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A Fundamental System of Real Solutions for Linear Differential Systems with Periodic Coefficients. (**)

Introduction.

In two previous papers [**2, 3**] (¹), we have considered questions of boundedness and stability of the solutions of differential systems of the form

$$\ddot{y}_j + \sigma_j^2 y_j + \lambda \sum_{1}^{n}{}_h \varphi_{jh}(t)\, y_h = 0 \qquad (j = 1, 2, \ldots, n)\,, \tag{1}$$

where (A) $\sigma_1, \sigma_2, \ldots, \sigma_n$ are distinct positive numbers, (B) λ is a « small » real parameter, (C) $\varphi_{jh}(t)$ are real periodic functions of period $T = 2\pi/\omega$,

$$\int_0^T \varphi_{jh}(t)\, dt = 0, \qquad \varphi_{jh}(t) = \sum_{-\infty}^{+\infty}{}_k c_{jhk} \cdot e^{ik\omega t}, \qquad \sum_{-\infty}^{+\infty}{}_k |\, c_{jhk} \,| < C, \qquad (j,\ h = 1, 2, \ldots, n)$$

(D) $m\omega \neq \sigma_j \pm \sigma_h$ $(j,\ h = 1, 2, \ldots, n;\ m = 1, 2, \ldots)$. As we have seen in [**2, 3**], conditions (A), (B), (C), (D) together with certain additional conditions on the matrix $\Phi(t) = \| \varphi_{jh}(t) \|$ assure that all the solutions of (1) are bounded in $(-\infty, +\infty)$ for λ sufficiently small in absolute value. The additional conditions are any of the following: either (α) $\Phi(t)$ is even $[\varphi_{jh}(t) = \varphi_{jh}(-t)$ $(j,\ h = 1, 2, \ldots, n)]$, or ($\beta$) $\Phi(t)$ is symmetric $[\varphi_{jh}(t) = \varphi_{hj}(t)$ $(j,\ h = 1, 2, \ldots, n)]$, or ($\gamma$) $\Phi(t) = \Phi_0(t) + \psi(t)$, where $\Phi_0 = \operatorname{diag}(\Phi_1, \ldots, \Phi_k)$ is the direct sum of

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blocks $\Phi_1, \ldots, \Phi_k$ each satisfying (α) or (β) and the elements of ψ on or above [or on or below] the blocks $\Phi_1, \ldots, \Phi_k$ are all zero.

In this paper, we shall consider only those systems (1) which satisfy (A), (B), (C), (D) along with any of the conditions (α), (β), or (γ), and we shall give explicit expressions for a fundamental system of real solutions of system (1) in the form of power series in λ. These expressions are analogous to the well known expressions of the solutions of the MATHIEU equation, and reduce themselves to these, when system (1) is the MATHIEU equation

$$\ddot{y} + \{\sigma^2 + \lambda \cdot \cos(2t)\} \cdot y = 0 . \tag{2}$$

We shall use consistently in the next sections the concepts and formulas given in [**2**]. We shall refer to them by quoting section and formula number.

§ 1. – Transformation of the fundamental system.

If in (1) we make the transformation

$$y_j = -(z_{2j-1} + z_{2j})/2, \qquad \dot{y}_j = -i\sigma_j \cdot (z_{2j-1} - z_{2j})/2, \qquad (j = 1, \ldots, n), \tag{1.1}$$

then (1) becomes

$$\left\{ \begin{aligned} \dot{z}_{2j-1} &= i\sigma_j\, z_{2j-1} + \frac{\lambda i}{2\sigma_j} \sum_1^n{}_h\, \varphi_{jh}(t) \cdot [z_{2h-1} + z_{2h}] \\ \dot{z}_{2j} &= -i\sigma_j\, z_{2j} - \frac{\lambda i}{2\sigma_j} \sum_1^n{}_h\, \varphi_{jh}(t) \cdot [z_{2h-1} + z_{2h}] \end{aligned} \right. \qquad (j = 1, \ldots, n) . \tag{1.2}$$

This is a system of the form

$$\dot{z}_j = \varrho_j\, z_j + \lambda \sum_1^{2n}{}_h\, \psi_{jh}(t) \cdot z_h \qquad (j = 1, \ldots, 2n). \tag{1.3}$$

