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Congruences of Curves in a Subspace of a non-Riemannian Space. (**)

In this paper I have considered a set of $m-n$ congruences of curves in a non-Riemannian space, which are such that through each point of a subspace, a curve of each congruence passes. With the help of these congruences the generalisations of the generalised equations of GAUSS and CODAZZI have been established in a non-Riemannian space.

1. - Notation (1).

Let us consider a space V_m referred to the coordinates y^α $(\alpha = 1, \ldots, m)$ and coeffiicients of symmetric connection $L^\alpha_{\beta\gamma}$ and a subspace or a sub-variety V_n defined by

$$\varphi^\sigma(y^1, \ldots, y^m) = 0 \qquad (\sigma = n + 1, \ldots, m) . \tag{1.1}$$

Let us further put

$$x^i = \varphi^i(y^1, \ldots, y^m), \qquad x^\sigma = \varphi^\sigma, \tag{1.2}$$

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(1) In what follows Latin indices take the values 1, ..., n; the letters at the beginning of the Greek alphabet the values 1, ..., m and those at the end the values $n+1$, ..., m. The indices on the left of the solidus (/) indicate the particular tensor and on the right the contravariance or the covariance of the tensor.

where the functions φ^i are arbitrary functions, except that the Jacobian of the φ's is not equal to zero. The equations (1.2) define, then, a coordinate system of which V_n is defined by $x^\sigma = 0$. The covariant vectors $v^\sigma/_a$ in V_m defined by

$$v^\sigma/_\alpha = \partial\varphi^\sigma/\partial y^\alpha = \partial x^\sigma/\partial y^\alpha, \tag{1.3}$$

are the covariant pseudo-normals to V_n. The components of a contravariant vector tangential to the curve of parameter x^σ at a point are given by

$$v_\sigma/^\alpha = \partial y^\alpha/\partial x^\sigma, \tag{1.4}$$

for a given value of σ. In consequence of (1.3) and (1.4) we have

$$v^\sigma/_\alpha \; v_\tau/^\alpha = \delta^\sigma_\tau\,, \tag{1.5}$$

and

$$v_\sigma/^\alpha \cdot \frac{\partial x^i}{\partial y^\alpha} = 0\,. \tag{1.6}$$

If a semi-colon (;) followed by Latin indices indicates repeated tensor derivatives with regard to the induced symmetric connection L^i_{jk} in V_n, and comma (,) followed by Greek indices denotes tensor derivatives with regard to the induced connection $L^\alpha_{\beta\gamma}$ in V_m, the following relations obtain (EISENHART [2], p. 173)

$$y^\alpha_{;\,ij} = w^\sigma/_{ij}\, v_\sigma/^\alpha, \tag{1.7}$$

$$w^\sigma/_{ij} = v^\sigma/_{\alpha,\beta}\, \frac{\partial y^\alpha}{\partial x^i}\, \frac{\partial y^\beta}{\partial x^j}\,. \tag{1.8}$$

2. – Congruences of curves.

Let us consider a set of $m - n$ congruences of curves in V_m, which are such that one curve of each congruence passes through each point of the subspace V_n. Let $\lambda_\tau/^\alpha$ be the components of a contravariant vector in the direction of a curve of a congruence through the point y^α of the subspace. The vector with components $\lambda_\tau/^\alpha$ is not, in general, in the direction of $v_\tau/^\alpha$ and, therefore, it may be specified by

$$\lambda_\tau/^\alpha = t_\tau/^l\, \frac{\partial y^\alpha}{\partial x^l} + c^\nu_\tau/\, v_\nu/^\alpha, \tag{2.1}$$

where

$$c_{\tau}^{\nu}/ = \lambda_{\tau}/^{\alpha} v^{\nu}/_{\alpha} \tag{2.2}$$

and

$$t_{\tau}/^{l} = \lambda_{\tau}/^{\alpha} \frac{\partial x^{l}}{\partial y^{\alpha}}. \tag{2.3}$$

