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W. H. FLEMING (*)

Irreducible Generalized Surfaces. (**)

1. - Introduction.

The notion of generalized surface is due to L. C. YOUNG [**10** c, e]. It was introduced to obtain desirable closure and compactness properties in the study of double integral problems of calculus of variations. For surfaces of finite topological type and sufficiently elementary boundary a remarkably complete theory was given in YOUNG's Memoir [**10** e]. It includes very general existence theorems for two-dimensional problems in parametric form, regular or not (1).

A generalized surface solution to a problem of minimum is, in the first analysis, merely an element of the completion in a certain topological vector space of a set of « elementary » surfaces. To obtain a meaningful theory, one must find a suitable representation for the solution. It has been known since YOUNG's work on generalized curves [**10** a] 20 years ago that one should expect a representation in terms of an ordinary surface and, attached to almost every point of the surface in place of the normal vector, a certain measure on the space of all possible unit normal vectors.

In the context of [**10** e] one wishes to find, for a wide enough class of

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(*) Address: Department of Mathematics, Brown University, Providence 12, Rhode Island, U.S.A..

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(1) With much more general boundary conditions or unrestricted topological types a quite different treatment is needed. See FLEMING and YOUNG [**5**].

generalized surfaces L, a micro-representation by a vector integral:

$$L = \iint_Q M_{uv}\, du\, dv, \tag{1.1}$$

where Q is a square. The key result of [**10** e] is the existence, for any connected generalized surface L, of a micro-representation whose carrier is a vector function on Q subject to identifications on the perimeter of the same topological type as L. The proof is based on DIRICHLET integral methods. It therefore relies ultimately on the existence of representations which are conformal in a certain generalized sense (specifically, MORREY's theorem) to establish a bound for the DIRICHLET integral in terms of the area.

There are two main innovations in the present paper. First, we apply conformal mapping for polyhedra of higher topological types, in a way similar to the author's recent extension [**4** a] of MORREY's theorem to two-manifolds. Second, the set of generalized surfaces of given topological type which admit a given set Γ of curves in space as boundary in the sense of [**10** e] is enlarged to include those which admit Γ as boundary in a certain limiting sense. We give general conditions (irreducibility) under which an element of the larger set necessarily belongs to the smaller and possesses a micro-representation. Various improvements are made in the results of [**10** e]. Boundary identifications are pointwise and piecewise linear, instead of merely arcwise. Less restrictive assumptions are made about the set Γ of boundary curves. The curves need not occupy two-dimensional measure 0, and many of the results apply to boundaries with multiple points.

The main definitions and results are summarized in § **3**. Paragraph **10** is devoted to applications to minimum problems, including existence theorems of a very general nature.

After this paper was written a paper by SIGALOV [**13** b] appeared, which also treats the problem of minimum for parametric surfaces of prescribed finite topological type. His results partly overlap [**10** e] and the present paper. However, they are limited to positive definite, semi-regular problems. The methods are extensions of the ones developed by SIGALOV to prove his earlier existence theorem [**13** a] for surfaces of the type of the 2-cell. The most important step in the proof is to find a uniformly convergent minimizing sequence of parametric representations, using a certain smoothing operation for surfaces. This same approach was also used by CESARI [**11**] and DANSKIN [**12**] in their basic existence theorems for the 2-cell case.

YOUNG's approach, which is continued in the present paper, is to replace uniform convergence by BEPPO LEVI convergence [**10** d, e] of parametric representations and weak convergence of certain associated linear functionals.

In this way one gets existence theorems for minimum problems which may have discontinuous solutions, and for non-regular problems. Under a weak normality assumption (much less restrictive than the conditions on the integrand imposed in [**11**], [**12**], [**13** a, b]) YOUNG proved [**10** e, 9.9] that any solution must possess a continuous representation. See §**10** for further remarks in this direction.

Like the present paper, SIGALOV [**13** b] does not confine himself to boundaries free of multiple points. The notions of surface improperly of given topological type and of limit boundary, §**3** below, have counterparts in [**13** b]. The same is true of the operation $\sim$ for boundaries with multiple points, considered in §**9**. SIGALOV uses as parameter domain a square with boundary identifications.

The identification of boundary segments in [**13** b] is essentially of the same sort (i.e., arcwise) as in [**10** e]. A device [**13** b, p. 90] which SIGALOV uses to achieve one-one pointwise identifications is valid only if the class of admissible parametric representations is enlarged to admit representations not absolutely continuous in TONELLI's sense.

2. – Preliminaries.

(a) *Generalized surfaces.* R^m is euclidean m-space and x a generic point of R^m. S^m is the set of all skew-symmetric $m \times m$ matrices $j = [j^{rs}]$ of rank 0 or 2. Let $|j| = [\sum_{r<s} (j^{rs})^2]^{1/2}$. For $a,\ b \in R^m$, let $a \times b = [a^r b^s - a^s b^r]$ (cross-product). F denotes the vector space of all continuous functions $f(x, j)$ on $R^m \times S^m$ satisfying the homogeneity condition $f(x, kj) = k\, f(x, j)$ for $k \geqslant 0$; and F_0 the subspace of all $f \in F$ satisfying the symmetry condition $f(x, -j) = f(x, j)$.

An *oriented* [*nonoriented*] *generalized surface* is any non-negative linear functional on $F[F_0]$. The *norm* (or *area*) $a(L)$ of a generalized surface L equals $L(i)$, where $i(x, j) = |j|$ for all (x, j). L is *situated* in a closed subset W of R^m if L has support in $W \times S^m$. The statement L_n *tends to* L shall mean all L_n are situated in the same compact set and $L(f) = \lim L_n(f)$ for all f (weak convergence, also called weak convergence by some mathematicians) (2). This

(2) The set of generalized surfaces was originally [**10** c] defined as the weak closure of the set of elementary surfaces, in analogy with YOUNG's earlier definition of generalized curve. By [**10** c, (8,3)] this set comprises all non-negative linear functionals on F. By restricting the topological types and boundaries of the approximating elementary surfaces, as done in the present paper, a much more restricted set of generalized surfaces is obtained.

defines a pseudo-topology on the space of generalized surfaces. For generalized surfaces situated in a fixed cube K, the pseudo-topology is equivalent to the metric topology defined by the McShane distance:

$$d(L, L') = \sup_{f \in \Phi} | L(f) - L'(f) |, \tag{2.1}$$

where Φ is the set of $f \in F$ (or F_0 in the nonoriented case) such that on the cartesian product of K with the unit sphere in S^m, $| f(x, j) | \leqslant 1$ and f satisfies a Lipschitz condition with Lipschitz constant 1. In this paper the word distance always refers to (2.1), a convention which does not agree with [**10** e]. As is well known, closed sets with uniformly bounded norms and carriers are compact.

(b) *Admissible domains.* The square with boundary identifications is replaced here by more general domains, which we proceed to define. By *cell* in a (u, v) plane let us mean a figure of the form $a \leqslant u \leqslant b$, $c \leqslant v \leqslant d$, where $a < b$, $c < d$, a and c are finite or $-\infty$, b and d are finite or $+\infty$. If a cell G is not finite, we consider G to have a single point at ∞. Let $G_1, \ldots, G_n$ be cells and suppose the perimeter of every G_i is subdivided into finitely many segments and half lines λ. Let Λ be the set of all λ of our subdivision. An identification process $\mathfrak{J}$ on Λ is termed *admissible* provided: (1) For every $\lambda \in \Lambda$, either λ is unidentified or λ is identified with exactly one other member of Λ; and (2) The identification of any pair $\lambda_1, \lambda_2 \in \Lambda$ is achieved by a linear homeomorphism of λ_1 onto λ_2 (in particular, half lines are identified with half lines).

The cells $G_1, \ldots, G_n$, Λ, and $\mathfrak{J}$ define a space B, which is called an *admissible domain* if $\mathfrak{J}$ is admissible and B is connected. B^* denotes the boundary (union of the unidentified λ) and $B^0 = B - B^*$ the interior. An admissible domain B is a compact two-manifold which is determined topologically by its orientability or not, the number s ($\geqslant 0$) of components of B^*, and the characteristic q. One has $q = s + 2h - 2$, $q = s + k - 2$, where h, k are the number of handles, cross-caps in the oriented, non-oriented cases, respectively. For these facts we refer to Seifert-Threlfall [**8**, Ch. 6]. The symbol τ will designate two things: first, the topological type of B; and second, if B is orientable whether B is oriented. In later paragraphs we shall use the notation $q(\tau)$, $s(\tau)$ for q, s, respectively.

The *distance* between two finite points $w, w' \in B$ is defined as the length of the shortest path in B joining w and w'. By *elementary path* in B we shall mean a connected polygon composed of vertical and horizontal line segments. A *cross-cut* in B is an arc with endpoints on B^* and otherwise disjoint from B^*.

(c) *Slit domains.* The class of admissible domains includes the parallel alit domains, the definition of which we take from [4 a]. Briefly, a slit domain A without boundary ($s = 0$) consists of a compactified plane with a finite number of horizontal slits, semi-infinite to the right. On the slits there is a piecewise linear identification process such that identified points have the same abscissa and ∞ belongs to an open 2-cell in A. A slit domain with boundary consists of a compactified upper half plane with a finite number of slits with identifications of a similar nature. The unidentified points on the slits and the real axis form the boundary A^*. For completeness, the full plane and upper half plane with no slits are regarded as slit domains.

We also take from [4 a] the definition of normalized slit domain, and convergence of a sequence of normalized slit domains with similarly arranged configurations of slits, vertices and edges. The limit of a convergent sequence A_n is again a slit domain, not necessarily homeomorphic with A_n. These concepts are all substantially the same as those used by SHIFFMAN [9] in connection with the PLATEAU problem.

(d) *BL and Dirichlet functions.* To simplify the notation the following definitions are given only when B is defined by cells $G_1, \ldots, G_n$ which are nonoverlapping subsets of a single (u, v) plane; moreover, if B is oriented its orientation shall agree in every G_i with the natural orientation of the plane. In particular, B may be any slit domain. The extensions to general admissible domains are fairly obvious; moreover, it turns out that everything can be reduced to the special case by a change of parameters.

The letter w denotes a generic point of B. A point w interior to some cell G_i is uniquely determined by its (u, v) coordinates. The correspondence between other points w and points (u, v) is determined by the identification process $\mathfrak{J}$, and is not generally biunique. We write $\int_B g(w)\, dw$ for the LEBESGUE integral of $g(u, v) \equiv g(w)$ over $G_1^0 \cup \ldots \cup G_n^0$, if it exists.

