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II. - A Cyclic Additivity Theorem of the Lebesgue Area. (**)

Introduction.

This paper is the second in a series of three papers, and deals with a cyclic additivity theorem of the LEBESGUE area, i.e., the theory developed in [**3**], will now be applied to the LEBESGUE area. In order to simplify references, the paper [**3**] will be referred to with the Roman numeral **I** followed, if necessary, by an Arabic numeral indicating the specific paragraph in **I**.

In his recent book, L. CESARI [**1**] introduced the concept of an admissible set A in E_2, where E_2 is the Euclidean plane. For the present it suffices to observe that such an admissible set need not be a PEANO space in E_2. For $(T,\ A)$ a continuous mapping from an admissble set $A \subset E_2$ into the Euclidean three space E_3, L. CESARI has defined the LEBESGUE area $L(T,$ A) of $(T,\ A)$ (see [**1**; 5.8]).

If A is a simply connected JORDAN region in E_2, and if $(T,\ A) = sf$, $f : A \to \mathfrak{M}$, $s : \mathfrak{M} \to E_3$ is an unrestricted factorization of $(T,\ A)$ in the sense of E. J. MICKLE and T. RADÓ [**2**], then from [**2**] the following cyclic additivity formula for $L(T,\ A)$ is available

$$(1) \qquad L(T,\ A) = \sum L(s\, r_C\, f,\ A), \qquad C \subset \mathfrak{M},$$

where r_C is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$.

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If an admissible set A is not a PEANO space in E_2, then a formula similar to (1) can be established by using the concept of an unrestricted factorization of (T, A) as introduced in **I.9**, i.e., $(T, A) = sf$, $f: A \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$. The main result of this paper will be

$$(2) \qquad L(T, A) = \sum L(s\, r_C\, f, G_C), \qquad C \in \mathfrak{K},$$

where r_C is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$, where $\mathfrak{K}$ is the class of proper cyclic elements associated with $(T, A) = sf$, and where G_C is the set associated with $C \in \mathfrak{K}$. For the above terminology the reader is referred to **I.12**. In order to establish (2) it will be sufficient to verify that $L(T, A)$ satisfies the conditions of **I.8**, with the collection of PEANO spaces P replaced by the collection of all finitely connected polygonal regions in E_2.

A Cyclic Additivity Theorem.

II.1. – A subset A of the Euclidean plane E_2 will be termed *admissible* provided one of the following cases holds. (a) A is a simply connected JORDAN region; (b) A is a finitely connected JORDAN region; (c) A is a finite union of disjoint regions of the type (a) or (b); (d) A is any open set in E_2; (e) A is any set open in a set of the type (a), (b), or (c). In particular A may be a *figure* F, i.e., a finite union of disjoint finitely connected polygonal regions. The reader is referred to [**1**; 5.1].

II.2. – In this paragraph we will give some definitions and those properties of the LEBESGUE area which are needed in the sequel.

(i) Given two continuous mappings (T, A), (T', A') from admissible sets A, A' into E_3 and $A \cap A' \neq 0$. For H a non-empty subset of $A \cap A'$, the number $d = d(T, T', H) = \underset{w \in H}{\text{l.u.b.}} \, | T(w) - T'(w) |$, satisfies the following properties: (a) $0 \leqslant d \leqslant \infty$; (b) $d(T, T', H) = d(T', T, H)$; (c) for any (T_i, A_i) $(i = 1, 2, 3)$ and $0 \neq H = A_1 \cap A_2 \cap A_3$, we have $d(T_1, T_3, H) \leqslant d(T_1, T_2, H) + d(T_2, T_3, H)$ (L. CESARI [**1**; 5.3]).

(ii) A sequence of continuous mappings (T_n, A_n) $(n = 1, 2, 3, \ldots)$ is said to converge to a continuous mapping (T, A), written $(T_n, A_n) \to (T, A)$ or simply $T_n \to T$, provided (a) $\{A_n\}$ is a sequence of admissible sets *invading* A, i.e., $A_n \subset A_{n+1} \subset A$, $A_n^0 \uparrow A^0$, where an upper ^{0} denotes the interior of a set; (b) $d(T, T_n, A_n) \to 0$ as $n \to \infty$ (L. CESARI [**1**; 5.3]).