We will now make use the formula for a fundamental system of solutions of (1.3) which is given in [**2**, formula (1.4)]. Let us put

$$\left\{ \begin{aligned} z^{(m)}_{11jh} &= z^{(m)}_{2j-1,2h-1}\,, & z^{(m)}_{12jh} &= z^{(m)}_{2j-1,2h} \\ z^{(m)}_{21jh} &= z^{(m)}_{2j,2h-1}\,, & z^{(m)}_{22jh} &= z^{(m)}_{2j,2h} \end{aligned} \right. \tag{1.4}$$

ad apply the substitutions given in [**2**, formulas (2.4)-(2.10)] to [**2**, formula (1.4)]. We then obtain:

$$(1.5)\qquad z_{11jh}^{(m)} = \delta_{jh}\cdot e^{i\tau_h t} + \sum_1^m{}_p \left(\frac{\lambda}{2}\right)^p \sum_1^n{}_{t_1,\dots,t_{p-1}} \sum_{-\infty}^{+\infty}{}_{k_1,\dots k_p} \sum_1^2{}_{u_1,\dots,u_{p-1}} (-1)^{u_1+\dots+u_{p-1}-p+1}\cdot$$

$$\cdot c_{jt_1k_1}^{(1,u_1,m)}\, c_{t_1t_2k_2}^{(u_1,u_2,m-1)} \dots c_{t_{p-1},hk_p}\cdot e^{i\cdot[(k_1+\dots+k_p)\omega+\tau_h]t}\cdot$$

$$\cdot\{\sigma_j\,\sigma_{t_1}\dots\sigma_{t_{p-1}}\cdot[-\tau_j+(k_1+\dots+k_p)\omega+\tau_h]\cdot$$

$$\cdot[(-1)^{u_1}\tau_{t_1}+(k_2+\dots+k_p)\omega+\tau_h]\dots[(-1)^{u_{p-1}}\tau_{t_{p-1}}+k_p\,\omega+\tau_h]\}^{-1},$$

where

$$\delta_{jh}=\begin{cases}0 & \text{if } j\neq h,\\ 1 & \text{if } j=h,\end{cases}\qquad c_{jhk}^{(u,v,m)}=\begin{cases}-d_{1,j}^{(m)} & \text{if } u=v=1,\ h=j,\ k=0,\\ -d_{2,j}^{(m)} & \text{if } u=v=2,\ h=j,\ k=0,\\ c_{jhk} & \text{if } k\neq 0,\end{cases}$$

and where $d_{1,j}^{(m)}$, $d_{2,j}^{(m)}$ are given in [**2**, formulas (2.9), (2.10)]. The convention is made in (1.5) that those summands for which the denominators vanish are excluded.

For the sake of brevity, we will not write explicitly the other three functions of (1.4) but indicate how they may be obtained from (1.5). If in $z_{11jh}^{(m)}$ we replace $c_{jt_1k_1}^{(1,u_1,m)}$ by $c_{jt_1k_1}^{(2,u_1,m)}$, then

$$z_{21jh}^{(m)} = -z_{11jh}^{(m)} + \delta_{jh}\cdot e^{i\tau_h t}.$$

If in $z_{11jh}^{(m)}$ we replace everywhere τ_h by $-\tau_h$, then

$$z_{12jh}^{(m)} = z_{11jh}^{(m)} - \delta_{jh}\cdot e^{-i\tau_h t}.$$

If in $z_{11jh}^{(m)}$ we replace $c_{jt_1k_1}^{(1,u_1,m)}$ by $c_{jt_1k_1}^{(2,u_1,m)}$ and τ_h by $-\tau_h$ everywhere, then

$$z_{22jh}^{(m)} = -z_{11jh}^{(m)} + 2\delta_{jh}\cdot e^{-i\tau_h t}.$$

Let us now put

$$(1.6)\qquad \begin{cases} y_{jh1}^{(m)} = -\dfrac{1}{2}\,(z_{11jh}^{(m)}+z_{21jh}^{(m)})\\[2ex] y_{jh2}^{(m)} = -\dfrac{1}{2}\,(z_{12jh}^{(m)}+z_{22jh}^{(m)})\end{cases}\qquad (j,\ h=1,\ \dots,\ n).$$