Let $x(w)$ denote a vector-valued function with domain B and values in R^m. If the vector partial derivatives x_u, x_v exist almost everywhere we write:

$$D(x, B) = (1/2) \int_B (x_u^2 + x_v^2)\, dw \qquad \text{(DIRICHLET integral)}, \tag{2.2}$$

$$a(x, B) = \int_B | x_u \times x_v |\, dw \qquad \text{(area integral)}.$$

In addition, for any line parallel to a coordinate axis or elementary path γ for which the indicated integral exists we write:

$$D_1(x, \gamma) = \int_\gamma [x_u^2\, du + x_v^2\, dv] \qquad \text{(simple DIRICHLET integral)}. \tag{2.3}$$

We call $x(w)$ a *BL* (BEPPO LEVI) *function* [**10** d] in B if: (1) for almost all lines l parallel to a coordinate axis the restriction of $x(w)$ to any finite segment λ of $l \cap G_i$ is absolutely continuous as a function of one variable, $i = 1, \dots, n$; and (2) $D(x, B)$ is finite. We write $\Theta(x, B)$ for the set of all elementary paths γ whose segments do not lie on exceptional lines l, such that $D_1(x, \gamma)$ is finite. The restriction of $x(w)$ to any $\gamma \in \Theta(x, B)$ is continuous, including across identified edges of B.

A continuous *BL* function is termed *Dirichlet*. A *BL* function whose restriction to B^* is continuous, *generalized Dirichlet* [**10** d]. Generalized DIRICHLET functions are needed to obtain a compactness property not possessed by DIRICHLET functions.

(e) *Parametric and micro-representations.* Let $x(w)$ be bounded and generalized DIRICHLET in B; and $j(w)$ the cross-product $x_u \times x_v$. Then $x(w)$ represents a *parametric surface* L, as follows:

$$L(f) = \int_B f[x(w), j(w)]\, dw, \qquad \text{all } f \in F \ [\text{or } F_0]. \tag{2.4}$$

The term *micro-representation* is used as in [**10** e, p. 7] except now the carrier $x(w)$ of a micro-representation M_w is a generalized DIRICHLET function (or a *BL* function in case B has no boundary) on some admissible domain B rather than a square. M_w is a micro-representation of the generalized surface L if:

$$L(f) = \int_B M_w(f)\, dw, \qquad \text{all } f \in F \ [\text{or } F_0]. \tag{2.5}$$

3. - Main definitions and theorems.

The first definition pertains to surfaces with no boundary $(s = 0)$.

Definition 1. $\mathfrak{L}(\tau)$ denotes the closure in the space of generalized surfaces of the set of all polyhedra P such that P has a piecewise linear representation on an admissible domain (defined by finite cells only) of type τ with no boundary. A generalized surface belonging to $\mathfrak{L}(\tau)$ for some τ is called *closed generalized surface of finite topological type.* If $q(\tau) = -2$, the elements of $\mathfrak{L}(\tau)$ are termed *generalized spheres.*

These definitions are essentially the ones given in [**10** e]. A representation theorem for closed generalized surfaces of finite topological type is given in § **11**.

Let $C_1, \dots, C_s$ be closed FRÉCHET curves in R^m, none of which reduces to a point. We write Γ for the set $\{ C_1, ..., C_s \}$. Γ is termed oriented if $C_1, ..., C_s$ are oriented curves. If $\Gamma = \{ C_1, ..., C_s \}$, $\Gamma' = \{C'_1, ..., C'_s\}$ and both are oriented or both nonoriented, $d(\Gamma, \Gamma')$ denotes the sum on i of the FRÉCHET distance between C_i and C'_i. Let B be an admissible domain of type τ whose boundary B^* has s components. A vector function $x(w)$ which is generalized DIRICHLET in B is said to be of *type* (τ, Γ) if, for every $i = 1, ..., s$, there is a component ω_i of B^* with $\omega_i \neq \omega_j$ for $i \neq j$ such that the restriction of $x(w)$ to ω_i represents C_i. Naturally, if τ is an oriented type both B and Γ are taken as oriented.

Definition 2. $\mathfrak{L}(\tau, \Gamma)$ denotes the closure in the space of generalized surfaces of the set of all parametric surfaces L such that L has a DIRICHLET parametric representation of type (τ, Γ).

Definition 3. $\mathfrak{L}_1(\tau, \Gamma)$ denotes the set of all generalized surfaces L such that $L = \lim L_n$, where L_n has DIRICHLET parametric representation $x_n(w)$ of type (τ, Γ_n) for $n = 1, 2, ...$, and $\lim d(\Gamma_n, \Gamma) = 0$.

Naturally, we shall consider only those Γ for which $\mathfrak{L}_1(\tau, \Gamma)$ is not empty. We say that L is of *finite topological type with boundary* Γ [*limit boundary* Γ] if $L \in \mathfrak{L}(\tau, \Gamma)$ [$L \in \mathfrak{L}_1(\tau, \Gamma)$] for some τ. We must be careful in referring to Γ as the boundary of L, since Γ is by no means unique [**10** c, § **6**]. However, a generalized surface can admit at most one boundary with no multiple points (it may of course admit no such boundary). For oriented types this was in effect proved in [**5** c, § **3**, Remark 2].

Limit boundaries were considered briefly by YOUNG in [**10** c], but no theory involving them was developed there. Their use allows us to make all approximations with polyhedra, and to apply a certain pinching process (§ **5**). The existence of generalized conformal representations is needed only for polyhedra; and hence neither MORREY's theorem nor its extension to 2-manifolds will be used. It is clear that $\mathfrak{L}(\tau, \Gamma) \subset \mathfrak{L}_1(\tau, \Gamma)$. We shall establish the not-so-obvious fact that under suitable restrictions on Γ, enough elements of $\mathfrak{L}_1(\tau, \Gamma)$ actually belong to $\mathfrak{L}(\tau, \Gamma)$.

Definition 4. We say that L is an *improper* element of $\mathfrak{L}_1(\tau, \Gamma)$ if L is the limit of some sequence of the sort described in Definition 3, such that for $n = 1, 2, ...,$ $x_n(w)$ is constant on an elementary path (§ **2** b) γ_n in its domain B_n satisfying one of the following:

(3.1) (a) $\gamma_n \subset B_n^0$ and γ_n is simple closed not null homotopic;

(b) γ_n is a cross-cut joining different components of B_n^*.

(c) γ_n is a cross-cut with both ends on the same component ω_{in} of B_n^* and the oscillation of $x_n(w)$ on neither arc of $\omega_{in} - \gamma_n$ tends to 0 as n tends to ∞.

An element of $\mathfrak{L}_1(\tau, \Gamma)$ which is not improper is termed *proper*. The improper elements are characterized in §'s **8** and **9**.

Definition 5. By least element of a set Ω of generalized surfaces we mean an $L \in \Omega$ such that $L_1 \leqslant L$ with $L_1 \in \Omega$ implies $L_1 = L$. An element of $\mathfrak{L}_1(\tau, \Gamma)$ which is both proper and least is called *irreducible*.

Our main theorem concerns the validity of the following statements.

(3.2) *The generalized surface $L_0 \in \mathfrak{L}_1(\tau, \Gamma)$ actually belongs to $\mathfrak{L}(\tau, \Gamma)$. Moreover, L_0 has a micro-representation whose carrier is a generalized Dirichlet vector-function of type (τ, Γ) with domain a parallel slit domain.*

(3.3) Theorem. *The statements* (3.2) *are valid in either of the following situations:*

I. *L_0 is irreducible, Γ is arbitrary.*

II. *L_0 is a least element of $\mathfrak{L}_1(\tau, \Gamma)$, Γ has no multiple points.*

Of course, to say that Γ has no multiple points means that its curves $C_1, \ldots, C_s$ are disjoint and simple closed. A proof for I is given in § **7**, and for II in § **8**.

The condition that Γ have no multiple points is to some extent unavoidable for the full validity of both parts of (3.2) for least elements, as a simple example in § **9** shows. However, the second part of (3.2) remains true for boundaries with a finite number of multiple points, provided the requirement that the carrier be generalized Dirichlet is relaxed. It is an open question what the most general conditions on Γ are in order that the first part of (3.2) be valid for all least elements of $\mathfrak{L}_1(\tau, \Gamma)$.

For comparison purposes, let us for the moment impose the setting of [**10** e]; namely, we consider only the set $\mathfrak{L}(\tau, \Gamma)$ and assume that Γ has no multiple points and occupies zero 2-dimensional Hausdorff measure. Then (3.3) implies that any $L \in \mathfrak{L}(\tau, \Gamma)$ is the sum of a generalized surface L_1 and an element L_2 of $\mathfrak{L}(\tau, \Gamma)$ with micro-representation whose carrier is of type (τ, Γ). In [**10** e] this appears in a stronger (3) form where L_1 is a generalized sphere. For it one needs the existence of a microrepresentation for all connected elements of $\mathfrak{L}(\tau, \Gamma)$, not just the irreducible ones. Very probably this latter result can also be proved with the present notion of micro-representation.

(3) Except for the weaker sort of boundary identifications in [**10** e].

The argument seems to involve substantially repetition of several deep parts of [**10** e], namely, §'s **3** and **7** and (8.2) (i), together with the extended MORREY theorem [**4** a].

4. – Conformal representation.

A polyhedron P is called nondegenerate of type (τ, Γ) if P has a light piecewise linear representation of type (τ, Γ) on an admissible domain defined by finite cells. Any orientable nondegenerate polyhedron or admissible domain may be regarded as a RIEMANN domain by introducing an analytic structure in the usual way [**2**, p. 66]. The nonorientable case is treated by introducing the orientable covering surface. When we say conformal mapping between nonoriented domains, we mean conformal under suitable choice of local orientations. We use in a fundamental way the theorem [**2**, p. 85] that every nondegenerate polyhedron and every admissible domain is the conformal image of a slit domain. The decisive feature of conformality here is the fact that it implies the equality of DIRICHLET integral and area.

(4.1) *Every polyhedron P nondegenerate of type (τ, Γ) has a Dirichlet representation $x(w)$ of type (τ, Γ) on a normalized slit domain A for which* $D(x, A) = a(x, A) = a(P)$.

(4.2) *Let $w = \varphi(\zeta)$ be a homeomorphism of an admissible domain B_1 into an admissible domain B, conformal in B_1^0. Then, for any $x(w)$ Dirichlet in B, $y(\zeta) = x[\varphi(\zeta)]$ is Dirichlet in B_1 and:*

(i) *the partial derivatives of $y(\zeta)$ are given almost everywhere by the usual formulas for composite differentiation;*

(ii) $D(y, B_1) \leqslant D(x, B)$.

