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Non Holonomic Subspaces. (**)

The object of the present paper is to introduce « Non Holonomic » (NH) curvature tensor, NH-RICCI tensor, the normal to NH-hypersurface and then to generalise the MAINARDI-CODAZZI relations in the NH-hypersurface.

1. - Non Holonomic Curvature tensor and NH-Ricci tensor.

Let V_i be the covariant components of a NH-vector, we have by NH-covariant Pfaffian (Pf) differentiation [1] (1):

$$V_{i,j} = \frac{\partial V_i}{w^j} - \left\{ {l \atop i\ j} \right\} V_l ,$$

where $\left\{ {l \atop i\ j} \right\}$ is the CHRISTOFFEL symbol of the NH-space.

The NH-covariant Pf-derivative of $V_{i,\,j}$ in the direction w^k is given by

$$V_{i,jk} = \frac{\partial^2 V_i}{w^j w^k} - \frac{\partial V_i}{w^l} \left\{ {l \atop j\ k} \right\} - \frac{\partial V_l}{w^j} \left\{ {l \atop i\ k} \right\} - V_l \frac{\partial}{w^k} \left\{ {l \atop i\ j} \right\} +$$

$$+ V_l \left\{ {l \atop b\ j} \right\} \left\{ {b \atop i\ k} \right\} + V_a \left\{ {a \atop i\ b} \right\} \left\{ {b \atop j\ k} \right\} .$$

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(1) Numbers in brackets refer to the References at the end of the paper.

Subtracting from this the corresponding equation found by interchanging j and k, viz.:

$$V_{i,kj} = \frac{\partial^2 V_i}{w^k w^j} - \frac{\partial V_i}{w^l}\left\{ {l \atop k\ j} \right\} - \frac{\partial V_l}{w^k}\left\{ {l \atop i\ j} \right\} - V_l \frac{\partial}{w^j}\left\{ {l \atop i\ k} \right\} +$$

$$+ V_l \left\{ {l \atop b\ k} \right\}\left\{ {b \atop i\ j} \right\} + V_a \left\{ {a \atop i\ b} \right\}\left\{ {b \atop k\ j} \right\},$$

we obtain

$$(1\cdot1) \qquad V_{i,jk} - V_{i,kj} = V_a\, R^a_{ijk},$$

where

$$(1\cdot2) \qquad R^a_{ijk} = \frac{\partial}{w^j}\left\{ {a \atop i\ k} \right\} - \frac{\partial}{w^k}\left\{ {a \atop i\ j} \right\} + \left\{ {a \atop b\ j} \right\}\left\{ {b \atop i\ k} \right\} - \left\{ {a \atop b\ k} \right\}\left\{ {b \atop i\ j} \right\},$$

or

$$R^a_{ijk} = \begin{vmatrix} \dfrac{\partial}{w^j} & \dfrac{\partial}{w^k} \\ \left\{ {a \atop i\ j} \right\} & \left\{ {a \atop i\ k} \right\} \end{vmatrix} + \begin{vmatrix} \left\{ {a \atop b\ j} \right\} & \left\{ {a \atop b\ k} \right\} \\ \left\{ {b \atop i\ j} \right\} & \left\{ {b \atop i\ k} \right\} \end{vmatrix}.$$

Since V_i is an arbitrary NH-covariant vector and the first member of (1·1) is a NH-covariant tensor of the third order, it follows that R^a_{ijk} is a mixed NH-tensor of the fourth order. It is called the NH-curvature tensor for the NH-metric g_{ij}. From the definition (1·2) it is clear that the tensor is skew symmetric in j and k, so that

$$(1\cdot3) \qquad R^a_{ijk} = - R^a_{ikj}$$

and from (1·2) it follows that

$$(1\cdot4) \qquad R^a_{ijk} + R^a_{jki} + R^a_{kij} = 0\,.$$

By successive NH-covariant Pf-derivative of V^i we obtain

$$(1\cdot5) \qquad V^i_{,jk} - V^i_{,kj} = - V^a\, R^i_{ajk}\,.$$

The NH-curvature tensor may be contracted in two different ways. One of these leads to a zero NH-tensor