(iii) If $A = F$ is a figure, then a continuous mapping (T, F) is called *quasi-linear* in F provided there is a finite subdivision S of F into triangles on each of which T is linear. We have now the following lemma. Given any continuous mapping (T, A), there exists a sequence (T_n, F_n) $(n = 1, 2, \ldots)$ of quasi-linear mappings from figures F_n into E_3 such that $(T_n, F_n) \rightarrow (T, A)$ (L. CESARI [**1**; 5.6 (iv)]).

Remark. Since $(T_n, F_n) \rightarrow (T, A)$, we have $F_n \subset F_{n+1} \subset A$, $F_n^0 \uparrow A^0$. Let there be given a compact subset K of A^0. Then $K \subset F_n^0$ for all n large.

(iv) Let (T, F) be a quasi-linear mapping, and let S be any finite subdivision of F into triangles t on each of which T is linear. The image under T of each $t \in S$ is a triangle (possibly degenerate) $\Delta \subset E_3$. If we denote by $|\Delta|$ the area of $\Delta \subset E_3$, then $a(T, F) = \sum_{t \in S} |\Delta|$, is termed *elementary area* of (T, F) (L. CESARI [**1**; 5.7]).

(v) Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_3. Denote by $\{\varphi\}$ the collection of all sequences $\varphi = [(T_n, F_n), (n = 1, 2, \ldots)]$ of quasi-linear mappings from figures F_n such that $T_n \rightarrow T$ and set

$$L(T, A) = \underset{\varphi \in \{\varphi\}}{\text{g.l.b.}} \ \liminf_{n \rightarrow \infty} a(T_n, F_n).$$

$L(T, A)$ is called the *Lebesgue area* of (T, A) and $0 \leqslant L \leqslant +\infty$ (L. CESARI [**1**; 5.8]).

(vi) If (T, A) is a continuous mapping from an admissible set A into E_3, and if A_n $(n = 1, 2, \ldots)$ is any sequence of admissible sets such that $A_n \subset A_{n+1} \subset A$, $A_n^0 \uparrow A^0$, then $L(T, A_n) \rightarrow L(T, A)$ as $n \rightarrow \infty$ (L. CESARI [**1**; 5.14 (iv)]).

(vii) If (T, A) is a continuous mapping whose graph $T(A)$ is contained in a finite system of straight lines, then $L(T, A) = 0$ (L. CESARI [**1**; 5.9]).

(viii) If (T_n, A_n) $(n = 0, 1, 2, \ldots)$ is a sequence of continuous mappings such that $T_n \rightarrow T_0$, then $L(T_0, A) \leqslant \liminf L(T_n, A_n)$ as $n \rightarrow \infty$ (L. CESARI [**1**; 5.10]).

(ix) If an admissible set A can be written as the union A_i $(i = 1, 2, \ldots)$ of disjoint admissible sets with the property that each interior point of A is interior to one A_i, then $L(T, A) = \sum L(T, A_i)$ (L. CESARI [**1**; 5.14 (ii)]).

(x) Let R_ν be a finitely connected JORDAN region (order of connectivity is ν), and let (T, R_ν) be a continuous mapping from R_ν into E_3. Then $T: \quad x = x(u, v),\ y = y(u, v),\ z = z(u, v), \quad (u, v) \in R_\nu$. Consider the plane mappings $T_1: \quad y = y(u, v), \quad z = z(u, v); \quad T_2: \quad x = x(u, v), \quad z = z(u, v); \quad T_3: \quad x = x(u, v),\ y = y(u, v); \quad (u, v) \in R_\nu$. Assume that E is a closed subset of R_ν such that $T_i(E)$ is of measure zero, $i = 1, 2, 3$. Let α be an open subset of R_ν, and let $\{\delta\}$ be the collection of components of $\alpha - E$. Then as a special case of L. CESARI [**1**; 21.4 (i)] there follows

$$V(T, \alpha) = V(T, \alpha - E) = \sum V(T, \delta), \tag{1}$$

$$V(T_i, \alpha) = V(T_i, \alpha - E) = \sum V(T_i, \delta) \qquad (i = 1, 2, 3), \tag{2}$$

where the above summations are carried over all $\delta \in \{\delta\}$. Here $V(T, \alpha)$ denotes the GEÖCZE area of (T, α), and $V(T_i, \alpha)$ the GEÖCZE area of the plane mapping (T_i, α) (see [**1**; 9.1]). For later application we state a corollary.