Tensor derivative of (2.1) with regard to the induced connection in V_n, is given by

$$\lambda_{\tau}/^{\alpha}_{;\,i} = t_{\tau}/^{l}_{;\,i} \frac{\partial y^{\alpha}}{\partial x^{l}} + t_{\tau}/^{l} y^{\alpha}_{;\,li} + c_{\tau}^{\nu}/_{;\,i} v_{\nu}/^{\alpha} + c_{\tau}^{\nu}/ v_{\nu}/^{\alpha}_{;\,i}. \tag{2.4}$$

Now (EISENHART [2], pp. 173, 175)

$$v_{\nu}/^{\alpha}_{;\,i} = l_{\nu}/^{j}_{i} \frac{\partial y^{\alpha}}{\partial x^{j}} + l_{\nu}^{\tau}/_{i} v_{\tau}/^{\alpha}, \tag{2.5}$$

where

$$l_{\nu}/^{j}_{i} = v_{\nu}/^{\alpha}_{;\,i} \frac{\partial x^{j}}{\partial y^{\alpha}} \tag{2.6}$$

and

$$l_{\nu}^{\tau}/_{i} = v_{\nu}/^{\alpha}_{;\,i} v^{\tau}/_{\alpha}. \tag{2.7}$$

By virtue of (1.7) and (2.5), the equation (2.4) assumes the form

$$\lambda_{\tau}/^{\alpha}_{;\,i} = p_{\tau}/^{l}_{i} \frac{\partial y^{\alpha}}{\partial x^{l}} + p_{\tau}^{\nu}/_{i} v_{\nu}/^{\alpha}, \tag{2.8}$$

where

$$p_{\tau}/^{l}_{i} = t_{\tau}/^{l}_{;\,i} + c_{\tau}^{\nu}/ l_{\nu}/^{l}_{i} \tag{2.9}$$

and

$$p^\nu_\tau/_i = t_\tau/^l\ w^\nu/_{li} + c^\nu_\tau/_{;\,i} + c^\mu_\tau/\ l^\nu_\mu/_i. \tag{2.10}$$

Tensor derivative of (2.8) again yields

$$\lambda_\tau/^\alpha_{;\,ij} = p_\tau/^l_{i;\,j} \frac{\partial y^\alpha}{\partial x^l} + p_\tau/^l_i\, y^\alpha_{;\,lj} + p^\nu_\tau/_{i;\,j}\, v_\nu/^\alpha + p^\nu_\tau/_i\, v_\nu/^\alpha_{;\,j} = \tag{2.11}$$

$$= (p_\tau/^l_{i;\,j} + p^\nu_\tau/_i\, l_\nu/^l_j) \frac{\partial y^\alpha}{\partial x^l} + (p^\nu_\tau/_{i;\,j} + p_\tau/^l_i\, w^\nu/_{lj} + p^\mu_\tau/_i\, l^\nu_\mu/_j)\, v_\nu/^\alpha.$$

Applying the conditions of integrability to (2.8), we get, by virtue of (2.8) and (2.11),

$$B^\alpha_{\beta\gamma\delta}\, \lambda_\tau/^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} + (p_\tau/^l_{i;\,j} + p^\nu_\tau/_i\, l_\nu/^l_j - p_\tau/^l_{j;\,i} - p^\nu_\tau/_j\, l_\nu/^l_i) \frac{\partial y^\alpha}{\partial x^l} + \tag{2.12}$$

$$+ (p^\nu_\tau/_{i;\,j} + p_\tau/^l_i\, w^\nu/_{lj} + p^\mu_\tau/_i\, l^\nu_\mu/_j - p^\nu_\tau/_{j;\,i} - p_\tau/^l_j\, w^\nu/_{li} - p^\mu_\tau/_j\, l^\nu_\mu/_i)\, v_\nu/^\alpha = 0,$$

where $B^\alpha_{\beta\gamma\delta}$ are the components of curvature tensor of V_m.