$$R^i_{ijk} = \frac{\partial}{w^j}\left(\frac{\partial}{w^k}\log\sqrt{g}\right) - \frac{\partial}{w^k}\left(\frac{\partial}{w^j}\log\sqrt{g}\right) + \left\{ {i \atop b\ j} \right\}\left\{ {b \atop i\ k} \right\} - \left\{ {i \atop b\ k} \right\}\left\{ {b \atop i\ j} \right\} = 0$$

by interchanging the dummy indices i and b. The other method of contraction leads to NH-Ricci tensor

$$(1\cdot6) \qquad R_{ij} = R^a_{ija} = \frac{\partial}{w^j}\left\{ {a \atop i\ a} \right\} - \frac{\partial}{w^a}\left\{ {a \atop i\ j} \right\} + \left\{ {a \atop b\ j} \right\}\left\{ {b \atop i\ a} \right\} - \left\{ {a \atop b\ a} \right\}\left\{ {b \atop i\ j} \right\}.$$

It is clear that R_{ij} is symmetric in the subscripts.

2. – Notation. Unit normal.

Let L_m $(m = n + 1)$ be a NH-manifold of dimensions m $(= n+1)$ and class not less than 4. The subspace L_n is called the NH-hypersurface of NH-enveloping space L_{n+1}. Let y^α $(\alpha = 1, \ldots, n+1)$ be the coordinates of a point in L_{n+1} and x^i those of the same point in L_n. Let the fundamental form for L_n be denoted by $g_{ij}\, w^i\, w^j$ and that for L_m by $G_{\alpha\beta}\, W^\alpha\, W^\beta$. We have

$$(2\cdot1) \qquad G_{\alpha\beta}\,\frac{W^\alpha}{w^i}\,\frac{W^\beta}{w^j} = g_{ij}\,.$$

As the functions y^α are invariants for transformations of the coordinates x^i in L_n, their first NH-covariant Pf-derivative with respect to L_n are the same as their NH-derivative with respect to the variables x^i. That is

$$y^\alpha_{,i} = \frac{W^\alpha}{w^i}\,.$$

Thus $(2\cdot1)$ becomes

$$(2\cdot1)' \qquad G_{\alpha\beta}\, y^\alpha_{,i}\, y^\beta_{,j} = g_{ij}\,.$$

For a fixed value of i, the vector of L_{n+1} whose NH-contravariant components are $y^\alpha_{,i}$, is tangential to the curve of parameter x^i in L_n. Consequently if

N^{a} are the NH-contravariant components of the unit vector normal to L_n, these must satisfy the relations

$$G_{\alpha\beta} N^{\beta} y^{\alpha}_{,i} = 0, \tag{2·2}$$

$$G_{\alpha\beta} N^{\alpha} N^{\beta} = 1. \tag{2·3}$$

3. – Generalised NH-covariant Pf-differentiation [2].

Let u_{α}, v^{β} be the components in L_m of two unit NH-vector fields which are Pf-parallel along C (a curve in L_n) with respect to L_m and similarly p^{i} the components in L_n of a unit NH-vector field which is Pf-parallel along C with respect to L_n. Then

$$\frac{\mathrm{d}u_{\alpha}}{\mathrm{d}s} - \left\{ \begin{matrix} \beta \\ \alpha \ \delta \end{matrix} \right\}_{G} u_{\beta} \frac{W^{\delta}}{\mathrm{d}s} = 0,$$

$$\frac{\mathrm{d}v^{\beta}}{\mathrm{d}s} + \left\{ \begin{matrix} \beta \\ \alpha \ \delta \end{matrix} \right\}_{G} v^{\alpha} \frac{W^{\delta}}{\mathrm{d}s} = 0,$$

$$\frac{\mathrm{d}p^{i}}{\mathrm{d}s} + \left\{ \begin{matrix} i \\ k \ j \end{matrix} \right\}_{g} p^{k} \frac{w^{j}}{\mathrm{d}s} = 0.$$

