Riv. Mat. Univ. Parma 8 (1957), 133-143

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Integral Representations of Laplace Type for Certain Hypergeometric Functions of Three Variables.

1. – Hypergeometric functions of three variables have been recently defined by me in one of my papers [**5**]. I have also obtained the elementary properties of these functions including the integral representations of EULER's type, the systems of partial differential equation, the relations between the functions and so on. In this paper I have obtained the single and double integrals of the LAPLACE type for these functions. It may be noted that single integral representations help us in the investigation of the general solution of the differential equations satisfied by them. These integrals also exhibit very elegantly the symmetry of the parameters and the variables in the given functions.

2. – Integral representations for F_E, F_G, F_K, F_M and F_N.

From the definition of F_E we have

$$F_E(a_1,\ a_1,\ a_1,\ b_1,\ b_2,\ b_2;\ c_1,\ c_2,\ c_3;\ x,\ y,\ z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1,\ m+n+p)\,(b_1,\ m)\,(b_2,\ n+p)}{(1,\ m)\,(1,\ n)\,(1,\ p)\,(c_1,\ m)\,(c_2,\ n)\,(c_3,\ p)}\, x^m\, y^n\, z^p\,.$$

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Replacing $(a_1, m+n+p)$ by $\frac{1}{\Gamma(a_1)}\int_0^\infty e^{-t}\, t^{a_1+m+n+p-1}\,dt$ and changing the order of integration and summation, we obtain

$$(1)\qquad F_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; x, y, z) =$$

$$= \frac{1}{\Gamma(a_1)}\int_0^\infty e^{-t}\, t^{a_1-1}\, {}_1F_1(b_1; c_1; xt)\cdot \Psi_2(b_2; c_2, c_3; yt, zt)\,dt,$$

$$\mathrm{Rl}\,(x+y+z+2\sqrt{yz})<1, \qquad \mathrm{Rl}\,(a_1)>0.$$

Similarly we obtain the integral representations:

$$(2)\qquad F_G(a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z) =$$

$$= \frac{1}{\Gamma(a_1)}\int_0^\infty e^{-t}\, t^{a_1-1}\, {}_1F_1(b_1; c_1; xt)\cdot \Xi_2(b_2, b_3; c_2; yt, zt)\,dt,$$

$$\mathrm{Rl}\,(x+y+z)<1, \qquad \mathrm{Rl}\,(a_1)>0;$$

$$(3)\qquad F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, z) =$$

$$= \frac{1}{\Gamma(a_2)}\int_0^\infty e^{-t}\, t^{a_2-1}\, {}_1F_1(b_2; c_2; yt)\cdot \Psi_1(b_1, a_1; c_1, c_3; x, zt)\,dt,$$

$$\mathrm{Rl}\{(1-x)(1-y)\}>\mathrm{Rl}\,(z), \qquad \mathrm{Rl}\,(a_2)>0;$$

$$(4)\qquad F_M(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z) =$$

$$= \frac{1}{\Gamma(b_1)}\int_0^\infty e^{-t}\, t^{b_1-1}\, {}_1F_1(a_1; c_1; xt)\cdot \Phi_1(a_2, b_2; c_2; zt, y)\,dt,$$

$$\mathrm{Rl}\,(x+z)<1, \qquad \mathrm{Rl}\,(b_1)>0;$$

and

(5) $$F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z) =$$

$$= \frac{1}{\Gamma(b_1)} \int_0^\infty e^{-t} t^{b_1-1} {}_1F_1(a_1; c_1; xt) \cdot \Xi_1(a_2, a_3, b_2, c_2; y, zt) \, dt,$$

$$\text{Rl } (x+z) < 1, \qquad \text{Rl } (b_1) > 0,$$

where F_G, F_K, F_M and F_N are the undermentioned hypergeometric function of three variables and Φ_1, Ψ_1, Ψ_2, Ξ_1 and Ξ_2 are the confluent hypergeometric functions of two variables defined by HUMBERT ([**2**], p. 325). We have:

(6) $$F_G(a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+n+p)(b_1, m)(b_2, n)(b_3, p)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n+p)} x^m y^n z^p,$$

(7) $$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)(a_2, n+p)(b_1, m+p)(b_2, n)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n)(c_3, p)} x^m y^n z^p,$$

(8) $$F_M(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)(a_2, n+p)(b_1, m+p)(b_2, n)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n+p)} x^m y^n z^p,$$

and

(9) $$F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; x, y, z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)(a_2, n)(a_3, p)(b_1, m+p)(b_2, n)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n+p)} x^m y^n z^p .$$

3. - Particular cases.