If φ is onto, then for every $x(w)$ generalized Dirichlet in B, there exists $y(\zeta)$ generalized Dirichlet in B_1 such that $y(\zeta) = x[\varphi(\zeta)]$ almost everywhere and (i) *and* (ii) *hold.*

Remarks. Naturally, if B has no boundary we read BL in place of generalized DIRICHLET everywhere in the last statement of (4.2). The hypothesis « onto » is then superfluous. We may also consider the case when B_1 and B are defined by finite cells only and φ is piecewise linear rather than conformal. Then (4.2) is still valid, and the proof is the same except a constant factor must be supplied in (ii).

Suppose in (4.2) that φ is onto. If M_w is a micro-representation on B of L with carrier $x(w)$, then φ transforms M_w into another micro-representation on B_1 of the same generalized surface L with carrier $y(\zeta)$. This is an exercise involving only the definitions and a well known change of variable theorem for double integrals.

Proof of (4.2). We suppose that B and B_1 are of the slightly special form described in § **2** (d). The extension to general admissible domains involves merely notational complications. Let $x(w)$ be DIRICHLET. If $G_1, \ldots, G_n, G'_1, \ldots, G'_m$ denote the cells defining B and B_1, respectively, let $G_{ij} = G'_j \cap \varphi^{-1}(G_i)$. Let $\varDelta$ be any interval contained in some G^0_{ij}. By [**10** b, (11.2)] the restriction of $y(\zeta)$ to $\varDelta$ is DIRICHLET, (i) holds in $\varDelta$, and $D(y, \varDelta) = D(x, \varphi(\varDelta))$. Since countably many such intervals cover almost all of B_1, (i) and (ii) follow by addition. It remains to show that $y(\zeta)$ is DIRICHLET on B_1. Clearly $y(\zeta)$ is continuous and $D(y, B_1)$ is finite. Let l be any line parallel to a coordinate axis such that $D_1(y, l)$ is finite and the restriction of $y(\zeta)$ to any segment of l lying in some G^0_{ij} is absolutely continuous. Let λ be any finite segment of l lying in some G'_j. Since φ is analytic in B^0_1, λ is the sum of the sets $\lambda \cap G^0_{ij}$ and a set having no limit point in B^0_1. By continuity of $y(\zeta)$ and finiteness of $D_1(y, \lambda)$ it follows that $y(\zeta)$ is absolutely continuous on λ. Since almost all lines l have the properties described, the proof that $y(\zeta)$ is DIRICHLET is complete.

Now suppose φ is onto and $x(w)$ generalized DIRICHLET. Let K by any figure in B which is either an interval in a single cell G_i or a pair of intervals in different cells joined along a common identified edge. Suppose the restriction of $x(w)$ to K^* is continuous. By a result of YOUNG [**10** d, (7.2)] given $\delta > 0$ there exists a set $W \subset K$ of measure $< \delta$ containing no point of K^* and a DIRICHLET function $x'(w)$ in K such that $x'(w) = x(w)$ for $w \in K - W$ and $D(x', K) \leqslant D(x, K)$.

Let S be the finite set in B_1 whose elements are the vertices and points at ∞ in B_1 and the φ^{-1} images of such points in B. Write $B_1 - S$ as the union of a locally finite set of nonoverlapping figures $\varDelta_1, \varDelta_2, \ldots$ each of the same nature as K above; let $E_n = \varDelta_1 \cup \ldots \cup \varDelta_n$. For $n = 1, 2, \ldots, \varphi(E_n)$ is contained in a set H_n which is the union of finitely many nonoverlapping figures K for which the restriction of $x(w)$ to K^* is continuous. Then there exists a DIRICHLET function $x_n(w)$ in H_n such that; (1) $x_n(w) = x(w)$ at all points of H^*_n and except for a subset W_n of H^0_n for which both W_n and $Y_n = \varphi^{-1}(W_n)$ have measure $< 2^{-n}$; (2) $D(x_n, H_n) \leqslant D(x, H_n) \leqslant D(x, B)$. Since φ is onto, Y_n contains no point of B^*_1.

Let $y_n(\zeta) = x_n[\varphi(\zeta)]$, $\zeta \in E_n$. By what is already proved, $y_n(\zeta)$ is DIRICHLET in E_n and $D(y_n, E_n) \leqslant D(x, B)$. Moreover, $y_n(\zeta) = x[\varphi(\zeta)]$ except in Y_n, and at every point of $E_n \cap B_1^*$. The following reasoning was used by YOUNG in proving a compactness property for BL functions on a square [**10** d, (5.4)] analogous to the one we now obtain for the sequence $y_n(\zeta)$. We sketch it and refer to [**10** d] for details. For almost all lines l parallel to a coordinate axis in the ζ-plane there is a subsequence of n (depending on l) for which $D_1(y_n, l)$ is bounded. This is by FATOU's lemma and boundedness of $D(y_n, E_n)$. Let V be the set of all such lines which in addition meet no finite point of S. For any finite part ω lying in B_1 of a line $l \in V$, $y_n(\zeta)$ is eventually defined on all of ω. Except for ζ in a null set $y_n(\zeta)$ is constant with value $x[\varphi(\zeta)]$ for $n = n(\zeta)$ large enough, and so is bounded. Then there is a countable dense subset V_1 and a fixed subsequence of n (again denoted by 1, 2, ...), such that for every $l \in V_1$: $(*)$ $D_1(y_n, l)$ is bounded and $y_n(\zeta)$ tends to a limit $y(\zeta)$ uniformly on any finite part of $l \cap G_j'$ $(j = 1, ..., m)$. Whe then extend $y(\zeta)$ by continuity to every line of the set V' corresponding to V for our subsequence, and to B_1^*. At the remaining points of B_1, $y(\zeta)$ is defined arbitrarily. For every $l \in V'$, there is a further subsequence of n (depending now on l) for which $D_1(y_n, l)$ is bounded and $(*)$ holds.

Moreover,

$$D(y, E_n) \leqslant \lim_m \inf D(y_m, E_n) \leqslant D(x, B) \qquad (n = 1, 2, \ldots).$$

It follows that $D(y, B_1) \leqslant D(x, B)$ and $y(\zeta)$ is generalized DIRICHLET.

Let $Z_n = \bigcup_{m \geq n} Y_m$. For almost all $\zeta \in E_n - Z_n$, $y_m(\zeta) = y(\zeta)$ for every $m > n$. Since Z_n has measure $< 2^{-n+1}$ this implies $y(\zeta) = x[\varphi(\zeta)]$ almost everywhere in B_1. At any point ζ which is a point of linear density of the intersection of $E_n - Z_n$ with both the vertical and the horizontal line through ζ and at which the partial derivatives of both $y_n(\zeta)$ and $y(\zeta)$ exist, these partial derivatives coincide. This happens almost everywhere in $E_n - Z_n$. Since the analogous statements involving the partial derivatives of $x(w)$ and $x_n(w)$ also hold, (i) of (4.2) follows from its validity for $x_n(w)$ and $y_n(\zeta)$. This completes the proof.

The following theorem solves the *converse* of the problem of micro-representation. Except for the use of (4.2) it is due to YOUNG.

(4.3) *If L has a micro-representation M_w whose carrier $x(w)$ is of type (τ, Γ), then $L \in \mathfrak{L}(\tau, \Gamma)$.*

Let $K_1, \ldots, K_r$ be non-overlapping admissible 2-cells whose union is the domain B of $x(w)$, such that the restriction of $x(w)$ to K_i^* is continuous for every $i = 1, \ldots, r$. To prove (4.3) it suffices to show that, for every i, the restriction of M_w to K_i represents a generalized surface which is the limit of parametric surfaces L_n with DIRICHLET representations $x_n(w)$ on K_i such that $x_n(w) =$

$= x(w)$ for all $w \in K_i^*$ and every n. But this follows from [**10** e, (2.2)] together with conformal mapping of a square onto K_i.

Representation on a square. With the following discussion the representations we shall obtain can be compared with [**10** e]. Let L have micro-representation M_w on an admissible domain B, with carrier $x(w)$. There exists a continuum k which is the union of B^* with finitely many elementary arcs $\gamma_1, \ldots, \gamma_m$ belonging to $\Theta(x, B)$, such that $A - k$ is an open 2-cell. There is a conformal mapping φ of the interior of the unit square Q onto $A - k$. Moreover, φ has a unique extension as a homeomorphism from Q to the admissible domain B_1 obtained from B by cutting along $\gamma_1, \ldots, \gamma_m$. By (4.2) and the Remark following it, L has a micro-representation on Q. Its carrier is subject to piecewise analytic boundary identifications by the SCHWARTZ reflection principle.

If B is defined by finite cells G_i only, one may use instead of φ a piecewise linear homeomorphism ψ of Q onto B_1, obtained by a well known elementary process [**8**, Ch. 6].

5. – A pinching process.

In this section we adapt to generalized surfaces a certain deformation process whereby the part of a surface near a given point x_0 is contracted into x_0. Processes of this sort have been used in connection with the PLATEAU problem [**2**, p. 154] and in the proof of cyclic additivity theorems for LEBESGUE area [**1** a, p. 69], [**1** b], [**7**, V. 2].

We may suppose $x_0 = 0$. Given $\varepsilon > 0$ let $\eta = \exp(-\varepsilon^{-1})$. Let:

$$(5.1) \qquad P(r) = \begin{cases} 1, & r \geqslant \eta \\ 1 + \{\log(\eta/r)/\log\eta\}, & \eta^2 \leqslant r \leqslant \eta \\ 0, & 0 \leqslant r \leqslant \eta^2. \end{cases}$$

Let $x(w)$ be DIRICHLET in an admissible domain B; suppose that the set of w with $|x(w)| = \eta$ or $|x(w)| = \eta^2$ has measure 0. Let $p(w) = P(|x(w)|)$, and $y(w)$ the product of the vector $x(w)$ by the scalar $p(w)$:

$$(5.2) \qquad y(w) = p(w) \cdot x(w).$$

The partial derivatives of $y(w)$ exist and are calculated almost everywhere by the elementary formula. Moreover:

$$D(y,\ B) \leqslant (1 + \varepsilon)^2\, D(x,\ B) \qquad [\mathbf{2},\ \text{p. } 155]\,. \tag{5.3}$$

It is not hard to show that $y(w)$ is Dirichlet. Moreover:

$$\begin{cases} y(w) = x(w) & \text{if} \quad |\, x(w)\,| \geqslant \eta \\ |\, y(w)\,| < \eta & \text{if} \quad |\, x(w)\,| < \eta\,. \end{cases} \tag{5.4}$$

(5.5) *Let L, L' be the parametric surfaces represented by $x(w)$, $y(w)$ respectively, where $y(w)$ is defined by* (5.2). *Then there exists a generalized surface L'' situated in the sphere with center x_0 and radius η such that the Mc Shane distance between L and $L' + L''$ is no more than $3\varepsilon \cdot D(x,\ B)$.*

Proof.

