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On the Mean Value of Integral Functions and Their Derivatives.

1. – If $f(z)$ is an integral function of order ϱ and lower order λ, the mean value of $f(z)$ defined for $|z| = r$ is

$$(1.1) \qquad I(r) = \frac{1}{2\pi} \int_0^{2\pi} | f(re^{i\theta}) | \, d\theta .$$

It is known ([3], p. 174) that it is an increasing function of r and $\log I(r)$ is a convex function of $\log r$. The object of the present paper is to study some of the properties of $I(r)$. In section two certain results obtained for logarithmic mean value have also been given.

Section I.

2. – Theorem 1.

Let

$$\max_{|z|=r} | f(z) | = M(r),$$

then, for $0 < r < R$,

$$(2.1) \qquad I(r) \leqslant M(r) \leqslant \frac{R+r}{R-r} I(R) .$$

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Proof: First part of the inequality follows from (1.1). From CAUCHY's formula, we have

$$2f(z) - f(0) = \frac{1}{2\pi i} \int\limits_{|\omega|=R} f(\omega) \frac{\omega + z}{\omega - z} \frac{d\omega}{\omega}$$

and

$$f(0) = \frac{1}{2\pi i} \int\limits_{|\omega|=R} \frac{f(\omega)}{\omega} \frac{R^2 + \omega\bar{z}}{R^2 - \omega\bar{z}} \, d\omega.$$

Consequently

$$2f(z) = \frac{1}{2\pi i} \int\limits_{|\omega|=R} \frac{f(\omega)}{\omega} \left[\frac{\omega + z}{\omega - z} + \frac{\bar{\omega} + \bar{z}}{\bar{\omega} - \bar{z}}\right] d\omega.$$

Let z be the value for which $|f(z)| = M(r)$, then

$$M(r) \leqslant \frac{R+r}{R-r} \cdot \frac{1}{2\pi} \int\limits_0^{2\pi} |f(Re^{i\varphi})| \, d\varphi$$

and hence the result.

From (2.1) we obtain the following corollaries:

Cor. 1.

$$\overline{\lim_{r\to\infty}} \frac{\log\log I(r)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = \begin{cases} \varrho \\ \lambda \end{cases}.$$

Cor. 2.

$$\overline{\lim_{r\to\infty}} \frac{\log I(r)}{N(r) \cdot \log r} \leqslant 1,$$

where $N(r)$ is the rank of the maximum term in the TAYLOR expansion of $f(z)$ about $z = 0$.

Cor. 3. For $0 \leqslant \varrho < \infty$, $r > r_0$,

$$I(r) \leqslant M(r) < I(r) \cdot r^{\varrho + \varepsilon}.$$

The result follows by putting $R = r + \{ r/\log I(r) \}$.

3. – Theorem 2.

If $I^{(1)}(r)$ be the mean value of $f'(z)$, the derivative of $f(z)$, then, for $r > r_0$,

$$I^{(1)}(r) \geqslant \frac{I(r)\cdot \log I(r)}{r\cdot \log r}.$$

Proof: We have

$$I^{(1)}(r) = \frac{1}{2\pi}\int_0^{2\pi} |\, f'(re^{i\theta})\,|\, \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \left| \lim_{\varepsilon\to 0+} \frac{f(re^{i\theta}) - f(re^{i\theta} - \varepsilon re^{i\theta})}{\varepsilon re^{i\theta}} \right| \mathrm{d}\theta$$

or

$$I^{(1)}(r) \geqslant \frac{1}{2\pi}\int_0^{2\pi} \lim_{\varepsilon\to 0+} \frac{|\, f(re^{i\theta})\,| - |\, f(re^{i\theta} - \varepsilon re^{i\theta})\,|}{\varepsilon r}\, \mathrm{d}\theta \geqslant \lim_{\varepsilon\to 0+} \frac{I(r) - I(r - r\varepsilon)}{\varepsilon r}.$$

Let $g(r) = \{\log I(r)/(\log r)\}$, then $g(r)$ is an increasing function of r, therefore,

$$I^{(1)}(r) \geqslant \lim_{\varepsilon\to 0+} \frac{r^{g(r)} - (r - r\varepsilon)^{g(r)}}{\varepsilon r} = g(r)\cdot r^{g(r)-1} = \frac{I(r)\cdot \log I(r)}{r\cdot \log r}.$$

Hence the result.

4. – In order to prove the next theorem, we need the following

Lemma. *Let $f(z)$ be an integral function of order ϱ $(< \infty)$, with zeros at $z_1, z_2, \ldots, z_n, \ldots$. Then, for large r,*

$$\left| \frac{f'(z)}{f(z)} \right| \leqslant O(r^{\varrho+\varepsilon-1}).$$

Proof.