The above five integrals give some very interesting special cases by reducing either of the two or both functions occuring in the integrand on the right. By taking special values for the parameters the ${}_1F_1$'s can be reduced by WHIT-

TAKER's function $M_{k,m}(x)$, generalised LAGUERRE polynomials $L_n^{(\alpha)}(x)$, WEBER's parabolic-cylinder function $D_n(x)$, HERMITE polynomials $H_n(x)$ and BESSEL function $I_a(x)$. Also in some cases the second function, involved in the integrands, which is a confluent hypergeometric function of two variables can be reduced to a WHITTAKER function in two variables $M_{k,a,b}(x, y)$, a product of two HERMITE polynomials, a BESSEL function etc., by taking special values of the variables and parameters.

In case we take special values for the variables we can obtain a number of known results for functions of two variables.

Consider the general integral given by

$$(1) \qquad h(a,\ b;\ x) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-t}\, t^{c-1}\, {}_1F_1(a;\ b;\ xt)\cdot \Phi\ \mathrm{dt},$$

where Φ is any one of the confluent hypergeometric functions Ψ_2, Ξ_2, Ψ_1, Φ_1, Ξ_1 and h is one of the functions F_E, F_G, F_K, F_M and F_N respectively.

We get on reducing the ${}_1F_1$ function the following special cases:

$$(2) \qquad h((1/2) - m - k,\ 2m + 1;\ x)) = \frac{x^{-m-½}}{\Gamma(c)} \int_0^\infty e^{-t\cdot[1-(x/2)]}\, t^{c-m-(3/2)}\, M_{k,m}(xt)\cdot \Phi\ \mathrm{dt},$$

$$(3) \qquad h(-n,\ 1 + k;\ x) = \frac{n!\cdot \Gamma(1 + k)}{\Gamma(1 + k + n)\cdot \Gamma(c)} \int_0^\infty e^{-t}\, t^{c-1}\, L_n^k(xt)\cdot \Phi\ \mathrm{dt},$$

$$(4) \qquad h(-n/2,\ -1/2;\ x) = \frac{2^{-n/2}}{\Gamma(c)} \int_0^\infty e^{-t\cdot[1-(x/2)]}\, t^{c-1}\, D_n(\sqrt{2xt})\cdot \Phi\ \mathrm{dt},$$

$$(5) \qquad h(k + (1/2),\ 2k + 1;\ 2x) = \frac{\Gamma(1 + k)}{\Gamma(c)}\,(x/2)^{-k} \int_0^\infty e^{-t\cdot(1-x)}\, t^{c-k-1}\, I_k(xt)\cdot \Phi\ \mathrm{dt}\,.$$

Since

$$(6) \qquad H_n(x) = 2^{n/2}\, e^{x^2/2}\, D_n(x\sqrt{2})$$

the integral **3**(4) can also be put in terms of $H_n(x)$.

Again reducing both the ${}_1F_1$ and the confluent hypergeometric functions of two variables, we get on using the relation

$$(7)\qquad M_{k,a,b}(x,\ y) = x^{a+(1/2)}\, y^{b+(1/2)}\, e^{-(x+y)/2}\, \Psi_2(a+b-k+1,\ 2a+1,\ 2b+1;\ x,\ y)$$

in **2**(1):

$$(8)\qquad F_E(a_1,\ a_1,\ a_1,\ (1/2)+m-k,$$

$$p+q-s+1,\ p+q-s+1;\ 2m+1,\ 2p+1,\ 2q+1;\ x,\ y,\ z) =$$

$$= \frac{x^{-m-(1/2)}\, y^{-p-(1/2)}\, z^{-q-(1/2)}}{\Gamma(a_1)} \int_0^\infty e^{-t\cdot[1-(x/2)-(y/2)-(z/2)]}\, t^{a_1-m-p-q-(5/2)}\,\cdot$$

$$\cdot M_{k,m}(xt)\cdot M_{s,p,q}(yt,\ zt)\, dt\,.$$

Also, using the following known relations (4),

$$(9)\qquad \Psi_2(a,\ c,\ c';\ x,\ x) = {}_3F_3(a,\ (c+c'-1)/2,\ [(c+c')/2],\ c,$$