Let $j = x_u \times x_v$ and $j' = y_u \times y_v = (p_u\ x + p\ x_u) \times (p_v\ x + p\ x_v)$. Using bilinearity of cross-products and the fact that $x \times x = 0$:

$$j' = p_u\ p\ (x \times x_v) + p\ p_v\ (x_u \times x) + p^2\ j\,. \tag{5.6}$$

Let Q_0, Q_1, Q_2 denote the set of w for which $|\, x(w)\,| < \eta^2$, $\eta^2 < |\, x(w)\,| < \eta$, and $|\, x(w)\,| > \eta$, respectively. By hypothesis, these sets comprise almost all of B. In Q_0 and Q_2 $p_u = p_v = 0$, while in Q_1 we have by elementary calculation:

$$|\, p_u\,| \leqslant \varepsilon\, |\, x_u\,| / |\, x\,|, \qquad |\, p_v\,| \leqslant \varepsilon\, |\, x_v\,| / |\, x\,|\,.$$

Then, since $|\, p\,| \leqslant 1$ and $|\, a \times b\,| \leqslant |\, a\,|\, |\, b\,|$ for any a, b:

$$|\, p_u\ p\ (x \times x_v)\,| \leqslant \varepsilon\, |\, x_u\,|\, |\, x_v\,|, \qquad |\, p_v\ p\ (x_u \times x)\,| \leqslant \varepsilon\, |\, x_u\,|\, |\, x_v\,|\,.$$

Therefore, by (5.6),

$$|\, j' - p^2\ j\,| \leqslant 2\varepsilon\, |\, x_u\,|\, |\, x_v\,| \leqslant \varepsilon\, [x_u^2 + x_v^2]\,. \tag{5.7}$$

Let K be any cube with center the origin containing the set $x(B)$. Consider any $f \in \Phi$ (§ **2** (a)). It is an exercise to show that, for all x, $x' \in K$ and j, $j' \in S^m$:

$$|\, f(x,\ j) - f(x',\ j)\,| \leqslant |\, x - x'\,|\, |\, j\,|, \qquad |\, f(x,\ j) - f(x,\ j')\,| \leqslant |\, j - j'\,|\,.$$

By (5.4) we then have, since $\eta < \varepsilon/2$,

$$(5.8)\qquad \left| \iint_B [f(x, j) - f(y, j)]\, dw \right| \leqslant \varepsilon \iint_B |j|\, dw = \varepsilon\, a(x, B) \leqslant \varepsilon \cdot D(x, B).$$

Using (5.7) and the fact that $p^2 f(y, j) = f(y, p^2 j)$:

$$(5.9)\qquad \left| \iint_B [p^2 f(y, j) - f(y, j')]\, dw \right| \leqslant 2\varepsilon \cdot D(x, B).$$

Define L'' by the formula:

$$(5.10)\qquad L''(f) = \iint_B (1 - p^2) \cdot f(y, j)\, dw, \qquad \text{all } f \in F \ [\text{or } F_0].$$

Clearly L'' is linear, non-negative. Thus, L'' is a generalized surface (not necessarily parametric). Since $p = 1$ if $|x| > \eta$, L'' is situated on the sphere $|x| \leqslant \eta$. Finally, the sum of the left sides of (5.8) and (5.9) without absolute value signs is $L(f) - L'(f) - L''(f)$. Therefore,

$$|L(f) - [L'(f) + L''(f)]| \leqslant 3\varepsilon \cdot D(x, B).$$

Since $f \in \Phi$ is arbitrary, this proves (5.5).

6. - Lemmas.

Our first statement is an approximation lemma. It is simple compared with other known ones, which unfortunately do not cover precisely the present situation.

(6.1) *Let L have* Dirichlet *parametric representation $x(w)$ of type (τ, Γ). Then L is the limit of nondegenerate polyhedra P_n of type (τ, Γ_n), where Γ_n tends to Γ.*

Proof.

First, we may ignore the requirement of nondegeneracy, since this can be achieved afterward by slight modifications of P_n. Second, it is enough to assume that the domain B of $x(w)$ is defined by finite cells and that each component of B^* belongs to $\Theta(x, B)$. If this is not so, we replace B by a subdomain B_1 with the required properties and the same topological type, such that $a(x, B - B_1)$ is small.

By the construction described in the last sentence of § **4**, L has another DIRICHLET representation $y(w)$ on a square Q with piecewise linear identifications on the perimeter π, such that $\pi \in \Theta(y, Q)$. By an elementary « sewing » process described in [**10** e, p. 35 bottom] there is a square Q' with $Q \subset Q'$, whose perimeter is subject to the same type of identifications as Q, and a DIRICHLET function $y'(w)$ piecewise linear on the perimeter of Q' and equal to $y(w)$ in Q, such that $a(y', Q' - Q)$ is small (4). Our conclusion now follows from [**10** e, (2.2)].

If $x(w)$ is BL in B, $l(\gamma)$ denotes for any $\gamma \in \Theta(x, B)$ the length of the curve represented by $x(w)$ on γ. If B is not a 2-cell or 2-sphere, let $\Omega(x, B)$ be the set of all simple closed elementary curves $\gamma \in \Theta(x, B)$ such that $\gamma \subset B^0$ and γ is not null homotopic, together with all elementary cross-cuts $\gamma \in \Theta(x, B)$ such that there is no arc λ of B^* with $\gamma \cup \lambda$ closed and null homotopic. Let

$$\varphi(x, B) = \inf_{\gamma \in \Omega(x, B)} l(\gamma). \tag{6.2}$$

The number $\varphi(x, B)$ is closely related to the inner diameter of a surface defined by SHIFFMAN [**9**], and to YOUNG's pinching constant [**10** e, p. 44].

A proof of the following statement is contained in that for [**4** a, Lemma 6].

(6.3) *Let A_n be a sequence of normalized slit domains, all of the same type τ, and $x_n(w)$ Dirichlet in A_n for $n = 1, 2, \ldots$. Suppose that $D(x_n, A_n)$ is bounded and $\liminf \varphi(x_n, A_n) > 0$. Then there is a subsequence for which A_n tends to a limit A, which is a slit domain of type τ.*

In the next lemma we suppose that τ is a type with non-null boundary (i.e., $s \geq 1$), but not the type of the 2-cell.

(6.4) *Let L be an irreducible element of $\mathfrak{L}_1(\tau, \Gamma)$. Let L_n be parametric with Dirichlet representation $x_n(w)$ of type (τ, Γ_n) on a normalized slit domain A_n, such that $D(x_n, A_n)$ is bounded, L_n tends to L, and Γ_n tends to Γ.*

Then there is a subsequence of n for which $\varphi(x_n, A_n)$ tends to a positive limit and the restrictions of $x_n(w)$ to A_n^* are equicontinuous (5).

Proof.

We argue by contradiction. Suppose $\varphi(x_n, A_n)$ tends to 0. Then there exists $\gamma_n \in \Omega(x_n, A_n)$ such that diam $x_n(\gamma_n)$ tends to 0, since diameter does not exceed

(4) The finiteness of the simple DIRICHLET integral $D_1(y, \pi)$ allows us to use this elementary procedure in place of the much deeper sewing theorem [**10** e, (3.2)].

(5) Equicontinuity will be defined as in [**4** a, § 6]. In effect, it means equicontinuity in every bounded part of A_n^* with respect to distance in A_n and also at ∞.

length. We may suppose that the sets $x_n(\gamma_n)$ tend to a point x_0, and may take $x_0 = 0$. Suppose first that γ_n is either simple closed or a cross-cut joining different components of A_n^*. Given $\varepsilon > 0$ let $\eta = \exp(-\varepsilon^{-1})$ and choose $N = N(\varepsilon)$ such that $x_N(\gamma_N)$ is contained in the η^2-neighborhood of x_0. We exclude the countable set of values of ε for which the set where $|x_n(w)|$ equals η or η^2 has positive measure for some n.

Let us apply the pinching process to $x_N(w)$, the corresponding quantities in (5.5) being denoted by $y_N(w)$, L_N', L_N''. Clearly $y_N(w)$ is constant on γ_N, and is of type (τ, Γ_N') where $d(\Gamma_N, \Gamma_N') < \varepsilon s$ (s the number of components of A_N^*). Let ε describe a sequence tending to 0, $N(\varepsilon)$ being chosen to tend to infinity. By (5.5) $L_N' + L_N''$ tends to L. For a subsequence of N, L_N' tends to a limit $L' \leqslant L$. L' is an improper element of $\mathfrak{L}_1(\tau, \Gamma)$, contrary to the hypothesis that L is irreducible.

Suppose next γ_n is a cross-cut with both endpoints on the same component $\omega_{1,n}$ of A_n^*. Let λ_{in}, λ_{2n} be the two arcs of $\omega_{in} - \gamma_n$. By choosing a subsequence we may suppose that diam $x_n(\lambda_{in})$ tends to a limit ϱ_i $(i = 1, 2)$. If both $\varrho_1 > 0$ and $\varrho_2 > 0$ the reasoning is as before, using (3.1) (c) in place of (3.1) (a) or (b). This is so even if $\gamma_n \notin \Omega(x_n, A_n)$, a remark we shall need for the second part of the proof. If, say, $\varrho_1 = 0$ we choose $N = N(\varepsilon)$ so that diam $x_N(\gamma_N \cup \lambda_{1N}) \leqslant \eta^2/2$. Now $\gamma_N \cup \lambda_{1N}$ is simple closed and not null homotopic; therefore, by continuity of $x_N(w)$ there exists a simple closed $\gamma_N' \in \Omega(x_N, A_N)$ such that diam $x_N(\gamma_N') \leqslant \eta^2$. By the previous reasoning we arrive again at a contradiction. This proves the first assertion of (6.4).