In the formula for $y_{jh2}^{(m)}$, replace everywhere k_i by $-k_i$ and use the property that

$$c_{jh,\,-k}^{(u,v,m)} = \overline{c}_{jhk}^{(3-u,3-v,m)}$$

for all meaninfull u, v, m, j, h, k [2, § 2]. In the so obtained expression, replace everywhere 3-u by u. The result is $\overline{y}_{jh1}^{(m)}$. Hence

$$y_{jh1}^{(m)} = \overline{y}_{jh2}^{(m)} \qquad (m = 0, 1, 2, \ldots).$$

Therefore

$$\frac{1}{2i}\left(y_{jh1}^{(m)} - y_{jh2}^{(m)}\right) = \operatorname{Im} y_{jh1}^{(m)} \quad \text{and} \quad \frac{1}{2}\left(y_{jh1}^{(m)} + y_{jh2}^{(m)}\right) = \operatorname{Re}\, y_{jh1}^{(m)}$$

are real, hence

$$\operatorname{Im} y_{jh1} = \lim_{m\to\infty} \operatorname{Im} y_{jh1}^{(m)} \quad \text{and} \quad \operatorname{Re} y_{jh1} = \lim_{m\to\infty} \operatorname{Re} y_{jh1}^{(m)}$$

$(j, h = 1, \ldots, n)$ form a fundamental system of real solutions for system (1).

§ 2. – A fundamental system of real solutions.

Expanding y_{jh1} (and also $d_{1,j}^{(2)}$ and $d_{2,j}^{(2)}$ by [2, formulas (2.9), (2.10)] in powers of λ, we have

$$(2.1) \qquad -2y_{jh1} = \delta_{jh}\cdot e^{i\tau_h t} + \frac{\lambda\tau_j}{\sigma_j}\sum_{-\infty}^{\infty}{}_k\, c_{jhk}\cdot e^{i\cdot[k\omega+\tau_h]t}\left\{(k\omega+\tau_h)^2-\tau_j^2\right\}^{-1} +$$

$$+\frac{\lambda^2\tau_j}{2\sigma_j}\sum_{1}^{n}{}_{t_1}\sum_{-\infty}^{+\infty}{}_{k_1,k_2}\sum_{1}^{2}{}_u \frac{(-1)^{u-1}}{\sigma_{t_1}} c_{jt_1k_1}\cdot c_{t_1hk_2}\cdot e^{i\cdot[(k_1+k_2)\omega+\tau_h]t}\left\{(-1)^u\tau_{t_1}+k_2\,\omega+\tau_h\right\}^{-1}\cdot$$

$$\cdot\left\{[(k_1+k_2)\omega+\tau_h]^2-\tau_j^2\right\}^{-1} - \frac{\lambda^3}{8\sigma_j^2}\sum_{1}^{n}{}_{t_1}\sum_{k_1+k_3=0}\sum_{-\infty}^{+\infty}{}_{k_2}\sum_{1}^{2}{}_u (-1)^{u-1}\, c_{jhk_2}\cdot c_{jt_1k_1}\cdot c_{t_1jk_3}\cdot$$

$$\cdot e^{i\cdot[k_2\omega+\tau_h]t}\left\{\frac{1}{\sigma_{t_1}\cdot[-\tau_j+k_2\,\omega+\tau_h]^2\,[(-1)^u\tau_{t_1}+k_3\,\omega+\tau_j]}+\right.$$

$$\left.+\frac{1}{\sigma_{t_1}\cdot[\tau_j+k_2\,\omega+\tau_h]^2\,[(-1)^u\tau_{t_1}+k_3\,\omega-\tau_j]}\right\}+$$

$$+\frac{\lambda^3\tau_j}{4\sigma_j}\sum_{1}^{n}{}_{t_1,t_2}\sum_{-\infty}^{+\infty}{}_{k_1,k_2,k_3}\sum_{1}^{2}{}_{u_1,u_2}(-1)^{u_1+u_2}\, c_{jt_1k_1}\cdot c_{t_1t_2k_2}\cdot c_{t_2hk_3}\cdot e^{i\cdot[(k_1+k_2+k_3)\omega+\tau_h]t}\cdot$$