Let $T^{\alpha}_{\beta i}$ be a NH-tensor field, defined along C which is a mixed NH-tensor of the second order in L_m and a NH-covariant vector in L_n. Then the product $u_{\alpha} w^{\beta} p^{i} T^{\alpha}_{\beta i}$ is a scalar invariant and along C it is a function of s. Its NH-intrinsic Pf-derivative with respect to s is also a scalar invariant. Obtaining the Pf-derivative of the product with respect to s and using the above equations of Pf-parallelism of the vectors, the NH-intrinsic Pf-derivative can be written as

$$u_{\alpha} v^{\beta} p^{i} \left[\frac{\mathrm{d}T^{\alpha}_{\beta i}}{\mathrm{d}s} + \left\{ \begin{matrix} \alpha \\ \gamma \ \delta \end{matrix} \right\}_{G} T^{\gamma}_{\beta i} \frac{W^{\delta}}{\mathrm{d}s} - \left\{ \begin{matrix} \gamma \\ \beta \ \delta \end{matrix} \right\}_{G} T^{\alpha}_{\gamma i} \frac{W^{\delta}}{\mathrm{d}s} - \left\{ \begin{matrix} k \\ i \ j \end{matrix} \right\}_{g} T^{\alpha}_{\beta k} \frac{w^{j}}{\mathrm{d}s} \right].$$

It follows from the quotient law that the expression in square brackets is a NH-tensor of the type $T^{\alpha}_{\beta i}$. Let us call it the Pf-intrinsic derivative of this NH-tensor with respect to s.

If C is any curve in L_n and the functions $T^{\alpha}_{\beta i}$ are defined through out the hypersurface. The above Pf-intrinsic derivative can be written as

$$\left[\frac{\partial T^{\alpha}_{\beta i}}{w^{j}} + \left\{ \begin{matrix} \alpha \\ \gamma \ \delta \end{matrix} \right\}_{G} T^{\gamma}_{\beta i}\, y^{\delta}_{,j} - \left\{ \begin{matrix} \gamma \\ \beta \ \delta \end{matrix} \right\}_{G} T^{\alpha}_{\gamma i}\, y^{\delta}_{,j} - \left\{ \begin{matrix} k \\ i \ j \end{matrix} \right\} T^{\alpha}_{\beta k} \right] \frac{w^{j}}{\mathrm{d}s}. \tag{3·1}$$

Now w^j/ds is an arbitrary, unit NH-vector in L_n. Since the direction of C is arbitrary, its coefficient in square brackets is a NH-tensor. We denote it by $T'_{\beta i;j}$. Thus

$$T^{\alpha}_{\beta i;j} = \frac{\partial T^{\alpha}_{\beta i}}{w^j} + \left\{ \begin{matrix} \alpha \\ \gamma \ \ \delta \end{matrix} \right\}_G T^{\gamma}_{\beta i}\, y^{\delta}_{,i} - \left\{ \begin{matrix} \gamma \\ \beta \ \ \delta \end{matrix} \right\}_G T^{\alpha}_{\gamma i}\, y^{\delta}_{,j} - \left\{ \begin{matrix} k \\ i \ \ j \end{matrix} \right\}_g T^{\alpha}_{\beta k}. \tag{3·2}$$

Let us call it the generalised covariant Pf-derivative of $T^{\alpha}_{\beta i}$ with respect to L_n or precisely as the Pf-tensor derivative of $T^{\alpha}_{\beta i}$ with respect to L_n.

The Pf-tensor derivative of the fundamental tensors $G_{\alpha\beta}$ and g_{ij} are both zero as is easily verified. These may therefore be treated as constants in Pf-tensor differention.

4. – Second Fundamental Form.

Since y^{α} is an invariant for the transformation of x's, its Pf-tensor derivative is the same as its NH-covariant Pf-derivative with respect to x's so that

$$y^{\alpha}_{;i} = y^{\alpha}_{,i} = \frac{\partial y^{\alpha}}{w^i} = \frac{W^{\alpha}}{w^i}. \tag{4·1}$$

The Pf-tensor derivative of this is

$$y^{\alpha}_{;ij} = \frac{\partial^2 y^{\alpha}}{w^i\, w^j} - \left\{ \begin{matrix} h \\ i \ \ j \end{matrix} \right\}_g y^{\alpha}_{,h} + \left\{ \begin{matrix} \alpha \\ \beta \ \ \gamma \end{matrix} \right\}_G y^{\beta}_{,i}\, y^{\gamma}_{,j}. \tag{4·2}$$