Let us deny equicontinuity on A_n^*, where n describes a subsequence for which A_n tends to a limit A of type τ [see (6.3)]. Then there exists $\varepsilon_0 > 0$, a further subsequence of n, and for n in this subsequence w_n, $w_n' \in A_n^*$, such that $|x_n(w_n) - x_n(w_n')| \geqslant \varepsilon_0$ and either both w_n and w_n' remain in a bounded part of the plane while their distance in A_n tends to 0 or else both w_n and w_n' tend to ∞. Let us apply reasoning used in a similar situation in [4 a, § 6], based on YOUNG's $\varepsilon - \delta$ gratings (one could use instead of this a lemma of SHIFFMAN [9]). There exist for all large n in our subsequence a cross-cut γ_n and an arc λ_{2n} of A_n^* with the same endpoints as γ_n, such that: (1) λ_{2n} contains both w_n and w_n'; (2) $l(\gamma_n)$ tends to 0; (3) $\gamma_n \cup \lambda_{2n}$ is null homotopic. (If w_n and w_n' remain bounded, λ_{2n} is a « small » arc; in the contrary case λ_{2n} contains ∞ and is in a « small » neighborhood of ∞). Since A_n is not a 2-cell $\gamma_n \cup \lambda_{1n}$ is not null homotopic, where λ_{1n} is the arc complementary to λ_{2n}. By reasoning above, diam $x_n(\lambda_{1n})$ is bounded away from 0. Since diam $x_n(\lambda_{2n}) \geqslant \varepsilon_0 > 0$, the quantities ϱ_1, ϱ_2 defined in the first part of the proof are both positive. This is impossible; hence we must have equicontinuity on A_n^*.

7. – Proof of part I of Theorem (3.3).

We can now establish part I of Theorem (3.3). In the proof we appeal to the preceding paragraphs and to a special microrepresentation theorem of YOUNG [**10** e, (5.1)] in which stringent boundary assumptions are made. The proof is divided into steps.

(i) Suppose first that τ is not the type of the 2-cell. Let L_0 be an irreducible element of $\mathfrak{L}_1(\tau, \Gamma)$. By definition and (6.1) L_0 is the limit of nondegenerate polyhedra P_n of type (τ, Γ_n), where Γ_n tends to Γ. By (4.1) every P_n has a DIRICHLET representation $x_n(w)$ of type (τ, Γ_n) on a normalized slit domain $\boldsymbol{A}_n$ with $D(x_n, \boldsymbol{A}_n) = a(P_n)$. Since $\lim P_n = L_0$ implies $\lim a(P_n) = a(L_0)$, $D(x_n, \boldsymbol{A}_n)$ is bounded. By choosing a subsequence, we may suppose that all $\boldsymbol{A}_n$ have similar configurations and that corresponding curves in Γ_n are represented on corresponding components of $\boldsymbol{A}_n^*$. By (6.3) and (6.4) there is a further subsequence (denoted again by 1, 2, ...) such that $\boldsymbol{A}_n$ tends to a limit $\boldsymbol{A}$ of type τ and the restrictions of $x_n(w)$ to $\boldsymbol{A}_n^*$ are equicontinuous.

(ii) Let V denote the set of all vertical and horizontal lines l such that: (1) l passes through no vertex of $\boldsymbol{A}$; (2) for $n = 1, 2, \dots$, any elementary path in $\boldsymbol{A}_n$ lying on l belongs to $\Theta(x_n, \boldsymbol{A}_n)$; and (3) there is a subsequence of n (depending on l) for which $D_1(x_n, l)$ is bounded. Since $D(x_n, \boldsymbol{A}_n)$ is bounded, V comprises almost all vertical and horizontal lines.

(iii) Let Δ be a rectangle in $\boldsymbol{A}$, containing no vertex [6], which consists either of a planar interval with at most one side on $\boldsymbol{A}^*$ or else a pair of intervals disjoint from $\boldsymbol{A}^*$ joined along a common identified edge. For all large n there corresponds to Δ in a natural way a rectangle $\Delta_n \subset \boldsymbol{A}_n$, such that Δ and Δ_n are superimposed outside the δ_n-neighborhood of the slits of $\boldsymbol{A}$ and the real$_n$ axis, where $\lim \delta_n = 0$. The definition of Δ_n is completed by following identifications in $\boldsymbol{A}_n$ rather than in $\boldsymbol{A}$. Details are given in [**4** a, § 5 e]. We denote by λ the relative boundary of Δ (= closure of $\Delta^* - \boldsymbol{A}^*$), and λ_n the relative boundary of Δ_n. If Δ and Δ' are nonoverlapping, then Δ_n and Δ_n' are nonoverlapping for all large n.

Let a, b denote the width and height of Δ, and G the interval $0 \leqslant u \leqslant a$, $0 \leqslant v \leqslant b$. Let h be a linear homeomorphism of G onto Δ which involves, at most, reflection of one part of Δ in a horizontal line (nonorientable case only), translations, and fitting two parts of Δ together along the identified edge. Let

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(6) More precisely, Δ shall contain no limit point of vertices of A_n (see [**4** a, §5]).

h_n be the linear homeomorphism of G onto Δ_n uniquely determined by the requirement that $h(w)$ and $h_n(w)$ are superimposed if w is any point of a horizontal edge of G such that $h(w) \notin A^*$. The points $h(w)$ and $h_n(w)$ have the same abscissa for all w, and $h(w) \in A^*$ if and only if $h_n(w) \in A_n^*$. The mapping h_n distorts the vertical scale by a factor k_n tending to 1.

(iv) Suppose now that λ, and hence also λ_n, lies on lines of V. Let $y_n(w) = x_n[h_n(w)]$. Then there is a factor p_n tending to 1 such that, if σ denotes $h^{-1}(\lambda)$:

$$D(y_n, G) \leqslant p_n D(x_n, \Delta_n), \tag{7.1 a}$$

$$D_1(y_n, \sigma) \leqslant p_n D_1(x_n, \lambda_n). \tag{7.1 b}$$

The left side of (7.1 a) is then bounded, and also the left side of (7.1 b) for a subsequence of n. For this subsequence the restrictions of $y_n(w)$ to σ are equicontinuous; and by (6.4) their restrictions to the remaining side of G (if any) are equicontinuous. By ASCOLI's theorem and [**10** d, (5.4)] there is a subsequence and a function $y(w)$ generalized DIRICHLET on G such that $y_n(w)$ tends to $y(w)$ in the BL sense in G and uniformly on G^*.

(v) Let L_n denote the parametric surface represented by $y_n(w)$ on G, also by the restriction of $x_n(w)$ to Δ_n. Since $a(L_n) \leqslant a(P_n)$ is bounded and all L_n are situated in a fixed sphere, L_n tends to a limit L for a further subsequence of n (still denoted by 1, 2, ...). Suppose Δ has no side on A^*. Then $D_1(y_n, G^*)$ is bounded; and by an important lemma of YOUNG [**10** e, (5.1)] L is the sum of a generalized sphere L'' and a generalized surface L' with a micro-representation m_w whose carrier is $y(w)$. Clearly $L' \leqslant L$.

If Δ has a side ω on A^* cover $G - h^{-1}(\omega)$ by a locally finite set of nonoverlapping intervals $\pi_1, \pi_2, \ldots,$ such that $D_1(y_n, \pi_r^*)$ is bounded for each $r = 1, 2, \ldots$ (possible according to the definition [**10** d] of BL convergence of y_n). Define analogously L_{rn}, and $L_r = \lim_n L_{rn}$ (further subsequence of n). Then $L_r = L_r' + L_r''$, where L_r'' is a generalized sphere and L_r' has a micro-representation m_w with carrier the restriction of $y(w)$ to π_r. This defines m_w almost everywhere in G. Now, for every N:

$$\sum_{r=1}^{N} L_r' \leqslant \sum_{r=1}^{N} L_r = \lim_n \sum_{r=1}^{N} L_{rn} \leqslant \lim_n L_n = L. \tag{7.2}$$

Hence, letting $L' = L'_1 + \ldots + L'_r + \ldots$, we have again $L' \leqslant L$; and L' has a micro-representation m_w.

(vi) Define

$$x(w) = y[h^{-1}(w)], \qquad M_w = m_{h^{-1}(w)}.$$

Then $x(w)$ is BL in Δ and its restriction to Δ^* is continuous. M_w is a micro-representation of L' with carrier $x(w)$:

$$L'(f) = \iint_{\Delta} M_w(f)\, dw, \qquad \text{all } f \in F \ [\text{or } F_0]. \tag{7.3}$$

Finally, if Δ and Δ' are nonoverlapping but have a line e in common, then the definition of $x(w)$ is consistent on e. Indeed, on any segment of e disjoint from the slits of A and the real axis, $x_n(w)$ tends uniformly to $x(w)$.

(vii) Now let A be covered except for vertices and ∞ by a locally finite set of nonoverlapping rectangles $\Delta^1, \Delta^2, \ldots$ of the sort described in (iii). Then $x(w)$ and M_w are defined in every Δ^j, hence except for a finite set of points of A. It is easily shown using (7.1 a) that $x(w)$ is BL in A, since $x(w)$ is already BL in every Δ^j and continuous on Δ^{j*}. By (7.3) and addition, M_w is a micro-representation of $L'_0 = L^{1'} + L^{2'} + \ldots$. By a calculation similar to (7.2), $L'_0 \leqslant L_0$.

For every j such that $\Delta^j \cap A^*$ is not void, the restriction of $x(w)$ to $\Delta^j \cap A^*$ represents a part of Γ which is the limit of the part of Γ_n represented on $\Delta^j_n \cap A^*_n$. From equicontinuity of $x_n(w)$ on A^*_n it follows that the restriction of $x(w)$ to A^* has a continuous extension to boundary vertices of A and ∞ which represents Γ (cf. reasoning in [4 a] and [**9**]). Thus, the carrier $x(w)$ of M_w is generalized DIRICHLET and of type (τ, Γ).

(viii) By (4.3) $L'_0 \in \mathfrak{L}(\tau, \Gamma)$. Since L_0 is a least element of $\mathfrak{L}_1(\tau, \Gamma)$, which contains $\mathfrak{L}(\tau, \Gamma)$, we must have $L'_0 = L_0$. This completes the proof of I, except for the type of the 2-cell.

(ix) In the remaining case all A_n and A are the upper half plane with no slits. Γ_n consists of a single curve C_n tending to a single curve C (not a point) comprising Γ. The previous reasoning applies provided we establish equicontinuity on A^*_n. Let $x_n(w)$ be normalized by the well-known 3-point condition. For example, let the x_n-images of the points 0, 1, ∞ on the real axis tend to three distinct limits. This guarantees that diam x_n (λ_{1n}) is bounded away from 0 in the final part of the proof of (6.4). From this equicontinuity follows as before.

8. – Proof of part II of Theorem (3.3).

Part II of Theorem (3.3), in which the boundary is assumed to have no multiple points, will be deduced from I by addition. Some partial results for boundaries with multiple points are described in § **9**. Statements (8.1) and (8.2) apply for arbitrary Γ, also for the case of no boundary after fairly obvious modifications.