Let E be a closed subset of R_ν such that $R_\nu - E = \alpha \mid \beta$, i.e., α, β are non-empty disjoint open subsets of $R_\nu - E$ whose union is $R_\nu - E$. If $|T_i(E)| = 0$ $(i = 1, 2, 3)$, then

$$V(T_i, R_\nu) = V(T_i, \alpha) + V(T_i, \beta) \qquad (i = 1, 2, 3), \tag{3}$$

$$V(T, R_\nu) = V(T, \alpha) + V(T, \beta). \tag{4}$$

(xi) Let (T, A) be a continuous mapping from an admissible set A into E_3. Then $V(T, A) = L(T, A)$ (L. CESARI [**1**; 24.1 (i)]).

II.3. – In this paragraph we shall briefly recall to the reader a cyclic additivity theorem due to E. J. MICKLE and T. RADÓ [**2**]. Let P be a PEANO space and let P^* be a metric space. Denote by $\mathfrak{S}$ the class of all continuous mappings from P into P^*. The following definition, already stated in **I**, will be repeated here for the sake of convenience.

Definition: An *unrestricted factorization* of a mapping $T \in \mathfrak{S}$ consists of a PEANO space $\mathfrak{M}$, called *middle space*, and two continuous mappings s, f such that $f: P \to \mathfrak{M}, \quad s: \mathfrak{M} \to P^*, \quad T = sf$.

Definition: Two mappings T', T'' in $\mathfrak{T}$ are termed a *partition* of a mapping $T \in \mathfrak{T}$ provided there are non-empty closed subsets E', E'' of P and a point $p_0^* \in P^*$ such that

(i) $E' \cup E'' = P$,

(ii) $T'(x) = \begin{cases} T(x), & x \in E' \\ p_0^*, & x \in E'', \end{cases} \qquad T''(x) = \begin{cases} T(x), & x \in E'' \\ p_0^*, & x \in E', \end{cases}$

for every $x \in P$.

Let $\Phi(T)$ be a real-valued, non-negative functional defined for each $T \in \mathfrak{T}$ satisfying the following conditions [for certain $T \in \mathfrak{T}$, $\Phi(T)$ may be $+\infty$].

(α) $\Phi(T)$ is lower semi-continuous in the following sense. If $T_n \in \mathfrak{T}$ ($n = 0, 1, 2, \ldots$) and $T_n \to T_0$ uniformly on P, then $\Phi(T_0) \leqslant \liminf \Phi(T_n)$ for $n \to \infty$.

(β) $\Phi(T)$ is additive under partition, i. e., if the mappings T', T'' constitute a partition of T, then $\Phi(T) = \Phi(T') + \Phi(T'')$.

(γ) If $T \in \mathfrak{T}$ admits of an unrestricted factorization whose middle space is a simple arc, then $\Phi(T) = 0$.

Remark: It should be noted that (γ) implies the statement: if $T \in \mathfrak{T}$ is constant, then $\Phi(T) = 0$. In [**2**] the following theorem is proved.

Theorem. Let $T = sf$, $f : P \to \mathfrak{M}$, $s : \mathfrak{M} \to P^*$ be an unrestricted factorization of a mapping $T \in \mathfrak{T}$. For C a proper cyclic element of $\mathfrak{M}$, let r_C be the monotone retraction from $\mathfrak{M}$ onto C. If a real-valued non-negative functional $\Phi(T)$ defined for each $T \in \mathfrak{T}$ satisfies the conditions (α), (β) and (γ), then

