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Stresses Due to a Certain Type of Nucleus of Strain in an Infinite Slab of Transversely Isotropic Material.

Introduction.

Some special problems of semi-infinite solid with transverse isotropy have been discussed by ELLIOT (1948) who has shown that the results can, in general, be expressed in terms of two functions and that in the case of axially symmetric stress distribution in terms of a single stress function. SEN (1954) obtained the result by a simpler method in case the elastic solid has only axisymmetric distribution of shearing stresses. The object of this paper is to find the distribution of stresses in an infinite slab of transversely isotropic material when a nucleus of strain in the form of a centre of rotation is situated inside it while one of its plane faces is free, the other being rigidly fixed.

1. - Method of solution.

We take the origin on the free boundary $z = 0$ of the infinite slab and the axis of z is drawn into the body at right angles to this plane. Assuming the axis of z to be the axis of elastic symmetry in a transversely isotropic material, we obtain the following stress-strain relations:

$$(1.1)\quad \begin{cases} \widehat{xx} = C_{11}\, e_{xx} + C_{12}\, e_{yy} + C_{13}\, e_{zz} \\ \widehat{yy} = C_{12}\, e_{xx} + C_{11}\, e_{yy} + C_{13}\, e_{zz} \\ \widehat{zz} = C_{13}\,(e_{xx} + e_{yy}) + C_{33}\, e_{zz} \\ \widehat{yz} = C_{44}\, e_{yz}, \qquad \widehat{zx} = C_{44}\, e_{zx} \\ \widehat{xy} = \{\, (C_{11} - C_{12})/2 \,\}\, e_{xy}, \end{cases}$$

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where C_{11}, C_{12}, ... are elastic constants. The equations of equilibrium in the absence of body forces are

$$(1.2)\qquad \begin{cases} \dfrac{\partial\widehat{xx}}{\partial x} + \dfrac{\partial\widehat{xy}}{\partial y} + \dfrac{\partial\widehat{xz}}{\partial z} = 0 \\[2ex] \dfrac{\partial\widehat{xy}}{\partial x} + \dfrac{\partial\widehat{yy}}{\partial y} + \dfrac{\partial\widehat{yz}}{\partial z} = 0 \\[2ex] \dfrac{\partial\widehat{xz}}{\partial x} + \dfrac{\partial\widehat{yz}}{\partial y} + \dfrac{\partial\widehat{zz}}{\partial z} = 0 \,. \end{cases}$$

Also, in the usual notation, if u, v, w are the components of displacement, then

$$(1.3)\qquad \begin{cases} e_{xx} = \dfrac{\partial u}{\partial x}\,, \qquad e_{yy} = \dfrac{\partial v}{\partial y}\,, \qquad e_{zz} = \dfrac{\partial w}{\partial z} \\[2ex] e_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\,, \qquad e_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\,, \qquad e_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\,. \end{cases}$$

For solving problems under consideration, let us assume

$$(1.4)\qquad u = -\frac{\partial\Phi}{\partial y}\,, \qquad v = \frac{\partial\Phi}{\partial x}\,, \qquad w = 0\,,$$

where Φ is a function of coordinates.

The stress components are now abtained as

$$(1.5)\qquad \begin{cases} \widehat{xx} = -A\,\dfrac{\partial^2\Phi}{\partial x\partial y}\,, \qquad \widehat{yy} = A\,\dfrac{\partial^2\Phi}{\partial x\partial y}\,, \qquad \widehat{zz} = 0 \\[2ex] \widehat{yz} = G\,\dfrac{\partial^2\Phi}{\partial x^2}\,, \qquad \widehat{zx} = -G\,\dfrac{\partial^2\Phi}{\partial y\partial z}\,, \qquad \widehat{xy} = (A/2)\left(\dfrac{\partial^2\Phi}{\partial x^2} - \dfrac{\partial^2\Phi}{\partial y^2}\right), \end{cases}$$

where

$$A = C_{11} - C_{12}\,, \qquad G = C_{44}\,.$$

By substituting the values of the stress components as obtained in relation (1.5) into the relations (1.2) we find that the third equation of (1.2) vanishes identically while the other two are satisfied if