$$\cdot\left\{\sigma_{t_1}\sigma_{t_2}[(-1)^{u}\tau_{t_1}+(k_2+k_3)\omega+\tau_h]\,[(-1)^{u_2}\tau_{t_2}+k_2\,\omega+\tau_h]\right\}^{-1}\cdot$$

$$\cdot\left\{[(k_1+k_2+k_3)\omega+\tau_h]^2-\tau_j^2\right\}^{-1}+O(\lambda^4).$$

In (2.1) make the following changes. Express all sums $\sum_{-\infty}^{+\infty}{}_k$ as sums of the form $\sum_1^{\infty}{}_k$. Replace c_{jhk} by $(a_{jhk} - i\, b_{jhk})/2$, $k > 0$, where a_{jhk}, b_{jhk} are the coefficients in the real FOURIER series of

$$\varphi_{jh}(t) = \sum_1^{\infty}{}_k \left\{ a_{jhk} \cdot \cos(k\omega t) + b_{jhk} \cdot \sin(k\omega t) \right\}.$$

From the so obtained expression, we will now extract the real and imaginary parts. Put

$$(2.2) \qquad \begin{cases} -2 \cdot \operatorname{Im} y_{jh1} = y_{jh} \\ -2 \cdot \operatorname{Re} y_{jh1} = y_{j,h+n}\,, \end{cases}$$

then the $2n$ vectors

$$[y_{jh} \quad (j = 1, \ldots, n)], \qquad [y_{j,h+n} \quad (j = 1, \ldots, n)], \qquad (h = 1, \ldots, n)$$

form a fundamental system of real solutions of system (1). Thus, we have:

$$(2.3) \qquad y_{jh} = \delta_{jh} \cdot \sin(\tau_h\, t) + \frac{\lambda\, \tau_j}{2\, \sigma_j} \sum_1^{\infty}{}_k \left\{ a_{jhk}\, q_k^0 - b_{jhk}\, p_k^1 \right\} +$$

$$+ \frac{\lambda^2\, \tau_j}{4\, \sigma_j} \sum_1^{n}{}_{t_1} \sum_1^{\infty}{}_{k_1,k_2} \frac{1}{\sigma_{t_1}} \left\{ A_0\, q^0_{-k_1,k_2} + A_1\, q^0_{k_1,k_2} - B_0\, p^1_{k_1,k_2} + B_1\, p^1_{k_1,-k_2} \right\} -$$

$$- \frac{\lambda^3}{32\, \sigma_j^2} \sum_1^{n}{}_{t_1} \sum_1^{\infty}{}_{k_2,k_3} \frac{1}{\sigma_{t_1}} \left\{ C_{1,0}\, q^0_{k_2,k_3} + C_{0,1}\, q^0_{k_2,-k_3} - D_{0,0}\, p^1_{k_2,k_3} + D_{1,1}\, p^1_{k_2,-k_3} \right\} +$$

$$+ \frac{\lambda^3 \tau_j}{8\, \sigma_j} \sum_1^{n}{}_{t_1,t_2} \sum_1^{\infty}{}_{k_1,k_2,k_3} \frac{1}{\sigma_{t_1}\, \sigma_{t_2}} \left\{ E_{1,1,1}\, q^0_{k_1,k_2,k_3} + E_{1,0,0}\, q^0_{-k_1,k_2,k_3} + \right.$$

$$+ E_{0,1,0}\, q^0_{k_1,-k_2,k_3} + E_{0,0,1}\, q^0_{k_1,k_2,-k_3} - F_{0,0,1}\, p^1_{k_1,k_2,k_3} -$$

$$\left. - F_{1,0,0}\, p^1_{-k_1,k_2,k_3} - F_{0,1,0}\, p^1_{k_1,-k_2,k_3} + F_{1,1,1}\, p^1_{k_1,k_2,-k_3} \right\} + O(\lambda^4).$$