$$\Phi(T) = \sum \Phi(s\, r_C\, f), \qquad C \subset \mathfrak{M}.$$

II.4. – Let us restrict now the Peano space P to be a finitely connected Jordan region R_ν in E_2, the order of connectivity being ν, and let us replace the metric space P^* by the Euclidean three space E_3. If $\mathfrak{T}$ denotes again the class of all continuous mappings T from R_ν into E_3, then the Lebesgue area $L(T, R_\nu)$ is a real-valued, non-negative functional of $T \in \mathfrak{T}$. The first objective will be to establish that $L(T, R_\nu)$ is cyclicly additive in the sense of the Theorem in **II.3**. The reader should observe that in case $\nu = 0$, i.e., R_ν is a simply connected JORDAN region, this result follows from [**2**], since then R_ν is a unicoherent PEANO space. However, for $\nu > 0$, R_ν is no longer unicoherent.

II.5. – We proceed now to verify that $L(T, R_\nu)$ satisfies the conditions (α), (β) and (γ) of **II.3**.

The conditions (α) is a consequence of the lower semi-continuity of $L(T, R_\nu)$ (**II.2** (viii)).

II.6. – Lemma: $L(T, R_\nu)$ satisfies the condition (β) of **II.3**.

Proof. Let T', T'' be a partition of a mapping $T \in \mathfrak{S}$. Then we have two non-empty closed subsets E', E'' of R_ν and a point $p_0 \in E_3$ such that $E' \cup E'' = R_\nu$ and

$$T'(w) = \begin{cases} T(w), & w \in E' \\ p_0, & w \in E'', \end{cases} \qquad T''(w) = \begin{cases} T(w), & w \in E'' \\ p_0, & w \in E'. \end{cases}$$

If we set $E = E' \cap E''$, then $E \neq 0$ and T is constant on E. The plane mappings T_i $(i = 1, 2, 3)$ introduced in **II.2** (x) then satisfy $| T_i(E) | = 0$.

Case 1: $E' \subset E''$. Then T' is constant on R_ν and hence $L(T', R_\nu) = 0$, where as $T'' = T$ on R_ν. Therefore, $L(T, R_\nu) = L(T', R_\nu) + L(T'', R_\nu)$.

Case 2: $E'' - E' \neq 0$, $E' - E'' \neq 0$. Then $R_\nu - E = [E' - (E' \cap E'')] \cup [E'' - (E' \cap E'')] = [E' - E''] \cup [E'' - E']$, and $(E' - E'') \cap (E'' - E') = 0$. The sets $E' - E''$, $E'' - E'$ are open in R_ν and hence they are admissible sets. From **II.2** (x), (xi) we conclude that $L(T, R_\nu) = L(T, E' - E'') + L(T, E'' - E')$. We assert that $L(T, E' - E'') = L(T', R_\nu)$. Since T' is constant on the closed set E'', the plane mappings T_i' $(i = 1, 2, 3)$ satisfy $| T_i'(E'') | = 0$ $(i = 1, 2, 3)$. Hence $L(T', R_\nu) = L(T', R_\nu - E'') = L(T', E' - E'') = L(T, E' - E'')$, since $T' = T$ on E'. Similarly, $L(T, E'' - E') = L(T'', R_\nu)$, and the proof of the Lemma is complete.

II.7. – Lemma: $L(T, R_\nu)$ satisfies the condition (γ) of **II.3**.

Proof. Assume that T admits of an unrestricted factorization $T = sf$. $f : R_\nu \to \mathfrak{M}$, $s : \mathfrak{M} \to E_3$ where $\mathfrak{M}$ is a simple arc. Without loss of generality we may assume that $\mathfrak{M}$ coincides with the unit interval I: $0 \leqslant x \leqslant 1$, For each n divide I into n sub-intervals by the points k/n $(k = 0, 1, \ldots, n)$. Define s_n as follows:

(1) $s_n(k/n) = s(k/n)$ $(k = 0, 1, \ldots, n)$,

(2) s_n is linear on each interval $[(k-1)/n, \ k/n]$ $(k = 1, \ldots, n)$.