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + k^2 \frac{\partial^2 \Phi}{\partial z^2} = 0, \tag{1.6}$$

where

$$k^2 = 2G/A .$$

2. – We consider a nucleus of strain at a point $(0, 0, c)$ defined by the relation

$$\Phi = P/R_1, \tag{2.1}$$

where P is a constant and

$$R_1^2 = x^2 + y^2 + (z - c)^2/k^2. \tag{2.2}$$

Corresponding stresses as obtained from relations (1.5) are

$$\widehat{yz}_1 = 3APx\,(z - c)/(2R_1^5), \qquad \widehat{zx}_1 = -3APy\,(z - c)/(2R_1^5), \qquad \widehat{zz}_1 = 0 . \tag{2.3}$$

So the nucleus of strain gives rise to the following stresses on the plane $z = 0$:

$$(\widehat{yz}_1)_{z=0} = -3APxc/(2R^5), \qquad (\widehat{zx}_1)_{z=0} = 3APyc/(2R^5), \qquad (\widehat{zz}_1)_{z=0} = 0, \tag{2.4}$$

where

$$R^2 = x^2 + y^2 + c^2/k^2. \tag{2.5}$$

Considering axially symmetrical coordinates we superimpose a stress system to nullify the stresses on the boundary $z=0$ as given in relation (2.4) such that

$$(\widehat{zz})_{z=0} = 0, \qquad (\widehat{rz})_{z=0} = 0, \qquad (\widehat{\theta z})_{z=0} = 3APcr/\{2(r^2 + c^2/k^2)^{5/2}\}. \tag{2.6}$$

The nucleus of strain gives rise to the following displacement components on the plane face $z = b + c$:

$$(2.7)\qquad \begin{cases} (u_\theta)_{z=b+c} = \left(\dfrac{\mathrm{d}\Phi}{\mathrm{d}r}\right)_{z=b+c} = -\dfrac{Pr}{(r^2 + b^2/k^2)^{3/2}} \\ (u_z)_{z=b+c} = 0, \qquad (u_r)_{z=b+c} = 0. \end{cases}$$

As the other plane face $z = b + c$ is rigidly fixed the displacement components as given in relations (2.7) must also vanish on the plane face $z = b + c$ by the superposition of a stress system.

3. – The equation (1.6) reduces to

$$(3.1)\qquad \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z_1^2} = 0$$

if $z = kz_1$.

As a solution of the equation (3.1) let us assume

$$(3.2)\qquad \Phi = \int_0^\infty \alpha\,[A_1\,e^{\alpha z/k} + A_2\,e^{-\alpha z/k}]\,J_0(\alpha r)\,\mathrm{d}\alpha\,.$$

So from relations (1.4) and (3.2) the displacement components are obtained as

$$(3.3)\qquad \begin{cases} u_\theta = \dfrac{\partial\Phi}{\partial r} = -\displaystyle\int_0^\infty \alpha^2\,[A_1\,e^{\alpha z/k} + A_2\,e^{-\alpha z/k}]\,J_1\,\alpha(r)\,\mathrm{d}\alpha \\ u_z = 0, \qquad u_r = 0\,. \end{cases}$$

Also we obtain the stresses from the relations (1.5) and (3.2)

$$(3.4)\qquad \begin{cases} \widehat{\theta z} = G\cdot\dfrac{\partial^2\Phi}{\partial r\partial z} = -G\cdot\displaystyle\int_0^\infty (\alpha^3/k)(A_1\,e^{\alpha z/k} - A_2\,e^{-\alpha z/k})\,J_1(\alpha r)\,\mathrm{d}\alpha \\ \widehat{zz} = 0, \qquad \widehat{rz} = 0\,. \end{cases}$$

Hence fron the conditions (2.6) and the relations (3.4) we get

$$(3.5) \qquad -\frac{G}{k}\int_0^\infty \alpha^3 (A_1 - A_2)\, J_1(\alpha r)\, d\alpha = \frac{3APcr}{2(r^2 + c^2/k^2)^{5/2}}.$$