$$c',\ c+c'-1,\ 4x)$$

and

$$(10)\qquad \Psi_1(1+a,\ -r,\ 1+a,\ 1+a;\ x,\ -y) =$$

$$= \frac{r!}{(1+a,\ r)}\,(1-x)^r\, e^{-x}\cdot L_r^{(a)}(xy/(x-1)),$$

where r is a positive integer in **2**(1) and **2**(3), we obtain

$$(11)\qquad F_E(a_1,\ a_1,\ a_1,\ b_1,\ b_2,\ b_2;\ c_1,\ c_2,\ c_3;\ x,\ y,\ x) =$$

$$= \frac{1}{\Gamma(a_1)} \int_0^\infty e^{-t}\, t^{a_1-1}\, {}_1F_1(b_1;\ c_1;\ xt)\,\cdot$$

$$\cdot\, {}_3F_3(b_2,\ (c_2+c_3-1)/2,\ (c_2+c_3)/2,\ c_2,\ c_3,\ c_2+c_3-1;\ 4yt)\, dt$$

and

(12) $$F_K(-r,\ a_2,\ a_2,\ 1+d,\ b_2,\ 1+d;\ (1+d),\ c_2,\ (1+d);\ x,\ y,\ -z) =$$

$$= \frac{e^{-x}\,(1-x)^r\, r!}{(1+d,\ r)\cdot\Gamma(a_2)} \int_0^\infty e^{-t}\, t^{a_2-1}\, {}_1F_1(b_2;\ c_2;\ yt)\cdot L_r^{(k)}\big(xzt/(x-1)\big)\, dt,$$

where the notation $(1+d)$ on the right in the definition of F_K denotes the presence of terms $(1+d,\ m)\,(1+d,\ p)$ and not $(1+d,\ m+p)$ according to our convention given in (5).

The **3**(11) and **3**(12) incidentally give us cases of reducibility of the function F_E and F_K, namely

(13) $$F_E(a_1,\ a_1,\ a_1,\ b_1,\ b_2,\ b_2;\ c_1,\ c_2,\ c_3;\ x,\ y,\ x) =$$

$$= {}_4F_3^{(2)}\left[\begin{matrix} a_1;\ b_1,\ d,\ e; & (c_2+c_3-1)/2, & (c_2+c_3)/2, & b_2; & x,\ 4y \\ c_1,\ d,\ e; & c_2+c_3-1, & c_2, & c_3 & \end{matrix}\right]$$

in the notation give by Burchnall and Chaundy ([**2**], p. 270) and

(14) $$F_K(-r,\ a_2,\ a_2,\ 1+d,\ b_2,\ 1+d;\ (1+d),\ c_2,\ (1+d);\ x,\ y,\ -z) =$$

$$= e^{-x}\,(1-x)^r\, F_2(a_2,\ b_2,\ -r;\ c_2,\ 1+d;\ y,\ xz/(x-1)) .$$

4. - Double integral representations for F_F, F_P, F_R.

Using similar methods as used in § **2**, we can show that the integrals for the above functions are of the type:

(1) $$F_F(a_1,\ a_1,\ a_1,\ b_1,\ b_2,\ b_1;\ c_1,\ c_2,\ c_2;\ x,\ y,\ z) =$$

$$= \frac{x^{-(c_1-1)/2}\cdot\Gamma(c_1)}{\Gamma(a_1)\cdot\Gamma(b_1)} \int_0^\infty\!\!\int_0^\infty e^{-(p+t)}\, t^{a_1-(c_1/2)-(1/2)}\, p^{b_1-(c_1/2)-(1/2)}\, J_{c_1-1}\big(2\sqrt{xpt}\big)\cdot$$

$$\cdot\Phi_3(b_2,\ c_2;\ yt,\ zpt)\, dp\, dt,$$

$$\mathrm{Rl}\,(y+z) < 1, \qquad \mathrm{Rl}\,(a_1) > 0, \qquad \mathrm{Rl}\,(b_1) > 0;$$