(8.1) *Let L have micro-representation M_w whose carrier $x(w)$ is of type (τ, Γ). Then L has another micro-representation m_w with carrier $y(w)$ again of type (τ, Γ), such that there is an open set G in the domain B of $y(w)$ on which $m_w = 0$ and $y(w)$ is constant with arbitrarily preassigned value x_0.*

Proof. By conformal mapping (§ **4**) we may assume $x(w)$ is defined on a slit domain A. Let $w_0 = (u_0, v_0)$ be a point of A, where u_0 is chosen so small and v_0 so large that the vertical and horizontal lines l_1, l_2 through w_0 meet no slit. In addition, w_0 is chosen so that the restriction of $x(w)$ to any finite part of l_i is absolutely continuous and $D_1(x, l_i)$ is finite, $i = 1, 2$. Let $g(t) = t$, $t < 0$, and $g(t) = \max(t - 1, 0)$, $t \geqslant 0$. Let $w' = \varphi(w)$ be the transformation of A onto itself defined by

$$u' = u_0 - g(u_0 - u), \qquad v' = v_0 + g(v - v_0).$$

Let $m_w = M_{\varphi(w)}$ except in the strips $u_0 - 1 < u < u_0$, $-\infty < v < \infty$ and $v_0 < v < v_0 + 1$, $-\infty < u < \infty$, and $m_w = 0$ in these strips. Let $y(w) = x[\varphi(w)]$ except in the open interval G' which is the intersection of the two strips. Let $y(w) = x_0$ on a smaller interval G concentric with G'; and let $y(w)$ be defined in the annulus $G' - G$ by linear interpolation on each radial line. One easily verifies the assertions of (8.1).

In (8.2) and (8.3) the following conventions concerning τ, τ', τ_1, τ_2 apply. These types are either all oriented or all nonoriented. If τ is nonoriented but orientable, the same is true for τ', τ_1, τ_2.

For any $\Gamma = \{C_1, \ldots, C_s\}$, $\Gamma' = \{C_1', \ldots, C_t'\}$ we write $\Gamma \cup \Gamma'$ for the set $\{C_1, \ldots, C_s, C_1', \ldots, C_t'\}$.

(8.2) *A generalized surface L has a micro-representation whose carrier is of type (τ, Γ) in either of the following situations:*

(a) *L has a micro-representation with carrier of type (τ', Γ), where $q(\tau') < q(\tau)$.*

(b) $L = L_1 + L_2$, *where* L_i *has a micro-representation whose carrier is of type* (τ_i, Γ_i) $(i = 1, 2)$, $\Gamma_1 \cup \Gamma_2 = \Gamma$, *and* $q(\tau_1) + q(\tau_2) + 2 \leqslant q(\tau)$. *We admit the possibility that* $\Gamma_1 = \Gamma$ *and* $s(\tau_2) = 0$.

Proof.

Suppose (a) holds. We apply (8.1) and follow the notation there. Let $\Delta \subset G$ be a square disjoint from B^*. Let B' be an admissible domain of type τ obtained by replacing Δ by a 2-manifold Δ' with the appropriate number of handles or cross-caps, whose boundary is identified with Δ^*. Let $z(w) = y(w)$, $m'_w = m_w$, $w \in B' - \Delta'$; and $z(w) = y(\Delta)$, $m'_w = 0$, $w \in \Delta'$. Then m'_w is a micro-representation of L with carrier $z(w)$ of type (τ, Γ).

If (b) holds, L_1 and L_2 have micro-representations whose carriers are constant with the same value x_0 on squares Δ and Δ_1 interior to their respective domains of definition. We discard Δ^0 and Δ_1^0, and identify Δ^* and Δ_1^* with each other, taking care to preserve orientation or orientability if possessed by τ. The resulting domain has characteristic $q(\tau_1) + q(\tau_2) + 2$. We have either the desired conclusion or case (a).

(8.3) *Suppose* Γ *is without multiple points*; *i.e.*, Γ *consists of disjoint simple closed curves. A necessary and sufficient condition that a generalized surface* L *be an improper element of* $\mathfrak{L}_1(\tau, \Gamma)$ *is that one of the following occur*: (1) $L \in \mathfrak{L}_1(\tau', \Gamma)$, *where* $q(\tau') < q(\tau)$; (2) $L = L_1 + L_2$, *where* $L_i \in \mathfrak{L}_1(\tau_i, \Gamma_i)$ $(i = 1, 2)$, $\Gamma_1 \cup \Gamma_2 = \Gamma$, $q(\tau_1) + q(\tau_2) + 2 \leqslant q(\tau)$; *or* (3) $L = L_1 + L_2$, *where* $L_1 \in \mathfrak{L}_1(\tau_1, \Gamma)$, $L_2 \in \mathfrak{L}(\tau_2)$, $q(\tau_1) + q(\tau_2) + 2 \leqslant q(\tau)$, *and* τ_2 *is not the type of the 2-sphere*.

Proof.

Necessity. Let L be the limit of L_n, where L_n has DIRICHLET parametric representation $x_n(w)$ of type (τ, Γ_n), $d(\Gamma_n, \Gamma)$ tends to 0, and $x_n(w)$ is constant on some elementary path γ_n in its domain B_n satisfying (3.1 a). Since Γ has no multiple points, we may disregard (3.1 b and c). Cut B_n along γ_n. This introduces a pair of new boundary components ω'_n, ω''_n or a single component ω_n according as γ_n separates every open set containing it or not. In the first instance, we identify ω'_n, ω''_n with the perimeters of 2-cells Z'_n, Z''_n, respectively, and set $x_n(w) = x_n(\gamma_n)$ in Z'_n and Z''_n. In the second we use instead a single 2-cell Z_n.

If γ_n does not separate B_n we have (1) upon passing to the limit. Otherwise we have (2) or (3); the passage to the limit in this case may be through a subsequence.

Sufficiency. This follows from (8.2), or more precisely from the constructions in its proof, applied to approximating parametric surfaces. Observe that in each case the elementary simple closed curve along which new identifications are made does not bound a 2-cell and hence (by a known theorem) is not null homotopic. The proof of sufficiency does not use the special nature of Γ.

If L is least, then so are L_1 and L_2 in (8.3) (2). In (3) L_2 must be 0 if L is least. From this it follows that every least element of $\mathfrak{L}_1(\tau, \Gamma)$ can be represented as a finite sum of irreducible generalized surfaces. We omit the details, which are analogous to [**10** e, (4.6)].

Proof of II. Suppose $q(\tau) = -1$. Then τ is the type of the 2-cell and Γ consists of a single simple closed curve C. Every element of $\mathfrak{L}_1(\tau, \Gamma)$ is proper; and every least element is irreducible, so that I applies. We proceed by induction on $q(\tau)$. Let L_0 be a least element of $\mathfrak{L}_1(\tau, \Gamma)$. If L_0 is proper, then L_0 is irreducible and I applies. Otherwise, we use (8.2) and (8.3) together with the induction hypothesis.

9. - Boundaries with multiple points.

There remains the question of extending the second part of Theorem (3.3), which concerns least elements. One may try to proceed from the irreducible case by addition, as in § **8**. For boundaries Γ with a finite number of multiple points, each of finite multiplicity, this can be done using a finite number of additions, subject to certain qualifications mentioned below. The proof involves considerations from surface area theory which we do not want to inject at this point, and is deferred to a later paper (7). The author does not know what can be done in the general case. One might hope for a countable decomposition analogous to CESARI's fine-cyclic decomposition for FRÉCHET surfaces of finite topological type [**1** b]. See also [**4** b] and [**6**]. For fine-cyclic elements one is dealing effectively with the case of a finite number of multiple points on the boundary. More precisely, in the middle space the boundary has finitely many multiple points.

An example. Let τ_0 denote the type of the nonoriented 2-cell. Let C_1, C_2 be simple closed curves in a plane with exactly one point p in common, and C the union of C_1 and C_2. Let G_i denote the interior of C_i, and L_i the parametric surface corresponding to G_i $(i = 1, 2)$. Suppose C_1 and C_2 chosen so that G_1 and G_2 are disjoint and no rectifiable curve in $G_1 \cup G_2 \cup C$ joins G_1 and G_2. For $i = 1, 2$, L_i has a DIRICHLET parametric representation $x_i(w)$ of type

(7) However, by formulating the problem differently we shall obtain an analogue of II. See (10.2) below.

(τ_0, C_i) on a square Q. By (9.1) below, $L_1 + L_2 \in \mathfrak{L}_1(\tau_0, C)$. It will be shown elsewhere that actually $L_1 + L_2 \in \mathfrak{L}(\tau_0, C)$, if C occupies 2-dimensional HAUSDORFF measure 0. But $L_1 + L_2$ can have no micro-representation whose carrier is generalized DIRICHLET of type (τ_0, C).

On the other hand, $L_1 + L_2$ has a parametric representation on Q with the following properties: (1) $x(w)$ is continuous on Q and represents C on Q^*; (2) there is a vertical line l on which $x(w)$ is constant with value p, such that C_1 and C_2 are represented on complementary arcs of Q^* with endpoints on l; and (3) $x(w)$ is DIRICHLET in any subinterval of Q disjoint from l. $D(x, Q)$ must be infinite. This example suggests how a representation for least elements can be obtained in case of boundaries with a finite number of multiple points, by widening the class of admissible carriers.

This paragraph will be concluded by extending the characterization (8.3) of the improper elements of $\mathfrak{L}_1(\tau, \Gamma)$.

Definition. The closed curve C splits into closed curves C' and C'' if C has a representation $x = f(\theta)$ on the unit circle g with $f(0) = f(\pi)$ such that, upon identifying the points 0 and π on g, C' and C'' are represented by $f(\theta)$ on the upper and lower halves of g, respectively.

We write $\Gamma \sim \Gamma'$ if Γ' is the set of curves obtained by splitting a single curve C_i of Γ, or vice versa. If Γ has s curves, then Γ' has $s+1$ curves in the first instance and $s-1$ curves in the second.

(9.1) *Statement* (8.3) *is valid for arbitrary* Γ *if we add to the conditions* (1), (2), (3) *the following*: (4) $L \in \mathfrak{L}_1(\tau, \Gamma')$, *where* $\Gamma' \sim \Gamma$ *and* $q(\tau') < q(\tau)$; *or* (5) $L = L_1 + L_2$, *where for* $i = 1, 2$, $L_i \in \mathfrak{L}_1(\tau_i, \Gamma_i)$, $\Gamma_1 \cup \Gamma_2 \sim \Gamma$, $q(\tau_1) + q(\tau_2) + 1 \leqslant q(\tau)$, *and* $s(\tau_1) + s(\tau_2) = s(\tau) + 1$.

Proof.