By using HANKEL inversion theorem, the equation (3.5) becomes (SNEDDON, 1951)

$$\alpha^2 G\,(A_2 - A_1)/k = (3/2)APc \int_0^\infty r^2 J_1(\alpha r)/(r^2 + c^2/k^2)^{5/2}\, dr = (1/2)APk\alpha\, e^{-c\alpha/k}.$$

So

$$(3.6) \qquad A_2 - A_1 = \{ APk^2/(2G\alpha) \} e^{-c\alpha/k} = (P/\alpha)e^{-c\alpha/k}.$$

Also, from the conditions (2.7) and the relations (3.3), we get

$$(3.7) \qquad -\int_0^\infty \alpha^2 \left[A_1\, e^{\alpha (b+c)/k} + A_2\, e^{-\alpha (b+c)/k}\right] J_1(\alpha r)\, d\alpha = Pr/(r^2 + b^2/k^2)^{3/2}.$$

By using HANKEL inversion theorem the relation (3.7) transforms into (SNEDDON, 1951)

$$(3.8) \qquad \alpha[A_1\, e^{\alpha (b+c)/k} + A_2\, e^{-\alpha (b+c)/k}] = -P \int_0^\infty r^2 J_1(\alpha r)/(r^2 + b^2/k^2)^{3/2}\, dr = -P\, e^{-b\alpha/k}.$$

The relations (3.6) and (3.8) are satisfied of we write

$$(3.9) \qquad \begin{cases} A_1 = -(P/\alpha)\, e^{-\alpha (b+c)/k} \cosh(c\alpha/k)/\cosh\{\alpha(b+c)/k\} \\ A_2 = \quad (P/\alpha) \sinh(b\alpha/k)/\cosh\{\alpha(b+c)/k\}. \end{cases}$$

Thus the stress distribution is given by the sum total of the stresses due to the nucleus of strain and that due to the superimposed system with the values of the constants as in (3.9) substituted .

The values of u_θ on $z = 0$ is given by

$$(3.10)\quad \begin{cases} (u_\theta)_{z=0} = -Pr/(r^2 + c^2/k^2)^{3/2} - \\ -\int_0^\infty [P\alpha/\cosh\{\alpha(b+c)/k\}][\sinh(b\alpha/k) - e^{-\alpha(b+c)/k}\cosh(c\alpha/k)]\, J_1(\alpha r)d\alpha. \end{cases}$$

To find the value of this expression (3.10) for different values of r numerically, we suppose

$$b = c = 1,$$

$$C_{11} = 2746, \quad C_{33} = 2409, \quad C_{12} = 980, \quad C_{13} = 674, \quad C_{44} = 666,$$

these being the values of the elastic constants in case of hexagonal crystals like beryl (cf. LOVE).

Here the constants are expressed in terms of an unit stress of 10^6 grammes weight per square centimetre. So

$$k = \sqrt{2G/A} = \sqrt{2C_{44}/(C_{11} - C_{12})} = 0{,}866 .$$

The results of calculation for $(u_\theta)_{z=0}$ are given in Table I:

Table I

$r =$	0	0,25	0,50	0,75	1,00	1,50	2,00
$-(u_\theta/P)_{z=0} =$	0	0,289	0,488	0,566	0,543	0,424	0,326

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References.

1. H. A. Elliot, *Three-dimensional stress distributions in hexagonal aeolotropic crystals*, Proc. Cambridge Philos. Soc. **44** (1948), 522-533.

2. A. E. H. Love, **A treatise on the mathematical Theory of Elasticity.** University Press, Cambridge 1927.

3. J. H. Michell, *The stress in an aelotropic elastic solid with an infinite plane*, Proc. Roy. Soc. London, Ser. A **32** (1900), 247-258.

4. B. B. Sen, Indian J. Theoret. Phys. **1** (1954), p. 3.

5. Ivan N. Sneddon, **Fourier Transforms.** McGraw-Hill, New York 1951.

Summary: **In this paper the distribution of stresses in an infinite slab of transversely isotropic materials has been obtained when a nucleus of strain in the form of a centre of rotation is situated inside it, one of its plane face being free and the other rigidly fixed. Numerical results have also been found.**