(2) $$F_P(a_1,\ a_2,\ a_1,\ b_1,\ b_1,\ b_2;\ c_1,\ c_2,\ c_2;\ -x,\ y,\ z) =$$

$$= \frac{x^{-(c_1-1)/2}\cdot\Gamma(c_1)}{\Gamma(a_1)\cdot\Gamma(b_1)} \int_0^\infty \int_0^\infty e^{-(p+t)}\, t^{a_1-(c_1/2)-(1/2)}\, p^{b_1-(c_1/2)-(1/2)}\, J_{c_1-1}\left(2\sqrt{xpt}\right)\cdot$$

$$\cdot\Phi_2(a_2,\ b_2,\ c_2;\quad yp,\ zt)\,dp\,dt,$$

$$\mathrm{Rl}\,(y) < 1,\qquad \mathrm{Rl}\,(z) < 1,\qquad \mathrm{Rl}\,(a_1) > 0,\qquad \mathrm{Rl}\,(b_1) > 0;$$

and

(3) $$F_R(a_1,\ a_2,\ a_1,\ b_1,\ b_2,\ b_1;\ c_1,\ c_2,\ c_2;\ x,\ y,\ z) =$$

$$= \frac{x^{-(c_1-1)/2}\cdot\Gamma(c_1)}{\Gamma(a_1)\cdot\Gamma(b_1)} \int_0^\infty \int_0^\infty e^{-(p+t)}\, t^{a_1-(c_1/2)-1}\, p^{b_1-(c_1/2)-(1/2)}\cdot J_{c_1-1}\left(2\sqrt{xpt}\right)\cdot$$

$$\cdot\,\Xi_2(a_2,\ b_2,\ c_2;\ y,\ zpt)\,dp\,dt,$$

$$\mathrm{Rl}\,(z) < 1,\qquad \mathrm{Rl}\,(a_1) > 0,\qquad \mathrm{Rl}\,(b_1) > 0,$$

where F_F, F_P and F_R are defined as:

(4) $$F_F(a_1,\ a_1,\ a_1,\ b_1,\ b_2,\ b_1;\ c_1,\ c_2,\ c_2;\ x,\ y,\ z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1,\ m+n+p)\,(b_1,\ m+p)\,(b_2,\ n)}{(1,\ m)\,(1,\ n)\,(1,\ p)\,(c_1,\ m)\,(c_2,\ n+p)}\, x^m\, y^n\, z^p,$$

(5) $$F_P(a_1,\ a_2,\ a_1,\ b_1,\ b_1,\ b_2;\ c_1,\ c_2,\ c_2;\ x,\ y,\ z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1,\ m+p)\,(a_2,\ n)\,(b_1,\ m+n)\,(b_2,\ p)}{(1,\ m)\,(1,\ n)\,(1,\ p)\,(c_1,\ m)\,(c_2,\ n+p)}\, x^m\, y^n\, z^p,$$

and

(6) $$F_R(a_1,\ a_2,\ a_1,\ b_1,\ b_2,\ b_1;\ c_1,\ c_2,\ c_2;\ x,\ y,\ z) =$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1,\ m+p)\,(a_2,\ n)\,(b_1,\ m+p)\,(b_2,\ n)}{(1,\ m)\,(1,\ n)\,(1,\ p)\,(c_1,\ m)\,(c_2,\ n+p)}\, x^m\, y^n\, z^p\,.$$

5. – Integral representations for F_S and F_T.

From the definition of the functions F_S and F_T defined by

(1) $$F_S(a_1,\ a_2,\ a_2,\ b_1,\ b_2,\ b_3;\ c_1,\ c_1,\ c_1;\ x,\ y,\ z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1,\ m)\,(a_2,\ n+p)\,(b_1,\ m)\,(b_2,\ n)\,(b_3,\ p)}{(1,\ m)\,(1,\ n)\,(1,\ p)\,(c_1,\ m+n+p)}\, x^m\, y^n\, z^p$$

and

(2) $$F_T(a_1,\ a_2,\ a_2,\ b_1,\ b_2,\ b_1;\ c_1,\ c_1,\ c_1;\ x,\ y,\ z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1,\ m)\,(a_2,\ n+p)\,(b_1,\ m+p)\,(b_2,\ n)}{(1,\ m)\,(1,\ n)\,(1,\ p)\,(c_1,\ m+n+p)}\, x^m\, y^n\, z^p,$$

we can deduce that

(3) $$F_S(a_1,\ a_2,\ a_2,\ b_1,\ b_2,\ b_3;\ c_1,\ c_1,\ c_1;\ x,\ y,\ z) = \frac{1}{\Gamma(a_1)\cdot\Gamma(a_2)} \int_0^\infty\!\!\int_0^\infty e^{-(p+t)}\, t^{a_1-1}\, p^{a_2-1}\, {}_3\Phi(b_1,\ b_2,\ b_3;\ c_1;\ xt,\ yp,\ zp)\, \mathrm{d}p\, \mathrm{d}t,$$