Taking the Pf-tensor derivative of $(2·1)'$ with respect to L_n we have

$$G_{\alpha\beta}\, y^{\alpha}_{;ik}\, y^{\beta}_{,j} + G_{\alpha\beta}\, y^{\alpha}_{,i}\, y^{\beta}_{;jk} = 0.$$

Let this be subtracted from the sum of two others obtained by interchanging i, j, k cyclically. Then, since $y^{\alpha}_{;ij}$ is symmetric in the subscripts we find

$$G_{\alpha\beta}\, y^{\alpha}_{;ij}\, y^{\beta}_{,k} = 0. \tag{4·3}$$

We see that $y^{\alpha}_{;ij}$ regarded as a NH-tensor of L_{n+1} is normal to L_n. It may therefore be expressed as

$$y^{\alpha}_{;ij} = \Omega_{ij}\, N^{\alpha}. \tag{4·4}$$

That the coefficients Ω_{ij} are the components of a symmetric NH-covariant tensor of the second order in L_n, is obvious, for the functions $y^{\alpha}_{;ij}$ are of this nature. It follows

$$\Omega_{ij} = y^{\alpha}_{;ij}\, G_{\alpha\beta}\, N^{\beta}. \tag{4·5}$$

5. – Pf-tensor derivative of the unit normal.

The unit normal N^{α} is a NH-contravariant vector in L_m whose Pf-tensor derivative with respect to L_n is given by

$$N^{\alpha}_{;i} = \frac{\partial N^{\alpha}}{w^{i}} + \left\{ \begin{matrix} \alpha \\ \beta \;\; \delta \end{matrix} \right\} N^{\beta}\, y^{\delta}_{,j}\,. \tag{5·1}$$

Taking Pf-tensor derivative of each side of (2.3) we have

$$G_{\alpha\beta}\, N^{\beta} N^{\alpha}_{;i} = 0, \tag{5·2}$$

which shows that $N^{\alpha}_{;i}$ regarded as a NH-vector in the y 's is orthogonal to the normal and therefore tangential to the hypersurface. It can be expressed as

$$N^{\alpha}_{;i} = A^{k}_{i}\, y^{\alpha}_{,k}\,. \tag{5·3}$$

Taking the Pf-tensor derivative of (2·2) with respect to L_n we find

$$G_{\alpha\beta}\, y^{\alpha}_{;ij}\, N^{\beta} + G_{\alpha\beta}\, y^{\alpha}_{,i}\, N^{\beta}_{;j} = 0$$

substituting from (4·4) and (5·3) we obtain, by virtue of (2·3) and (2·1)′,

$$\Omega_{ij} = -\, G_{\alpha\beta}\, y^{\alpha}_{,i}\, y^{\beta}_{,k}\, A^{k}_{j} = -\, g_{ik}\, A^{k}_{j}\,.$$

Multiplying by g^{ih} and summing for i we deduce

$$\Omega_{ij}\, g^{ih} = -\, A^{h}_{j}. \tag{5·4}$$

Thus (5·3) becomes

$$N^{\alpha}_{;i} = -\, \Omega_{ij}\, g^{jk}\, y^{\alpha}_{,k}\,. \tag{5·5}$$

This is the Pf-tensor derivative of N^{α}.

Since $y^{\alpha}_{,i}$ are components of a NH-covariant vector in L_n, it follows

$$(5\cdot 6) \qquad y^{\alpha}_{,ijk} - y^{\alpha}_{,ikj} = y^{\alpha}_{,p} R^{p}_{ijk} = y^{\alpha}_{,p} g^{ph} R_{hijk}\,.$$

Now

$$y^{\alpha}_{;ij} = \Omega_{ij} N^{\alpha}$$

by (4·4), hence

$$y^{\alpha}_{;ijk} = \Omega_{ij,k} N^{\alpha} + \Omega_{ij} N^{\alpha}_{;k}\,.$$

Also

$$y^{\alpha}_{;ij} = y^{\alpha}_{,ij} + \left\{ {\alpha \atop \beta\ \gamma} \right\} y^{\beta}_{,i}\, y^{\gamma}_{,j}\,.$$