If these equations be multiplied by $\partial x^h/\partial y^\alpha$ and $v^\nu/_\alpha$ and be summed, we have the respective equations

$$B^\alpha_{\beta\gamma\delta}\, \lambda_\tau/^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial x^h}{\partial y^\alpha} + p_\tau/^h_{i;\,j} - p_\tau/^h_{j;\,i} + p^\nu_\tau/_i\, l_\nu/^h_j - p^\nu_\tau/_j\, l_\nu/^h_i = 0, \tag{2.13}$$

$$B^\alpha_{\beta\gamma\delta}\, \lambda_\tau/^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} v^\nu/_\alpha + p^\nu_\tau/_{i;\,j} + p_\tau/^l_i\, w^\nu/_{lj} + p^\mu_\tau/_i\, l^\nu_\mu/_j - \tag{2.14}$$

$$- (p^\nu_\tau/_{j;\,i} + p_\tau/^l_j\, w^\nu/_{li} + p^\mu_\tau/_j\, l^\nu_\mu/_i) = 0.$$

The equations (2.13) *and* (2.14) *are the generalisations of the generalised equations of Gauss and Codazzi in a non-Riemannian space* (EISENHART **[2]**, p. 176).

From (2.8), we have

$$\lambda_\tau/^\alpha_{;\,i}\ v^\nu/_\alpha = p^\nu_\tau/_i, \tag{2.15}$$

and

$$\lambda_\tau/^{\alpha}_{;\,i} \frac{\partial x^l}{\partial y^\alpha} = p_\tau/^l_i \,. \tag{2.16}$$

Therefore:

The equations (2.13) *and* (2.14) *are the equations giving the expressions for* $p^\nu_\tau/_i$, *defined by* (2.15).

In case the vectors with components $\lambda_\tau/^\alpha$ are in the direction of $v_\tau/^\alpha$, we have from (2.15) and (2.16)

$$p^\nu_\tau/_i = l^\nu_\tau/_i, \qquad p_\tau/^l_i = l_\tau/^l_i \;;$$

and the equations (2.13) and (2.14) assume the forms:

$$B^\alpha_{\beta\gamma\delta}\, v_\tau/^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial x^h}{\partial y^\alpha} + l_\tau/^h_{i;\,j} - l_\tau/^h_{j;\,i} + l^\nu_\tau/_i\, l_\nu/^h_j - l^\nu_\tau/_j\, l_\nu/^h_i = 0$$

and

$$B^\alpha_{\beta\gamma\delta}\, v_\tau/^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j}\, v^\nu/_\alpha + l^\nu_\tau/_{i;\,j} - l_\tau/^\nu_{j;\,i} + l_\tau/^l_i\, w^\nu/_{lj} - l_\tau/^l_j\, w^\nu/_{li} +$$
$$+ l^\mu_\tau/_i\, l^{\,\nu}_{\mu}/_j - l^\mu_\tau/_j\, l^\nu_\mu/_i = 0$$

which are the generalised equations of Gauss and Codazzi (Eisenhart [**2**], p. 176).

Let us now consider the case when $m = n + 1$. V_n will then reduce to a hypersurface and the equations (2.13) and (2.14) will assume the forms

$$B^\alpha_{\beta\gamma\delta}\, \lambda^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial x^h}{\partial y^\alpha} + p^h_{i;\,j} - p^h_{j;\,i} + p_i\, l^h_j - p_j\, l^h_i = 0$$

and

$$B^\alpha_{\beta\gamma\delta}\, \lambda^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j}\, v_\alpha + p_{i;\,j} - p_{j;\,i} + p^l_i\, w_{lj} - p^l_j\, w_{li} + p_i\, l_j - p_j\, l_i = 0,$$

where λ^a are the components of a contravariant vector in the direction of a curve of a congruence of curves, which are such that one curve of each congruence passes through each point of the hypersurface and

$$p_i = \lambda^a_{;i} v_a, \qquad p^h_i = \lambda^a_{;i} \frac{\partial x^h}{\partial y^a}, \qquad w_{ij} = v_{a,\beta} \frac{\partial y^a}{\partial x^i} \frac{\partial y^\beta}{\partial x^j},$$

v_a being the components of the covariant pseudonormal to V_n.

3. – Fundamental equations.