$$\mathrm{Rl}(x) < 1, \qquad \mathrm{Rl}\,(y+z) < 1, \qquad \mathrm{Rl}\,(a_1) > 0, \qquad \mathrm{Rl}\,(a_2) > 0,$$

and

(4) $$F_T(a_1,\ a_2,\ a_2,\ b_1,\ b_2,\ b_1;\ c_2,\ c_1,\ c_1;\ x,\ y,\ z) = \frac{1}{\Gamma(a_2)\cdot\Gamma(b_1)} \int_0^\infty\!\!\int_0^\infty e^{-(p+t)}\, t^{a_2-1}\, p^{b_1-1}\, {}_3\Phi'(a_1,\ b_2;\ c_1;\ xp,\ yt,\ zpt)\, \mathrm{d}p\, \mathrm{d}t,$$

$$\mathrm{Rl}\left\{(1-x)\,(1-y)\right\} \geqq \mathrm{Rl}\,(z), \qquad \mathrm{Rl}\,(b_1) > 0,$$

where

$$ {}_3\Phi(a_1,\ a_2,\ a_3;\ c_1;\ x,\ y,\ z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1,\ m)\,(a_2,\ n)\,(a_3,\ p)}{(1,\ m)\,(1,\ n)\,(1,\ p)\,(c_1,\ m+n+p)}\, x^m\, y^n\, z^p,$$

and ${}_3\Phi'$ is a confluent form of ${}_3\Phi$ defined by

$$\lim_{\varepsilon \to \infty} {}_3\Phi(a_1, a_2, \varepsilon; c_1; x, y, z/\varepsilon) = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)(a_2, n)}{(1, m)(1, n)(1, p)(c_1, m+n+p)} x^m y^n z^p.$$

6. – We now give some interesting double integral representations for F_E and F_K whose single integral representations have already been given in § **2**.

We have from the definition of these functions

(1) $$F_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; x, -y, -z) = \frac{1}{\Gamma(a_1)\cdot\Gamma(b_2)} \int_0^\infty \int_0^\infty e^{-(p+t)} p^{a_1-1} t^{b_2-1} {}_1F_1(b_1; c_1; xp) \cdot {}_0F_1(c_2; -ypt) \cdot {}_0F_1(c_3; -zpt) \, dp \, dt$$

and

(2) $$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, -z) = \frac{1}{\Gamma(a_2)\cdot\Gamma(b_1)} \int_0^\infty \int_0^\infty e^{-(p+t)} p^{b_1-1} t^{a_2-1} {}_1F_1(a_1; c_1; xp) \cdot {}_1F_1(b_2; c_2; yt) \cdot {}_0F_1(c_3; -zpt) \, dp \, dt,$$

$$\mathrm{Rl}\,(x) < 1, \qquad \mathrm{Rl}\,(y) < 1, \qquad \mathrm{Rl}\,(b_1) > 0, \qquad \mathrm{Rl}\,(a_2) > 0 .$$

Special Cases.

By giving special values to the parameters we obtain a number of integrals involving the product of Bessel functions and a Whittaker function, the product of Weber's parabolic cylinder functions and a Bessel function and so on. Some of the interesting integrals are the following:

(1) $$F_E(a_1, a_1, a_1, (1/2)+m-k, b_2, b_2;\; 2m+1, 1+p, 1+q; x, -y, -z) = x^{-m-(1/2)} y^{-p} z^{-q} \frac{\Gamma(1+p)\cdot\Gamma(1+q)}{\Gamma(a_1)\cdot\Gamma(b_2)} \int_0^\infty \int_0^\infty e^{-s-t\cdot[1-(x/2)]} s^{b_2-(p/2)-(q/2)-1} \cdot t^{a_1-(p/2)-(q/2)-m-(3/2)} \cdot M_{k,m}(xt) \cdot J_p\left(2\sqrt{yst}\right) \cdot J_q\left(2\sqrt{zst}\right) \, ds \, dt$$

and

(2) $$F_K(-m,\ a_2,\ a_2,\ b_1,\ -n,\ b_1;\ 1+k,\ 1+p,\ 1+q;\ x,\ y,\ -z) =$$

$$= \frac{n!\, m!\, \Gamma(1+p)\cdot\Gamma(1+q)\cdot\Gamma(1+k)}{\Gamma(a_2)\cdot\Gamma(b_1)\cdot\Gamma(1+k+m)\cdot\Gamma(1+p+n)}\, z^{-q} \int_0^\infty \int_0^\infty e^{-(s+t)}\, s^{b_1-(q/2)-1}\, t^{a_2-(c/2)-1}\,\cdot$$

$$\cdot L_m^{(k)}(xt)\cdot L_n^{(p)}(yt)\cdot J_q\left(2\sqrt{zpt}\right)\ \mathrm{d}p\ \mathrm{d}t,$$

where m and n are positive integers.