Necessity. We must supplement the proof of (8.3) by considering the case when γ_n is a cross-cut. Let B_n' be the domain, of characteristic one less than $q(\tau)$, obtained by cutting B_n along γ_n. B_n' is admissible if γ_n does not separate B_n, and splits into two disjoint admissible domains in the contrary case. Upon passing to the limit (possibly through a subsequence of n) we obtain (4) or (5) respectively.

Sufficiency. It has already been shown that (1), (2), or (3) is sufficient. For (4) or (5) we in effect replace cutting along an arc by the reverse operation of pasting two arcs together. To avoid tedious repetitions we give the details only for (4), in the case when Γ' is obtained by splitting a curve C_i of Γ into parts C_i', C_i''. Then $L = \lim L_n$, where L_n has DIRICHLET parametric representation $x_n(w)$ of type (τ', Γ_n') such that $d(\Gamma_n', \Gamma')$ tends to 0. We may assume that

$x_n(w)$ is defined on a slit domain A_n of type τ'. Now $x_n(w)$ represents on components ω'_{in}, ω''_{in} of A_n^* curves C'_{in}, C''_{in} tending to C'_i, C''_i. Let $f(\theta)$ be a representation of C_i on the unit circle, as described in the definition above (9.1). For $n = 1, 2, \ldots$ there is a homeomorphism h'_n of ω'_{in} onto the upper half circle g' (with endpoints identified), such that $| x_n(w) - f[h'_n(w)] | < n^{-1}$, all $w \in \omega'_{in}$. We may suppose that the inverse image w'_n of the endpoints of g' is not a vertex of A_n and that the vertical line l'_n through w'_n does not belong to the exceptional set described in § **2** (d). This is possible since such points are dense in ω'_{in}. Let w''_n, l''_n be defined correspondingly. By a variant of the procedure used to prove (8.1), we may thicken l'_n and l''_n to form vertical strips and thereby obtain a new representation $\widehat{x_n}(w)$, again of type (τ', Γ'_n), on a slit domain $\widehat{A}_n$ constant on the segments λ'_n, λ''_n of $\widehat{A}^*_n$ corresponding to w'_n, w''_n. Let K_n be a square with vertical sides identified with λ'_n, λ''_n, respectively, and horizontal sides unidentified. Then $\widehat{A}_n$ and K_n define an admissible domain B_n of characteristic $q(\tau') + 1$ whose boundary has s components. The identifications may be arranged so as to preserve orientation or orientability, if possessed by τ'. We extend $\widehat{x_n}(w)$ to K_n by linear interpolation on each horizontal line. Then $\widehat{x_n}(w)$ again represents L_n on B_n, and on B_n^* represents a set of curves Γ_n tending to Γ. Passing to the limit, $L \in \mathfrak{L}_1(\widehat{\tau}, \Gamma)$ where $q(\widehat{\tau}) = q(\tau') + 1 \leqslant \leqslant q(\tau)$. By (8.2 a), $L \in \mathfrak{L}_1(\tau, \Gamma)$.

The remaining cases are similar.

10. - Applications.

Let us write $\mathfrak{L}$, $\mathfrak{L}_1$ for $\mathfrak{L}(\tau, \Gamma)$, $\mathfrak{L}_1(\tau, \Gamma)$, respectively; i.e., $\mathfrak{L}$, $\mathfrak{L}_1$ consist of those generalized surfaces of given type τ and boundary, or limit boundary, Γ. We consider the problem $L(f_0) =$ minimum in either $\mathfrak{L}$ or $\mathfrak{L}_1$, for fixed $f_0 \in F$ (or $f_0 \in F_0$ if τ is nonoriented). Let μ, μ_1 denote the respective greatest lower bounds, whether attained or not. From the elementary point of view we may prefer the problem μ, since inf $L(f_0)$ in $\mathfrak{L}$ is (almost by definition) the same as in the subset of $\mathfrak{L}$ consisting of parametric surfaces with DIRICHLET representation of type (τ, Γ). However, the problem μ_1 has certain advantages.

Let $\mathfrak{L}_2$ denote the set of improper elements of $\mathfrak{L}_1$, and $\mu_2 = \inf L(f_0)$ among all $L \in \mathfrak{L}_2$. We call the problem μ_1 *properly posed if* $\mu_2 > \mu_1$ (8).

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(8) A condition of this sort due to DOUGLAS, played a decisive role in the solution of the PLATEAU problem for higher topological types. See COURANT [**2**, Chap. 4], DOUGLAS [**3**], SHIFFMAN [**9**].

(10. 1) *Let f_0 positive semi-definite* (i.e., $f_0 \geqslant 0$). *Suppose either:* (a) *the problem μ_1 is properly posed (Γ arbitrary); or* (b) *Γ has no multiple points (μ_1 not necessarily properly posed).*

Then $\mu = \mu_1$. Moreover, if there is a solution in $\mathfrak{L}_1$ there is as solution in $\mathfrak{L}$ which has a micro-representation whose carrier is of type (τ, Γ).

Proof. Clearly $\mu \geqslant \mu_1$ since $\mathfrak{L} \subset \mathfrak{L}_1$. For every $L \in \mathfrak{L}_1$ there exists a least element $L' \in \mathfrak{L}_1$ with $L' \leqslant L$. Since $f_0 \geqslant 0$, $L'(f_0) \leqslant L(f_0)$. Thus, $\mu_1 = \inf L(f_0)$ among all least elements of $\mathfrak{L}_1$; and in case (a), among all proper, least (i.e., irreducible) elements. The conclusion now follows from Theorem (3.3).

Theorem (10.1) does not assert the existence of a minimum. However, conditions for this are already known. The minimum is attained in $\mathfrak{L}_1$ if and only if a minimizing sequence with bounded supports and areas exists [**10** c, (5.2) Principle of Minimum]. If one assumes in advance bounded supports, then such a minimizing sequence exists if f_0 is positive definite. If Γ occupies 2-dimensional HAUSDORFF measure 0, positive definiteness can be replaced by weaker conditions [**5** a, § 13]. If $m \mid j \mid \leqslant f_0(x, j) \leqslant M \mid j \mid$ for all (x, j), where $0 < m \leqslant M < \infty$, then a minimizing sequence with bounded supports exists. This follows from a lemma [**10** e, (9.4) and (9.5)] due to CESARI and YOUNG.

The theory includes solutions in the ordinary sense to problems of classical interest. Suppose either (a) or (b) of (10.1); moreover, suppose that μ is attained, and that f_0 satisfies the weak normality condition $M(f_0) > 0$ for every micro-sphere $M \neq 0$. Let L_0 be a solution with micro-representation as in (10.1). We apply a result of YOUNG [**10** e, (9.9)] together with dissection of the parameter domain and conformal transformations as in the proof of (4.3). L_0 has another micro-representation M_w whose carrier $x(w)$ is continuous (therefore DIRICHLET) and of type (τ, Γ). If f_0 satisfies a weak regularity condition [**10** e, p. 54 top], in particular if f_0 is convex in the variable j, then the parametric surface represented by the carrier $x(w)$ furnishes a minimum. In particular, the existence theorems of CESARI, DANSKIN, and SIGALOV (see [**11**], [**12**], [**13** a]) appear as very special cases.

Finally, let us show that the condition $f_0 \geqslant 0$ in (10.1) is not so restrictive as may seem. If τ is a nonoriented type, then this condition is necessary in order that μ be finite. Indeed, suppose $f_0(x_0, j_0) < 0$ for some (x_0, j_0). Since τ is nonoriented, f_0 is symmetric in j; hence, $f_0(x_0, -j_0) = f(x_0, j_0)$. Then there exists a polyhedral sphere P (pair of small discs, back to back) such that $P(f_0) < 0$. Now NP is a polyhedral sphere for every positive integer N. If L is DIRICHLET of type (τ, Γ), so is $L + NP$ for every N, by (8.2). Therefore $\mu = -\infty$.

Suppose next τ is oriented. Let us call f_0 *essentially positive semi-definite* if $f_0 + \varphi \geqslant 0$ for some exact [**5** a] φ in F. If Γ occupies 2-dimensional HAUSDORFF measure 0, then the problem μ for f_0 is equivalent to the problem μ for $f_0 + \varphi$ in the sense that $L(f_0)$ differs from $L(f_0 + \varphi)$ by the constant factor $L(\varphi)$ for all $L \in \mathfrak{L}$. This is proved in [**5** c, Lemma 1]. Results of [**5** a, § 5] show that essential positive definiteness comes reasonably near to being necessary that μ be finite. More precisely, it would be necessary if we allowed discontinuous φ and comparison surfaces of arbitrary oriented topological type with boundary Γ.

Boundaries with finitely many multiple points. For this class of boundaries we can get an existence theorem of the same degree of generality as for no multiple points, provided the minimum problem is posed somewhat differently. Our result (10.3 below) is analogous to Theorem II of SIGALOV [**13** b].

The discussion is limited to the oriented case; the nonoriented case is quite similar.

Let K be the carrier of a boundary Γ with finitely many multiple points, oriented by the orientations of the curves of Γ. The set K is a finite linear graph in m-space R^m; and only finitely many points of K are covered more than once by curves of Γ. *Let us regard any two such boundaries Γ and Γ' with the same carrier as equivalent.* Thus, instead of Γ and τ we shall prescribe an oriented finite linear graph K and a number h of handles. Let $\mathfrak{L}^*(h, K)$ [$\mathfrak{L}_1^*(h, K)$] be the union of all sets $\mathfrak{L}(\tau, \Gamma)$ [$\mathfrak{L}_1(\tau, \Gamma)$], where $h(\tau) = h$ and Γ has carrier K. In view of (8.2 a) we might equally well say $h(\tau) \leqslant h$ in this definition. $\mathfrak{L}^*(h, K)$ and $\mathfrak{L}_1^*(h, K)$, each being the union of finitely many closed sets, are closed.

(10.2) *Every least element L_0 of $\mathfrak{L}_1^*(h, K)$ belongs to $\mathfrak{L}^*(h, K)$; and L_0 has a micro-representation whose carrier is of type (τ, Γ), where Γ has carrier K and τ is a type with h handles* [i.e., $h(\tau) = h$].

Let μ^*, μ_1^* denote the respective g.l.b.'s of $L(f_0)$ in $\mathfrak{L}^*(h, K)$, $\mathfrak{L}_1^*(h, K)$. The following is obtained from (10.2) by repeating the discussion following (10.1):

(10.3) *Let f_0 be positive semi-definite. Then $\mu^* = \mu_1^*$. If there exists a minimizing sequence in $\mathfrak{L}_1^*(h, K)$ with bounded supports and areas, then there is a solution in $\mathfrak{L}^*(h, K)$ which possesses a micro-representation of the sort described in* (10.2).