Substituting the values of $\lambda_\tau/^a$, $p_\tau/^l_i$ and $p^\nu_\tau/_i$ from (2.1), (2.9) and (2.10) in (2.13) and (2.14), and using the relation

$$t_\tau/^h_{;ij} - t_\tau/^h_{;ji} = - t_\tau/^l B^h_{lij},$$

we get after simplification and re-arranging the terms

$$(3.1) \qquad t_\tau/^l \left[B^a_{\beta\gamma\delta} \frac{\partial y^\beta}{\partial x^l} \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial x^h}{\partial y^a} + l_\nu/^h_j w^\nu/_{li} - l_\nu/^h_i w^\nu/_{lj} - B^h_{lij} \right] +$$

$$+ c^\nu_\tau/ \left[B^a_{\beta\gamma\delta} v_\nu/^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial x^h}{\partial y^a} + l_\nu/^h_{i;j} - l_\nu/^h_{j;i} + l^\mu_\nu/_i l_\mu/^h_j - l^\mu_\nu/_j l_\mu/^h_i \right] = 0$$

and

$$(3.2) \qquad t_\tau/^l \left[B^a_{\beta\gamma\delta} v^\nu/_a \frac{\partial y^\beta}{\partial x^l} \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} + w^\nu/_{li;j} - w^\nu/_{lj;i} + l^\nu_\mu/_j w^\mu/_{li} - l^\nu_\mu/_i w^\mu/_{lj} \right] +$$

$$+ c^\mu_\tau / \left[B^a_{\beta\gamma\delta} v_\mu/^\beta v^\nu/_a \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} + l^\nu_\mu/_{i;j} - l^\nu_\mu/_{j;i} + \right.$$

$$\left. + w^\nu/_{lj} l_\mu/^l_i - w^\nu/_{li} l_\mu/^l_j + l^\varrho_\mu/_i l^\nu_\varrho/_j - l^\varrho_\mu/_j l^\nu_\varrho/_i \right] = 0 .$$

Since the equations (3.1) and (3.2) hold for arbitrary $\lambda_\tau/^a$, they hold for arbitrary values of $c^\nu_\tau/$ and $t_\tau/^l$. Hence the expressions within parentheses of the equations (3.1) and (3.2) are identically zero. We, therefore, have the fundamental equations for V_n:

$$B^h_{lij} = B^\alpha_{\beta\gamma\delta} \frac{\partial y^\beta}{\partial x^l} \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial x^h}{\partial y^\alpha} + l_\nu/^h_j\, w^\nu/_{li} - l_\nu/^h_i\, w^\nu/_{lj}, \tag{3.3}$$

$$B^\alpha_{\beta\gamma\delta}\, v_\nu/^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial x^h}{\partial y^\alpha} + l_\nu/^h_{i;\,j} - l_\nu/^h_{j;\,i} + l^\mu_\nu/_i\, l_\mu/^h_j - l^\mu_\nu/_j\, l_\mu/^h_i = 0, \tag{3.4}$$

$$B^\alpha_{\beta\gamma\delta}\, v^\nu/_\alpha \frac{\partial y^\beta}{\partial x^l} \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} + w^\nu/_{li;\,j} - w^\nu/_{lj;\,i} + l^\nu_\mu/_j\, w^\mu/_{li} - l^\nu_\mu/_i\, w^\mu/_{lj} = 0, \tag{3.5}$$

$$B^\alpha_{\beta\gamma\delta}\, v_\mu/^\beta\, v^\nu/_\alpha \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} + l^\nu_\mu/_{i;\,j} - l^\nu_\mu/_{j;\,i} + \tag{3.6}$$

$$+ w^\nu/_{lj}\, l_\mu/^l_i - w^\nu/_{li}\, l_\mu/^l_j + l^\varrho_\mu/_i\, l^\nu_\varrho/_j - l^\varrho_\mu/_j\, l^\nu_\varrho/_i = 0.$$

The four equations (3.3), (3.4), (3.5) *and* (3.6) *are the generalised equations of Gauss and Mainardi-Codazzi for a subspace V_n embedded in a non-Riemannian space V_m first obtained by Voss and Ricci* (EISENHART [**1**], p. 168).

It may be mentioned that these equations have been obtained here in a different but simpler way.

References.

1. L. P. EISENHART, *Riemannian Geometry*, University Press, Princeton 1926.

2. L. P. EISENHART, *Non-Riemannian Geometry*, University Press, Princeton 1927.