The corresponding expression for $y_{j,h+n}$ is

$$y_{j,h+n} = \delta_{jh}\cdot\cos(\tau_h\, t) + \frac{\lambda\,\tau_j}{2\,\sigma_j}\sum_1^{\infty}{}_k\left\{a_{jhk}\,p_k^0 + b_{jhk}\,q_k^1\right\} + \tag{2.4}$$

$$+\frac{\lambda^2\,\tau_j}{4\,\sigma_j}\sum_1^n{}_{t_1}\sum_1^{\infty}{}_{k_1,k_2}\frac{1}{\sigma_{t_1}}\left\{A_0\,p^0_{-k_1,k_2} + A_1\,p^0_{k_1,k_2} + B_0\,q^1_{k_1,k_2} - B_1\,q^1_{k_1,-k_2}\right\} -$$

$$-\frac{\lambda^3}{32\,\sigma_j}\sum_1^n{}_{t_1}\sum_1^{\infty}{}_{k_1,k_2}\frac{1}{\sigma_{t_1}}\left\{C_{1,0}\,p^0_{k_2,k_3} + C_{0,1}\,p^0_{k_2,-k_3} + D_{0,0}\,q^1_{k_2,k_3} - D_{1,1}\,q^1_{k_2,-k_3}\right\} +$$

$$+\frac{\lambda^3\,\tau_j}{8\,\sigma_j}\sum_1^n{}_{t_1,t_2}\sum_1^{\infty}{}_{k_1,k_2,k_3}\frac{1}{\sigma_{t_1}\,\sigma_{t_2}}\left\{E_{1,1,1}\,p^0_{k_1,k_2,k_3} + E_{1,0,0}\,p^0_{-k_1,k_2,k_3} +\right.$$

$$+E_{0,1,0}\,p^0_{k_1,-k_2,k_3} + E_{0,0,1}\,p^0_{k_1,k_2,-k_3} + F_{0,0,1}\,q^1_{k_1,k_2,k_3} +$$

$$\left.+F_{1,0,0}\,q^1_{-k_1,k_2,k_3} + F_{0,1,0}\,q^1_{k_1,-k_2,k_3} - F_{1,1,1}\,q^1_{k_1,k_2,-k_3}\right\} + O(\lambda^4).$$

The symbols used in (2.3), (2.4) are defined as follows:

$$p^{\gamma}_{l_1,l_2,\ldots,l_s} = P_{l_1,l_2,\ldots,l_s} + (-1)^{\gamma}\,P_{-l_1,-l_2,\ldots,-l_s},$$

$$q^{\gamma}_{l_1,l_2,\ldots,l_s} = Q_{l_1,l_2,\ldots,l_s} + (-1)^{\gamma}\,Q_{-l_1,-l_2,\ldots,-l_s},$$

$$P_k = [\cos\{(k\omega + \tau_h)\cdot t\}]/[(k\omega + \tau_h)^2 - \tau_j^2], \qquad Q_k = P_k\cdot\tan\{(k\omega + \tau_h)\cdot t\},$$

$$P_{k_1,k_2} = \frac{\tau_{t_1}\cdot\cos\{[(k_1+k_2)\omega + \tau_h]\cdot t\}}{\{(k_2\omega + \tau_h)^2 - \tau_{t_1}^2\}\{[(k_1+k_2)\omega + \tau_h]^2 - \tau_j^2\}},$$

$$Q_{k_1,k_2} = P_{k_1,k_2}\cdot\tan\{[(k_1+k_2)\omega + \tau_h]\cdot t\},$$

$$P_{k_2,k_3} = \tau_t\cdot\left(\frac{1}{[-\tau_j + k_2\,\omega + \tau_h]^2\cdot[(k_3\,\omega + \tau_j)^2 - \tau_{t_1}^2]} +\right.$$

$$\left.+\frac{1}{[\tau_j + k_2\,\omega + \tau_h][(k_3\,\omega - \tau_j)^2 - \tau_{t_1}^2]}\right)\cdot\cos\{(k_2\,\omega + \tau_h)\cdot t\},$$