Proof of (10.2).

Let us make the following preliminary observations. Let L_0 be any least element of $\mathfrak{L}_1^*(h, K)$. Then $L_0 \in \mathfrak{L}_1(\tau_0, \Gamma_0)$ where $h(\tau_0) = h$ and Γ_0 has carrier

K, and L_0 is a least element of $\mathfrak{L}_1(\tau_0, \Gamma_0)$. If L_0 is irreducible the conclusion of (10.2) follows from Theorem (3.3). Otherwise, L_0 is an improper element of $\mathfrak{L}_1(\tau_0, \Gamma_0)$, and we shall apply (9.1).

We proceed by induction on the pair $(q(\tau_0), K)$, ordered lexicographically, where order on graphs K is set inclusion. If $q(\tau_0) = -1$ and K is a simple closed curve, then L_0 cannot be improper. In the general case, suppose first (9.1) (1) or (4) holds. Then $L_0 \in \mathfrak{L}_1(\tau', \Gamma')$ where $q(\tau') < q(\tau_0)$ and either $\Gamma' = \Gamma$ or $\Gamma' \sim \Gamma$. Since « if and only if » holds in (9.1) and L_0 is least in $\mathfrak{L}_1(\tau_0, \Gamma_0)$, L_0 must be a least element of $\mathfrak{L}_1(\tau', \Gamma')$. Since the operation $\sim$ increases the number of boundary curves by at most one, $s(\tau') \geqslant s(\tau_0) - 1$. Then from the formula $q = s + 2h - 2$ and the relation $q(\tau') < q(\tau_0)$ we have $h(\tau') \leqslant h(\tau_0) = h$. By the induction hypothesis, L_0 has a micro-representation whose carrier is of type (τ'', Γ), where $h(\tau'') = h(\tau') \leqslant h$ and Γ has carrier K. Let τ be the oriented type with $h(\tau) = h$ and $s(\tau) = s(\tau'')$. By (8.2 a), applied with τ'' in place of τ' there, L_0 has a micro-representation whose carrier is of type (τ, Γ). By (4.3) $L_0 \in \mathfrak{L}(\tau, \Gamma) \subset \mathfrak{L}^*(h, K)$.

Since L_0 is least, in (9.1) (3) we must have $L_2 = 0$; this case reduces to (1). Suppose (9.1) (2) or (5) holds. Then $L_0 = L_1 + L_2$, as indicated there. For $i=1, 2$, L_i is a least element of $\mathfrak{L}_1(\tau_i, \Gamma_i)$. Since $q \geqslant -1$ for any type with boundary, we must have $q(\tau_i) \leqslant q(\tau)$ for $i = 1, 2$ in view of the relations on q in (2) and (5). Taking account also of the corresponding relations on s, $h(\tau_1) + h(\tau_2) \leqslant h(\tau_0) = h$. By the induction hypothesis L_i has a micro-representation whose carrier is of type (τ_i', Γ_i'), where $h(\tau_i') = h(\tau_i)$ and Γ_i' has the same carrier as Γ_i $(i = 1, 2)$. Let $\Gamma = \Gamma_1' \cup \Gamma_2'$, and τ the type with $h(\tau) = h$ and $s(\tau) = s(\tau_1') + s(\tau_2')$. We have

$$q(\tau) = s(\tau) + 2h - 2 \geqslant$$

$$\geqslant s(\tau_1') + s(\tau_2') + 2[h(\tau_1') + h(\tau_2')] - 2 = q(\tau_1') + q(\tau_2') + 2 .$$

Our conclusion now follows from (8.2 b) and (4.3) as before.

11. – Closed generalized surfaces.

A definition of irreducible closed generalized surface could be given, analogous to Definition 5 (§ **3**) for surfaces with boundary. We shall not do this, since with little extra effort a more general representation theorem will be obtained. Let us recall Definition 1 (§ **3**) of the sets $\mathfrak{L}(\tau)$. To avoid trivialities, we assume in the following that $L \neq 0$. The same conventions as in § **8** apply regarding τ, τ', τ_1, τ_2.

Definition. L is an *improper* element of $\mathfrak{L}(\tau)$ if either $L \in \mathfrak{L}(\tau')$ where $q(\tau') < q(\tau)$, or $L = L_1 + L_2$ where $L_i \in \mathfrak{L}(\tau_i)$ $(i = 1, 2)$, $q(\tau_1) + q(\tau_2) + 2 \leqslant q(\tau)$ and neither L_1 nor L_2 is a generalized sphere. A proper element L is termed *connected* if L is not of the form $L_1 + L_2$ where $L_1 \neq 0$ is a generalized sphere and $L_2 \neq 0$ is in $\mathfrak{L}(\tau)$ [**10** e, p. 31] (9).

Let $\Phi(x, B)$ be as in (6.2). For every $\varepsilon > 0$ let $\chi(x, B; \varepsilon) = \inf l(Z^*)$ among all 2-cells $Z \subset B$ whose boundary Z^* belongs to $\Theta(x, B)$ for which $a(x, Z) \geqslant \varepsilon$. We shall take for granted the following statement, which is an easy consequence of the definitions and a process of harmonic interpolation used often by YOUNG. Note especially [**10** e, (6.4) (i)].

(11.1) *Let L be a connected element of $\mathfrak{L}(\tau)$. Then, for any sequence L_n tending to L such that L_n has Dirichlet parametric representation $x_n(w)$ on an admissible domain B_n of type τ for $n = 1, 2, \ldots$, we have* $\liminf \varphi(x_n, B_n) > 0$ *and* $\liminf \chi(x_n, B_n; \varepsilon) > 0$ *for every* $\varepsilon > 0$.

The main result of this paragraph is:

(11.2) *Let L_0 be a connected element of $\mathfrak{L}(\tau)$. Then L_0 has a micro-representation whose carrier is a* BL *function on a parallel slit domain of type τ.*

This is a strengthened version of [**10** e, (6.1)]. When combined with (8.2) above and [**10** e, (4.4) and (4.6)], it leads by addition to a complete solution of the representation problem for closed generalized surfaces of finite topological type.

Proof of (11.2). Suppose first τ is not type of the 2-sphere. The reasoning will follow § 7. Let P_n be a sequence of nondegenerate polyhedra of type τ tending to L_0, and $x_n(w)$ a DIRICHLET representation of P_n on a normalized slit domain A_n of type τ with $D(x_n, A_n)$ bounded. By (6.3) and (11.1) we may assume that A_n tends to a limit A of type τ. Cover A except for vertices and ∞ by rectangles $\Delta^1, \Delta^2, \ldots$ as in § 7 (vii). The corresponding rectangle Δ_n^j is defined in A_n for $n \geqslant n_j$, where we may assume $n_1 \leqslant n_2 \leqslant \ldots$. On Δ_n^j, $x_n(w)$ represents a parametric surface L_n^j, which we may assume tends to a limit L^j for every $j = 1, 2, \ldots$.

Let us show that:

$$L_0 = L^1 + L^2 + \ldots . \tag{11.3}$$

(9) For $q(\tau) = 2$, $\mathfrak{L}(\tau)$ is the set of generalized spheres; every element of $\mathfrak{L}(\tau)$ is proper in this case. Thus, a generalized sphere L is connected if L is not expressable as a non-trivial sum of generalized spheres.

By the same calculation used in § 7, $L_0 \geqslant L^1 + L^2 + \dots$. For every N, $L_0 - (L^1 + \dots + L^N)$ is the limit of the parametric surface represented by $x_n(w)$ on the set $E_n^N = A_n - (\Delta_n^1 \cup \dots \cup \Delta_n^N)$.

Therefore,

$$(11.4) \qquad a[L_0 - (L^1 + \dots + L^N)] = \lim_n a(x_n, E_n^N) \qquad (N = 1, 2, \dots).$$

To obtain (11.3) it suffices to show that the right side of (11.4) tends to 0 with N. For this we appeal to the following facts, contained in slightly different form in the proof of [**4** a, Lemma 6], which are basically an application of $\varepsilon - \delta$ gratings and the notion of convergence of slit domains. Let $r - 1$ be the number of vertices of the limit domain A. Then given $\delta > 0$ there exist $\sigma(\delta) > 0$, an integer $n(\delta)$, and for every $n \geqslant n(\delta)$ disjoint 2-cells $Z_{1n}, \dots, Z_{rn}$ with the following properties: (1) $Z_{in}^* \in \Theta(x_n, A_n)$ and $l(Z_{in}^*) < \delta$ $(i = 1, \dots, r)$; (2) every vertex of A_n is interior to some Z_{in}, $1 \leqslant i \leqslant r - 1$, and distant from Z_{in}^* by at least $\sigma(\delta)$; and (3) $\infty \in Z_{rn}$ and no point of Z_{rn}^* lies outside the circle with center the origin and radius $[\sigma(\delta)]^{-1}$. In view of (2) and (3) there exists $N = N(\delta)$ such that $E_n^N \subset Z_{1n} \cup \dots \cup Z_{rn}$ for all large n. For $\varepsilon = \max_i a(Z_{in})$, $\chi(x_n, A_n; \varepsilon) < \delta$. Then, by (11.1),

$$S = \lim_n \inf \sum_{i=1}^{r} a(Z_{in})$$

tends to 0 with δ. However,

$$\lim_n a(x_n, E_n^N) \leqslant S, \qquad \text{for } N = N(\delta),$$

from which (11.3) follows.

By the reasoning of § 7 (v), $L^j = L^{j'} + L^{j''}$, where $L^{j'}$ has a micro-representation on Δ^j and $L^{j''}$ is a generalized sphere (the case where Δ^j has a side on A^* doesn't exist here!). Then $L_0' = \sum L^{j'}$ has a micro-representation on A and $L_0'' = \sum L^{j''}$ is a generalized sphere by [**10** c, § 6]. By (4.3), $L_0' \in L(\tau)$. If $L_0' = 0$, then $L_0 = L_0''$ is a generalized sphere, contrary to hypothesis. Since L_0 is connected we must have $L_0'' = 0$ and $L_0 = L_0'$, which completes the proof.

Proof for type of the 2-sphere. All A_n and A are the compactified plane with no slits. By a magnification in A_n we may assume that $a(x_n, Q) \geqslant (1/2) \cdot a(x_n, A_n)$, where Q is the unit circle. The proof then proceeds as before.

Remark. In [**10** e, (6.1)] for 2-spheres the carrier is defined on a square Q and is constant on Q^*. To obtain such a representation from (11.2), we first arrange by (8.1) that the carrier is constant in a neighborhood of some finite point w_0 of A and then invert the plane about w_0.

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