Therefore

$$y^{\alpha}_{;ijk} = \left[y^{\alpha}_{,ij} + \left\{ {\alpha \atop \beta\ \gamma} \right\} y^{\beta}_{,i}\, y^{\gamma}_{,j} \right]_{,k} + \left\{ {\alpha \atop \beta\ \gamma} \right\} y^{\gamma}_{,k}\, y^{\beta}_{;ij} =$$

$$= y^{\alpha}_{,ijk} + \frac{\partial}{\partial x^k} \left\{ {\alpha \atop \beta\ \gamma} \right\} y^{\beta}_{,i}\, y^{\gamma}_{,j} + \left\{ {\alpha \atop \beta\ \gamma} \right\} [y^{\beta}_{,ik}\, y^{\gamma}_{,i} + y^{\beta}_{,i}\, y^{\gamma}_{,jk}] + \left\{ {\alpha \atop \beta\ \gamma} \right\} y^{\gamma}_{,k}\, y^{\beta}_{;ij}\,.$$

Equating the two values of $y^{\alpha}_{;ijk}$ we get

$$\Omega_{ij,k} N^{\alpha} - \Omega_{ij}\, \Omega_{kh}\, g^{hp}\, y^{\alpha}_{,p} = y^{\alpha}_{,ijk} + \left\{ {\alpha \atop \beta\ \gamma} \right\} (y^{\beta}_{,ik}\, y^{\gamma}_{,j} + y^{\beta}_{,i}\, y^{\gamma}_{,jk}) +$$

$$+ \left\{ {\alpha \atop \beta\ \gamma} \right\} y^{\gamma}_{,k} \left(y^{\beta}_{,ij} + \left\{ {\beta \atop \delta\ \varepsilon} \right\} y^{\delta}_{,i}\, y^{\varepsilon}_{,j} \right) + y^{\beta}_{,i}\, y^{\gamma}_{,j}\, y^{\varepsilon}_{,k} \frac{\partial}{\partial y^{\varepsilon}} \left\{ {\alpha \atop \beta\ \gamma} \right\}.$$

Interchanging j and k we obtain

$$\Omega_{ik,j} N^{\alpha} - \Omega_{ik}\, \Omega_{jh}\, g^{hp}\, y^{\alpha}_{,p} = y^{\alpha}_{,ikj} + \left\{ {\alpha \atop \beta\ \gamma} \right\} \Big[y^{\gamma}_{,k}\, y^{\beta}_{,ij} + y^{\beta}_{,i}\, y^{\gamma}_{,kj} + y^{\gamma}_{,j}\, y^{\beta}_{,ik} +$$

$$+ \left\{ {\beta \atop \delta\ \varepsilon} \right\} y^{\delta}_{,i}\, y^{\varepsilon}_{,k}\, y^{\gamma}_{,j} \Big] + y^{\beta}_{,i}\, y^{\gamma}_{,k}\, y^{\varepsilon}_{,j} \frac{\partial}{\partial y^{\varepsilon}} \left\{ {\alpha \atop \beta\ \gamma} \right\}.$$

Subtracting we get

$$y^{\alpha}_{,ijk} - y^{\alpha}_{,ikj} = N^{\alpha}(\Omega_{ij,k} - \Omega_{ik,j}) + y^{\alpha}_{,p}\, g^{ph}(\Omega_{ik}\,\Omega_{hj} - \Omega_{ij}\,\Omega_{kh}) +$$

$$+ \left\{ {\alpha \atop \beta\ \gamma} \right\} \left[y^{\beta}_{,ij}\, y^{\gamma}_{,k} + y^{\beta}_{,i}\, y^{\gamma}_{,kj} + y^{\gamma}_{,j}\, y^{\beta}_{,ik} + \left\{ {\beta \atop \delta\ \varepsilon} \right\} y^{\gamma}_{,j}\, y^{\delta}_{,i}\, y^{\varepsilon}_{,k} \right] -$$