7. – The above integrals can easily give the LAPLACE transforms of all the hypergeometric functions.

Following the usual notations if

$$\varphi(p) = p \int_0^\infty e^{-ps} f(s)\ \mathrm{d}s$$

we will write

$$\varphi(p) \doteqdot f(s),$$

and if

$$\varphi(p,\ q) = pq \int_0^\infty \int_0^\infty e^{-ps-qt} f(s,\ t)\ \mathrm{d}s\ \mathrm{d}t$$

we will write

$$\varphi(p,\ q) \mathrel{\ddot{\ddot{=}}} f(s,\ t)\,.$$

Using the above symbolism we obtain from **2**(3) for instance.

(1) $$\Gamma(a_2)\cdot p^{1-a_2}\cdot F_K(a_1,\ a_2,\ a_2,\ b_1,\ b_2,\ b_1;\ c_1,\ c_2,\ c_3;\ x,\ y/p,\ z/p) \doteqdot$$

$$\doteqdot s^{a_2-1}\cdot {}_1F_1(b_2;\ c_2;\ sy)\cdot \Psi_1(b_1,\ a_1,\ c_1,\ c_3;\ x,\ zs)\,.$$

Fron **4**(2) we obtain

(2) $$\Gamma(a_1)\cdot\Gamma(b_1)\cdot x^{(c_1-1)/2}\, p^{1-b_1}\, q^{1-a_1}\cdot F_P(-\, x/(pq),\ y/p,\ y/p,\ z/q) \doteqdot$$

$$\doteqdot \Gamma(c_1)\cdot s^{b_1-(c_1/2)-(1/2)}\, t^{a_1-(c_1/2)-(1/2)}\cdot J_{c_1-1}(2\sqrt{xst})\cdot \Xi_2(a_2,\ b_2;\ c_2;\ ys,\ zt).$$

8. – It may be mentioned that the conditions of convergence of the integrals given in §§ **2**, **4** and **6** have been obtained from the following estimates of the functions for large values of the variables:

(1) $\Psi_2(b_2\ \ c_2,\ c_3;\ y,\ z) \sim C\, e^{x+y+2}\sqrt{xy},$ $\quad |y|, |z| \to \infty,$

(2) $\Xi_2(b_2,\ b_3,\ c_2;\ y,\ z) \sim C\, e^{y+z},$ $\quad |y|, |z| \to \infty,$

(3) $\Psi_1(b_1,\ a_1,\ c_1,\ c_3;\ x,\ z) \sim C\, e^{z/(1-z)}$ $\quad |z| \to \infty,$

(4) $\Phi_1(a_2,\ b_2,\ c_2;\ y,\ z) \sim C\, e^{y},$ $\quad |z| \to \infty,$

(5) $\Xi_1(a_2,\ a_3,\ b_2,\ c_2;\ y,\ z) \sim C\, e^{z},$ $\quad |z| \to \infty,$

(6) $\Phi_3(b_2,\ c_2;\ y,\ z) \sim C\, e^{y+2\sqrt{z}},$ $\quad |y|, |z| \to \infty,$

(7) $\Phi_2(a_2,\ b_2;\ c_2;\ y,\ z) \sim C\, e^{2\sqrt{z}},$ $\quad |z| \to \infty,$

(8) ${}_3\Phi(b_1,\ b_2,\ b_3;\ c_3;\ x,\ y,\ z) \sim C\, e^{x+y+z}$ $\quad |x|, |y|, |z| \to \infty,$

(9) ${}_3\Phi'(a_1,\ b_2,\ c_1;\ x,\ y,\ z) \sim C e^{x+y+2\sqrt{z}},$ $\quad |x|, |y|, |z| \to \infty.$

I am grateful to Dr. R. P. AGARWAL for his kind guidance in the preparation of this paper and to the Government of India for a research grant.

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