^i}{ds}\frac{ds}{d\bar{s}} = \left(\xi^i_{(1)} + \varepsilon \frac{d\lambda^i}{ds}\right)\left(1 - \varepsilon \frac{\psi}{ds}\right) = \xi^i_{(1)} + \varepsilon\left(\frac{d\lambda^i}{ds} - \frac{\psi}{ds}\xi^i_{(1)}\right).$$

Comparing (3·2) with (3·1), it follows that the condition that a curve C may deform into a curve C' is

$$\frac{d\lambda^i}{ds} - \frac{\psi}{ds}\xi^i_{(1)} = \xi^i_{(1),k}\lambda^k - \xi^l_{(1)}\lambda^i_{,l} = \frac{\partial \xi^i_{(1)}}{\partial x^k}\lambda^k - \xi^l_{(1)}\frac{\partial \lambda^i}{\partial x^l},$$

i.e.

$$(3\cdot 3)\qquad 2\frac{d\lambda^i}{ds} = \frac{\partial \xi^i_{(1)}}{\partial x^k}\lambda^k + \xi^i_{(1)}\frac{\psi}{ds}.$$

Thus (3·3) *is the condition that a curve* C *in* V_n *may deform into a curve* C' *when the space* V_n *admits a one-parameter group of motions generated by an infinitesimal transformation* (1·1).

For deformation along the tangent, condition (3·3) reduces to

$$\frac{d\lambda^i}{ds} = \xi^i_{(1)}\frac{\psi}{ds},$$

i.e.

$$\bar{\xi}^i_{(1)} = \xi^i_{(1)}.$$

In what follows, we shall assume that (3·3) is satisfied, i.e. the curve C is deformed into the curve C'.

4. – Parallel tangent deformation.

If $\xi^i_{(1)}$ is the unit tangent vector to the curve C, at the point P in V_n, then the unit tangent vector $\bar{\xi}^i_{(1)}$ to the deformed curve C' at the corresponding point P' in $\bar{V}_n$ is given by the relation (3·2), i.e.

$$(4\cdot 1)\qquad \bar{\xi}^i_{(1)} = \xi^i_{(1)} + \varepsilon\left(\frac{d\lambda^i}{ds} - \frac{\psi}{ds}\xi^i_{(1)}\right).$$

Let $\bar{\bar{\xi}}^i_{(1)}$ be a vector at a point P', parallel to the vector $\xi^i_{(1)}$ at P, so that to the first order of smallness

$$\bar{\bar{\xi}}^i_{(1)} = \xi^i_{(1)} - \left\{ {i \atop j\,k} \right\} \xi^k_{(1)} (\bar{x}^j - x^j) = \xi^i_{(1)} - \varepsilon\, \lambda^k\, \xi^j_{(1)} \left\{ {i \atop j\,k} \right\}, \tag{4·2}$$

where $\left\{ {i \atop j\,k} \right\}$ are the CHRISTOFFEL symbols of the second kind for V_n at P.

Now $\bar{\xi}_{(1)} - \bar{\bar{\xi}}^i_{(1)}$ is an infinitesimal vector, which we shall denote by $\delta\xi^i_{(1)}$. The condition that the tangent to the curve be displaced parallelly during the deformation is that

$$\Delta \xi^i_{(1)} \equiv \lim_{\delta s \to 0} \frac{\delta \xi^i_{(1)}}{\delta s} = 0\,,$$

where

$$\mathrm{d}s = \sqrt{g_{ij} (\bar{x}^j - x^j)(\bar{x}^i - x^i)} = \varepsilon\lambda\,,$$

λ denoting the magnitude of the vector with contravariant components λ^i. Subtracting (4·2) from (4·1) we get

$$\delta\xi^i_{(1)} = \varepsilon \left(\frac{\mathrm{d}\lambda^i}{\mathrm{d}s} + \left\{ {i \atop j\,k} \right\} \xi^j_{(1)} \lambda^k - \frac{\psi}{\mathrm{d}s}\, \xi^i_{(1)} \right).$$

Hence the condition for parallel tangent deformation is

$$\frac{\mathrm{d}\lambda^i}{\mathrm{d}s} + \left\{ {i \atop j\,k} \right\} \xi^i_{(1)} \lambda^k - \frac{\psi}{\mathrm{d}s}\, \xi^i_{(1)} = 0\,,$$

i.e.

$$\mathrm{D}\lambda^i = \frac{\psi}{\mathrm{d}s}\, \xi^i_{(1)}, \tag{4·3}$$

where D now stands for the derived vector with regard to the curve C.

Condition (4·3) can be written in the form

$$\begin{cases} \mathrm{D}\lambda^i\, \xi_{(\alpha) i} = 0 & (\alpha = 2,\ 3, \ldots,\ n), \\ \mathrm{D}\lambda^i\, \xi_{(1) i} = \dfrac{\psi}{\mathrm{d}s}\,. \end{cases} \tag{4·4}$$

Thus the equations (4·4) *are the conditions that the tangents to a curve may be displaced parallelly, when the space V_n admits a one-parameter group of motions generated by an infinitesimal deformation* (1·1).

5. – Alternative forms for the condition of parallel tangent deformation.