$$Q_{k_2,k_3} = P_{k_2,k_3}\cdot\tan\{(k_2\,\omega + \tau_h)\cdot t\},$$

$$P_{k_1,k_2,k_3} = \frac{\tau_{t_1}\,\tau_{t_2}\cdot\cos\{[(k_1+k_2+k_3)\omega + \tau_h]\cdot t\}}{\{[(k_1+k_2+k_3)\omega + \tau_h]^2 - \tau_j^2\}\,\{[(k_2+k_3)\omega + \tau_h]^2 - \tau_{t_1}^2\}\,\{(k_2\,\omega + \tau_h)^2 - \tau_{t_2}^2\}},$$

$$Q_{k_1,k_2,k_3} = P_{k_1,k_2,k_3} \cdot \tan\{[(k_1 + k_2 + k_3)\omega + \tau_h] \cdot t\},$$

$$A_\alpha = a_{jt_1k_1} \cdot a_{t_1hk_2} + (-1)^\alpha b_{jt_1k_1} \cdot b_{t_1hk_2},$$

$$B_\alpha = a_{jt_1k_1} \cdot b_{t_1hk_2} + (-1)^\alpha a_{t_1hk_2} \cdot b_{jt_1k_1},$$

$$C_{\alpha,\beta} = a_{jhk_2} \cdot a_{jt_1k_3} \cdot a_{t_1jk_3} + a_{jhk_2} \cdot b_{jt_1k_3} \cdot b_{t_1jk_3} + \\ + (-1)^\alpha a_{jt_1k_3} \cdot b_{jhk_2} \cdot b_{t_1jk_3} + (-1)^\beta a_{t_1jk_3} \cdot b_{jhk_2} \cdot b_{jt_1k_3},$$

$$D_{\alpha,\beta} = a_{jhk_2} \cdot a_{jt_1k_3} \cdot b_{t_1jk_3} + a_{jhk_2} \cdot a_{t_1jk_3} \cdot b_{jt_1k_3} + \\ + (-1)^\alpha a_{jt_1k_3} \cdot a_{t_1jk_3} \cdot b_{jhk_2} + (-1)^\beta b_{jhk_2} \cdot b_{jt_1k_3} \cdot b_{t_1jk_3},$$

$$E_{\alpha,\beta,\gamma} = a_{jt_1k_1} \cdot a_{t_1t_2k_2} \cdot a_{t_2hk_3} + (-1)^\alpha a_{jt_1k_1} \cdot b_{t_1t_2k_2} \cdot b_{t_2hk_3} + \\ + (-1)^\beta a_{t_1t_2k_2} \cdot b_{jt_1k_1} \cdot b_{t_2hk_3} + (-1)^\gamma a_{t_2hk_3} \cdot b_{jt_1k_1} \cdot b_{t_1t_2k_2},$$

$$F_{\alpha,\beta,\gamma} = a_{jt_1k_1} \cdot a_{t_1t_2k_2} \cdot b_{t_2hk_3} + (-1)^\alpha a_{t_1t_2k_2} \cdot a_{t_2hk_3} \cdot b_{jt_1k_1} + \\ + (-1)^\beta a_{jt_1k_1} \cdot a_{t_2hk_3} \cdot b_{t_1t_2k_2} + (-1)^\gamma b_{jt_1k_1} \cdot b_{t_1t_2k_2} \cdot b_{t_2hk_3},$$

The characteristic exponents $i\tau_j$ are given by

$$\tau_j = \sigma_j + \frac{\lambda^2}{8\,\sigma_j} \sum_1^n{}_t \sum_1^\infty{}_k (a_{jtk} \cdot a_{tjk} + b_{jtk} \cdot b_{tjk}) \cdot \\ \cdot \left\{ \frac{1}{(k\omega + \sigma_j)^2 - \sigma_t^2} + \frac{1}{(k\omega - \sigma_j)^2 - \sigma_t^2} \right\} + O(\lambda^3). \tag{2.5}$$

Formula (2.5) is obtained immediately from [**2**, formulas (2.9), (2.11)].

§ 3. - Examples.

We give here some immediate applications of the theorems mentioned in the Introduction and of the formulas (2.3), (2.4), (2.5), where for the sake of brevity, we write only the terms in λ. Terms in λ^2, λ^3, could be obtained directly from (2.3), (2.4), (2.5). Further terms could be obtained by using the procedure of § **1**, § **2**.