In case $s(k/n) = s[(k-1)/n]$ for some k, let s_n be constant on $[(k-1)/n, \ k/n]$. Then s_n is continuous on I and, setting $T_n = s_n f$, we have that $T_n(R_\nu)$ is contained in a finite system of straight lines. Therefore, by **II.2** (vii), $L(T_n, R_\nu) = 0$. Since $s_n \to s$ uniformly on I, we have also $T_n \to T$ uniformly on R_ν. Hence $L(T, R_\nu) \leqslant \liminf L(T_n, R_\nu) = 0$. This completes the proof.

II.8. – In view of **II.5**, **II.6**, **II.7**, we have the following theorem (see **II.3**).

Theorem. *Let (T, R_ν) be a continuous mapping from a finitely connected Jordan region R_ν into E_3. If $T = sf$, $f : R_\nu \to \mathfrak{M}$, $s : \mathfrak{M} \to E_3$ is an unrestricted factorization of T, and if r_C denotes the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$, then*

$$L(T, R_\nu) = \sum L(s r_C f, R_\nu), \qquad C \subset \mathfrak{M}.$$

Remark. Let (T, R_ν) be a continuous mapping from R_ν into E_3, and let $T = lm$, $m : R_\nu \Longrightarrow \mathfrak{M}$, $l : \mathfrak{M} \to E_3$ be a monotone-light factorization of T. If $\nu = 0$, i.e., if R_ν is a simply connected JORDAN region, then it follows from T. RADÓ [**4**; V.2.54] that $L(T, R_0) > 0$ if and only if $\mathfrak{M}$ contains at least one proper cyclic element. A similar result for the case $\nu > 0$ is not true.

II.9. – Let $\mathfrak{A}'$ be the class of admissible sets $A \subset E_2$ (see **II.1**) and let $\mathfrak{F}$ be the collection of all finitely connected polygonal regions in E_2. Let us denote by $\mathfrak{A}$ the class of sets generated by $\mathfrak{F}$ in the sense of **I.1**, i.e., a set A is a member of $\mathfrak{A}$ if either of the following holds:

(i) A is a figure (**II.1**).

(ii) If A is not a figure, then there exists a sequence of figures $\{F_n\}$ such that for each n, $F_n \subset A^0$, and for any compact subset K of A^0, there is an integer $\bar{n} = n(K)$ with the property that $K \subset F_n^0 \subset A^0$ for all $n \geqslant \bar{n}$.

In view of **II.2** (iii) we have the inclusion $\mathfrak{A}' \subset \mathfrak{A}$. In applying the results of **I**, the fact that there are sets in $\mathfrak{A}$ which are not admissible does not cause any difficulty (see **II.10**).

II.10. – Let $\mathfrak{S}$ and $\mathfrak{A}$ be given as in **II.9**, and denote by $(\mathfrak{T}, \mathfrak{A})$ the class of all continuous mappings (T, A) from $A \in \mathfrak{A}$ into E_3. If $A \in \mathfrak{A}$ is not admissible, then the non-empty set A^0 is admissible (**II.1**), and we define $L(T, A) = L(T, A^0)$. This definition obtains its justification from the fact that for every admissible set A, $L(T, A) = L(T, A^0)$ (see [**1**; 5.9 (ii)]).

The LEBESGUE area $L(T, A)$ satisfies the conditions of **I.8** which are listed below.

(α) $L(T, A)$ is real-valued and non-negative. For certain $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, we may have $L(T, A) = +\infty$.

(β) For every $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, where A is a finite union of disjoint finitely connected JORDAN regions $R_1, \ldots, R_n$,

$$L(T, A) = \sum_{i=1}^{n} L(T, R_i)$$

(see **II.2** (ix)).

(γ) For $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, and $\{F_n\}$ any sequence of figures satisfying **II.9** (ii),

$$L(T, A) = \lim L(T, F_n), \qquad \text{as} \qquad n \to \infty.$$

(δ) For A', A'' two sets in $\mathfrak{A}$ for which $A'' \subset A'$ and for $(T, A') \in (\mathfrak{T}, \mathfrak{A})$,

$$L(T, A'') \leqslant L(T, A').$$

(ε) Let R be a finitely connected JORDAN region, and let (T, R) be a continuous mapping from R into E_3. If $(T, R) = sf$, $f: R \to \mathfrak{M}$, $s: \mathfrak{M} \to E_3$ is an unrestricted factorization of (T, R) (**II.3**), then

$$L(T, R) = \sum L(s\, r_C\, f, R), \qquad C \subset \mathfrak{M},$$

where r_C denotes the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$ (see **II.8**).