$$- \left\{ {\alpha \atop \beta\ \gamma} \right\} \left[y^{\beta}_{,ik}\, y^{\gamma}_{,j} + y^{\beta}_{,i}\, y^{\gamma}_{,jk} + y^{\gamma}_{,k}\, y^{\beta}_{,ij} + \left\{ {\beta \atop \delta\ \varepsilon} \right\} y^{\delta}_{,i}\, y^{\varepsilon}_{,j}\, y^{\gamma}_{,k} \right] +$$

$$+ y^{\beta}_{,i}\, y^{\gamma}_{,j}\, y^{\varepsilon}_{,k} \frac{\partial}{\partial y^{\varepsilon}} \left\{ {\alpha \atop \beta\ \gamma} \right\} - y^{\beta}_{,i}\, y^{\gamma}_{,k}\, y^{\varepsilon}_{,j} \frac{\partial}{\partial y^{\varepsilon}} \left\{ {\alpha \atop \beta\ \gamma} \right\} =$$

$$= N^{\alpha}(\Omega_{ij,k} - \Omega_{ik,j}) + y^{\alpha}_{,p}\, g^{ph}(\Omega_{ik}\,\Omega_{hj} - \Omega_{ij}\,\Omega_{kh}) +$$

$$+ y^{\gamma}_{,i}\, y^{\delta}_{,j}\, y^{\varepsilon}_{,k} \left[\left\{ {\alpha \atop \beta\ \delta} \right\} \left\{ {\beta \atop \gamma\ \varepsilon} \right\} - \left\{ {\alpha \atop \beta\ \varepsilon} \right\} \left\{ {\beta \atop \gamma\ \delta} \right\} + \frac{\partial}{\partial y^{\delta}} \left\{ {\alpha \atop \gamma\ \varepsilon} \right\} - \frac{\partial}{\partial y^{\varepsilon}} \left\{ {\alpha \atop \gamma\ \delta} \right\} \right]$$

by suitably changing the dummy indices. Or

$$y^{\alpha}_{,ijk} - y^{\alpha}_{,ikj} = N^{\alpha}(\Omega_{ij,k} - \Omega_{ik,j}) + y^{\alpha}_{,p}\, g^{ph}(\Omega_{ik}\,\Omega_{hj} - \Omega_{ij}\,\Omega_{kh}) + \overline{R}^{\alpha}_{\gamma\delta\varepsilon}\, y^{\gamma}_{,i}\, y^{\delta}_{,j}\, y^{\varepsilon}_{,k},$$

where $\overline{R}^{\alpha}_{\gamma\delta\varepsilon}$ are NH-RIEMANN symbols for the NH-tensor $G_{\alpha\beta}$ evaluated at points of the NH-hypersurface.

We can write (5·6) as

$$(5\cdot 7) \qquad y^{\alpha}_{,p}\, g^{ph}[R_{hijk} - (\Omega_{hj}\,\Omega_{ik} - \Omega_{hk}\,\Omega_{ij})] -$$

$$- N^{\alpha}(\Omega_{ij,k} - \Omega_{ik,j}) - \overline{R}^{\alpha}_{\gamma\delta\varepsilon}\, y^{\gamma}_{,i}\, y^{\delta}_{,j}\, y^{\varepsilon}_{,k} = 0.$$

Multiplying (5·7) by $G_{\alpha\beta}\, N^{\beta}$ and summing with respcet to α we obtain, by virtue of (2·2) and (2·3),

$$\Omega_{ij,k} - \Omega_{ik,j} + \overline{R}_{\beta\gamma\delta\varepsilon}\, N^{\beta} \frac{W^{\gamma}}{w^{i}} \frac{W^{\delta}}{w^{j}} \frac{W^{\varepsilon}}{w^{k}} = 0\,.$$

Similarly multiplying (5.7) by $G_{\lambda\beta}\, y^{\beta}_{;l}$ and summing with respect to α, we have

$$R_{lijk} = \Omega_{lj}\,\Omega_{ik} - \Omega_{lk}\,\Omega_{ij} + \overline{R}_{\beta\gamma\delta\varepsilon}\,\frac{W^{\beta}}{w^{l}}\,\frac{W^{\gamma}}{w^{i}}\,\frac{W^{\delta}}{w^{j}}\,\frac{W^{\varepsilon}}{w^{k}}\,.$$

These are the generalised MAINARDI-CODAZZI relations in the NH-system.

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