3.1. – The MATHIEU equation. In case system (1) is the MATHIEU equation (2), the expressions (2.3), (2.4), (2.5) reduce themselves to

$$y_{11} = \sin(\tau t) + \frac{\lambda\tau}{8\sigma}\left\{\frac{\sin[(2+\tau)\cdot t]}{\tau+1} + \frac{\sin[(2-\tau)\cdot t]}{\tau-1}\right\} + O(\lambda^2),$$

$$y_{12} = \cos(\tau t) + \frac{\lambda\tau}{8\sigma}\left\{\frac{\cos[(2+\tau)\cdot t]}{\tau+1} + \frac{\cos[(2-\tau)\cdot t]}{\tau-1}\right\} + O(\lambda^2),$$

where $\tau = \sigma - \lambda^2[16\sigma\cdot(\sigma^2-1)]^{-1} + O(\lambda^4)$. With appropriate change in notation, these are the formulas given in [**4**, p. 20].

3.2. – Consider the differential system

$$\text{(3.1)} \qquad \begin{cases} \ddot{y}_1 + \sigma_1^2 y_1 = \lambda\cdot\sin(\omega t)\cdot y_2 \\ \ddot{y}_2 + \sigma_2 y_2 = \lambda\cdot\sin(\omega t)\cdot y_1 \end{cases}$$

which satisfies conditions (A), (B), (C), (D) and (β) of the Introcution. By CESARI's results [**1**], mentioned in [**2**], all solutions of (3.1) are bounded. From (2.3), (2.4), (2.5), we obtain the following fundamental system of solutions of (3.1):

$$\begin{cases} y_{11} = \sin(\tau_1 t) + O(\lambda^2) \\ y_{21} = -\dfrac{\lambda\,\tau_2}{2\,\sigma_2}\left\{\dfrac{\cos[(\omega+\tau_1)\cdot t]}{(\tau_2-\omega-\tau_1)(\tau_2+\omega+\tau_1)} - \dfrac{\cos[(\omega-\tau_1)\cdot t]}{(\tau_2+\omega-\tau_1)(\tau_2-\omega+\tau_1)}\right\} + O(\lambda^2), \end{cases}$$

$$\begin{cases} y_{12} = -\dfrac{\lambda\,\tau_1}{2\,\sigma_1}\left\{\dfrac{\cos[(\omega+\tau_2)\cdot t]}{(\tau_1-\omega-\tau_2)(\tau_1+\omega+\tau_2)} - \dfrac{\cos[(\omega-\tau_2)\cdot t]}{(\tau_1+\omega-\tau_2)(\tau_1-\omega+\tau_2)}\right. + O(\lambda^2) \\ y_{22} = \sin(\tau_2 t) + O(\lambda^2), \end{cases}$$

$$\begin{cases} y_{13} = \cos(\tau_1 t) + O(\lambda^2) \\ y_{23} = \dfrac{\lambda\,\tau_2}{2\,\sigma_2}\left\{\dfrac{\sin[(\omega+\tau_1)\cdot t]}{(\tau_2-\omega-\tau_1)(\tau_2+\omega+\tau_1)} + \dfrac{\sin[(\omega-\tau_1)\cdot t]}{(\tau_2+\omega-\tau_1)(\tau_2-\omega+\tau_1)}\right\} + O(\lambda^2), \end{cases}$$

$$\begin{cases} y_{14} = \dfrac{\lambda\,\tau_1}{2\,\sigma_1}\left\{\dfrac{\sin[(\omega+\tau_2)\cdot t]}{(\tau_1-\omega-\tau_2)(\tau_1+\omega+\tau_2)} + \dfrac{\sin[(\omega-\tau_2)\cdot t]}{(\tau_1+\omega-\tau_2)(\tau_1-\omega+\tau_2)}\right\} + O(\lambda^2) \\ g_{24} = \cos(\tau_2 t) + O(\lambda^2), \end{cases}$$

where $\tau_i = \sigma_i + O(\lambda^2)$ $(i = 1, 2)$.

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