Proof. It is only necessary to establish (γ). Since $F_n \subset A^0$, there follows $L(T, F_n) \leqslant L(T, A)$, and hence:

(1) $$\limsup_{n\to\infty} L(T, F_n) \leqslant L(T, A) .$$

Next we observe that by **I.2** we have a subsequence $\{F_{n_i}\}$ satisfying the condition **II.9** (ii) and $F_{n_i} \subset F^0_{n_{i+1}}$ $(i = 1, 2, 3, \ldots)$. Then $F^0_{n_i} \uparrow A^0$, and hence from **II.2** (vi):

(2) $$L(T, A) = \lim L(T, F_{n_i}) .$$

For a fixed i, there is by **II.9** (ii) an integer $\bar{n}_i$ such that $F_{n_i} \subset F^0_n$ for all $n \geqslant \bar{n}_i$. Since $L(T, F_{n_i}) \leqslant L(T, F_n)$, $n \geqslant \bar{n}_i$, there follows $L(T, F_{n_i}) \leqslant \liminf_{n\to\infty} L(T, F_n)$. Since this relation is valid for every i, we obtain

(3) $$\lim L(T, F_{n_i}) \leqslant \liminf_{n\to\infty} L(T, F_n) .$$

(1), (2) and (3) yield the desired relation.

II..11 – Let $(\mathfrak{T}, \mathfrak{A})$ be given as in **II.10**. For the sake of completeness we state again the definition of an unrestricted factorization given in **I.9**.

Definition. An unrestricted factorization of a mapping $(T, A) \in \in (\mathfrak{T}, \mathfrak{A})$ consists of a PEANO space $\mathfrak{M}$, a subset $\mathfrak{M}^*$ of $\mathfrak{M}$, and two continuous mappings s, f such that:

(1) $f : A \to \mathfrak{M}^*$,

(2) $s : \mathfrak{M}^* \to E_3$,

(3) $T = sf$.

We shall write $(T, A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$.

Remark. Apart from the Remarks 1, 2 of **I.9**, the following additional observation should be made. Every $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ admits of a *trivial* unrestricted factorization of the following form. Let us denote by $\bar{E}_2$ the compact Euclidean plane E_2, i.e., the one-piont compactification of E_2

by adjoining ∞ to E_2. Thus $\overline{E}_2$ is a homeomorphic image of the unit sphere, and $\overline{E}_2$ is a PEANO space. If we denote by i the identity mapping, then $(T, A) = Ti$, $i : A \Longrightarrow A$, $T : A \to E_3$, $A \subset \overline{E}_1$.

II.12. – Let $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, and let $(T, A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of (T, A). As in **I.12**, let $\mathfrak{K}$ be the class of proper cyclic elements associated with $(T, A) = sf$, and let G_C be the set associated with $C \in \mathfrak{K}$, i.e., G_C is the union of all components G of A satisfying $r_C\, f(G) \subset \mathfrak{M}^*$, and $\mathfrak{K}$ is the class of proper cyclic elements C of $\mathfrak{M}$ for which there is at least one such component G of A.

In view of **I.15** and **II.10**, we can state our main result:

Theorem. *Let (T, A) be a continuous mapping from a set $A \in \mathfrak{A}$ into E_3* (see **II.9**). *Let $(T, A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$, be an unrestricted factorization of (T, A). If for C a proper cyclic element of $\mathfrak{M}$, we denote by r_C the monotone retraction from $\mathfrak{M}$ onto C, then we have the following cyclic additivity formula*

$$L(T, A) = \sum L(s\, r_C\, f,\ G_C), \qquad C \in \mathfrak{K},$$

where $\mathfrak{K}$ is the class of proper cyclic elements associated with $(T, A) = sf$, and where G_C is the set associated with $C \in \mathfrak{K}$.

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