λ^i being a vector field in V_n, can be expressed linearly in terms of any n vectors, which do not lie in the same geodesic surface. Let these n vectors be a vector $\xi^i_{(1)}$ tangent to the curve C and $n-1$ vectors $\xi^i_{(\alpha)}$ $(\alpha = 2, 3, \ldots, n)$ which are the first normal, the second normal, ..., the $(n-1)^{\text{th.}}$ normal respectively to the curve, so that

$$\lambda^i = \sum_{a=1}^{n} c_{a/}\, \xi^i_{(a)}\,, \tag{5·1}$$

where $c_{p/} = \lambda^i \xi_{(p)i}$. The condition (4·3) now assumes the form

$$\sum_{a=1}^{n} \left(\frac{dc_{a/}}{ds}\, \xi^i_{(a)} + c_{(a)} D\, \xi^i_{(a)} \right) = \frac{\psi}{ds}\, \xi^i_{(1)}\,, \tag{5·2}$$

i.e.

$$\sum_{a=1}^{n} \left(\frac{dc_{a/}}{ds}\, \xi^i_{(a)} + c_{(a)}\, \xi^i_{(a),k}\, \xi^k_{(1)} \right) = \frac{\psi}{ds}\, \xi^i_{(1)}\,. \tag{5·2'}$$

Multiplying (5·2′) by $\xi_{(\beta)i}$ $(\beta = 1, 2, \ldots, n)$ and summing for i, we get

$$\sum_{a=1}^{n} \left(\frac{dc_{a/}}{ds}\, \xi^i_{(a)}\, \xi_{(\beta)i} + c_{a/}\, \xi^i_{(a),k}\, \xi^k_{(1)}\, \xi_{(\beta)i} \right) = \frac{\psi}{ds}\, \xi^i_{(1)}\, \xi_{(\beta)i}\,,$$

i.e.

$$\frac{dc_{\beta/}}{ds} + \sum_{a=1}^{n} c_{a/}\, y_{a\beta 1} = \frac{\psi}{ds}\, \delta^{\beta}_{1}\,,$$

which can be written in the form

$$(5\cdot3)\qquad \begin{cases} \dfrac{\mathrm{d}c_{1/}}{\mathrm{d}s} + \sum\limits_{\alpha=1}^{n} c_{\alpha/}\, y_{\alpha 11} = \dfrac{\psi}{\mathrm{d}s} \\ \dfrac{\mathrm{d}c_{\beta/}}{\mathrm{d}s} + \sum\limits_{\alpha=1}^{n} c_{\alpha/}\, y_{\alpha\beta 1} = 0 \qquad (\beta = 2,\ 3,\ \dots,\ n), \end{cases}$$

where y_{lhk} are the RICCI's coefficients of rotation (C. E. WEATHERBURN [**3**], p. 98).

Thus (5·3) *is the alternative form for* (4·3).

From (5·3) we have:

If $c_1 =$ *constant and the curve* C *is a geodesic, then the first variation of the arc is zero.*

If all the c' *are constants with* $c_1 = 0$ *and all the congruences of the orthogonal ennuple are normal, then the deformation is parallel tangent deformation.*

Also from FRENET's formulae we have

$$\mathrm{D}\xi^i_{(\alpha)} = k_\alpha\, \xi^i_{(\alpha+1)} - k_{\alpha-1}\, \xi^i_{(\alpha-1)} \qquad (\alpha = 1,\ 2,\ \dots,\ n;\ k_0 = 0;\ k_n = 0),$$

where $k_1, k_2, \dots, k_n$ are the first, second, ..., $(n-1)^{\text{th.}}$ curvatures respectively, of the curve C relative to the Riemannian space V_n.

Therefore the condition (5·2) can be expressed as

$$\sum_{\alpha=1}^{n} \left\{ \frac{\mathrm{d}c_{\alpha/}}{\mathrm{d}s}\, \xi^i_{(\alpha)} + c_{\alpha/} (k_\alpha \xi^i_{(\alpha+1)} - k_{\alpha-1}\, \xi^i_{(\alpha-1)}) \right\} = \frac{\psi}{\mathrm{d}s}\, \xi^i_{(1)}\,.$$

Since $k_0 = k_n = 0$, this equation assumes the form

$$(5\cdot4)\qquad \sum_{\alpha=1}^{n} \left(\frac{\mathrm{d}c_{\alpha/}}{\mathrm{d}s} + k_{\alpha-1}\, c_{\alpha-1/} - k_\alpha\, c_{\alpha+1/} \right) \xi^i_{(\alpha)} = \frac{\psi}{\mathrm{d}s}\, \xi^i_{(1)}\,.$$

Multiplying (5·4) by $\xi_{(1)i}$ and summing for i, we get

$$\frac{\psi}{\mathrm{d}s} = \frac{\mathrm{d}c_{1/}}{\mathrm{d}s} - k_1\, c_{2/}.$$

Hence the condition (5·4) reduces to

$$\sum_{\alpha=2} \left(\frac{\mathrm{d}c_{\alpha/}}{\mathrm{d}s} + k_{\alpha-1}\, c_{\alpha-1/} - k_\alpha\, c_{\alpha+1/} \right) \xi^i_{(\alpha)} = 0\,.$$

Therefore the condition that λ^i be a parallel tangent deformation is

$$(5\cdot5)\qquad \left\{\begin{aligned} \frac{dc_{2/}}{ds} &= k_2\, c_{3/} - k_1\, c_{1/} \\ \frac{dc_{3/}}{ds} &= k_3\, c_{4/} - k_2\, c_{2/} \\ &\cdots\cdots\cdots\cdots \\ \frac{dc_{n/}}{ds} &= \qquad - k_{n-1}\, c_{n-1/}\,. \end{aligned}\right.$$

Thus (5·5) *is another form of the condition of parallel tangent deformation.*

If all the c' are constants, then (5·5) is the condition that the curve be a generalised helix (H. A. HAYDEN [**1**]).

References.

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