Set

$$f(z) = z^k\, e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n + \ldots + (z/z_n)^p/p},$$

where p is the genus of the canonical product, $Q(z)$ a polynomial of degree $\leqslant \varrho$ and k is a positive integer. Logarithmic derivation leads to

$$\frac{f'(z)}{f(z)} = \frac{k}{z} + Q'(z) + \sum_{n=1}^{\infty} \left\{ \frac{1}{z - z_n} + \frac{1}{z_n} + \dots + \frac{z^{p-1}}{z_n^p} \right\} =$$

$$= \frac{k}{z} + Q'(z) + \sum_{n=1}^{\infty} \frac{1}{z - z_n} \left(\frac{z}{z_n}\right)^p,$$

where $Q'(z)$ is a polynomial of degree $\leqslant \varrho - 1$. We put

$$\Phi(z) = \sum_{n=1}^{\infty} \frac{1}{z - z_n} \left(\frac{z}{z_n}\right)^p = \sum\nolimits_1 + \sum\nolimits_2,$$

where

$$\sum\nolimits_1 = \sum_{|z_n| < (1-\delta)r} \frac{1}{z - z_n} \left(\frac{z}{z_n}\right)^p, \qquad \sum\nolimits_2 = \sum_{(1+\delta)r < |z_n|} \frac{1}{z - z_n} \left(\frac{z}{z_n}\right)^p,$$

$0 < \delta < 1$, such that no zero of $f(z)$ lies in the annulus $(1 - \delta)r \leqslant |z_n| \leqslant (1 + \delta)r$, which is always possible by suitably choosing δ and r. Now

$$\sum\nolimits_1 \leqslant \sum_{r_n < (1-\delta)r} \left(1 - \frac{r_n}{r}\right)^{-1} \frac{r^{p-1}}{r_n^p}.$$

Since $r_n/r < 1 - \delta$, therefore, $1 - (r_n/r) > \delta$. Hence

$$\sum\nolimits_1 \leqslant O\left(r^{p-1} \sum_{r_n < (1-\delta)r} r_n^{-p}\right) \leqslant O\left(r^{\varrho+\varepsilon-1} \sum_{r_n < (1-\delta)r} r_n^{-\varrho-\varepsilon}\right) = O(r^{\varrho+\varepsilon-1}).$$

Further,

$$\sum\nolimits_2 \leqslant \sum_{(1+\delta)r < r_n} \left(1 - \frac{r}{r_n}\right)^{-1} \frac{r^p}{r_n^{p+1}}.$$

Since $r/r_n < 1/(\delta + 1)$, therefore, $1 - (r/r_n) > \delta/(\delta + 1)$ and hence

$$\sum\nolimits_2 \leqslant O\left(r^p \sum_{(1+\delta)r < r_n} \frac{1}{r_n^{p+1}}\right).$$

If $p = \varrho - 1$, then $\sum_2 \leqslant O(r^{\varrho-1})$. If $p > \varrho - 1$, then for arbitrary small positive ε, $\varrho + \varepsilon < p + 1$. Hence

$$\sum_2 \leqslant O\Big(r^p \sum_{(1+\delta)r < r_n} r_n^{-p-1+\varrho+\varepsilon}\, r_n^{-\varrho-\varepsilon}\Big) \leqslant O\Big(r^{\varrho+\varepsilon-1} \sum_{(1+\delta)r<r_n} r_n^{-\varrho-\varepsilon}\Big) = O(r^{\varrho+\varepsilon-1}).$$

Since

$$|\, f'(z)/f(z) \,| \leqslant (k/r) + |\, Q'(z) \,| + |\, \textstyle\sum_1 | + |\, \textstyle\sum_2 |,$$

the result follows:

Theorem 3.

$$\overline{\lim_{r\to\infty}} \frac{\log\{\, r\cdot[I^{(s)}(r)/I(r)]^{1/s} \,\}}{\log r} = \varrho,$$

where $I^{(s)}(r)$ is the mean value of $f^{(s)}(z)$, the s-th derivative of $f(z)$, for $|\, z \,| = r$.

Proof.

We have

$$I^{(1)}(r) = \frac{1}{2\pi}\int_0^{2\pi} |\, f'(re^{i\theta}) \,|\, \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |\, f(re^{i\theta}) \,|\, |\, f'(re^{i\theta})/f(re^{i\theta}) \,|\, \mathrm{d}\theta \leqslant I(r)\cdot r^{\varrho+\varepsilon-1},$$

for $r > r_0$. Writing this inequality for s-th derivative, we get

$$I^{(s)}(r) \leqslant I^{(s-1)}(r)\cdot r^{\varrho+\varepsilon-1}.$$

Giving s the values 1, 2, 3, etc. and multiplying together, we get

$$I^{(s)}(r) < I(r)\cdot r^{(\varrho+\varepsilon-1)s},$$

and hence we have

$$\overline{\lim_{r\to\infty}} \frac{\log\{\, r\cdot[I^{(s)}(r)/I(r)]^{1/s} \,\}}{\log r} \leqslant \varrho. \tag{4.1}$$

Further, we have

$$I^{(1)}(r) \geqslant \{ I(r) \cdot \log I(r) \}/(r \cdot \log r) .$$

If we proceed with $I^{(s)}(r)$ and $I^{(s-1)}(r)$ instead of $I^{(1)}(r)$ and $I(r)$ as in Theorem 2, we obtain

$$I^{(s)}(r) \geqslant \{ I^{(s-1)}(r) \cdot \log I^{(s-1)}(r) \}/(r \cdot \log r) .$$

From the above inequality we easily obtain

$$\underline{\lim}_{r \to \infty} \frac{\log\{r \cdot [I^{(s)}(r)/I(r)]^{1/s}\}}{\log r} \geqslant \varrho . \tag{4.2}$$

The result follows on combining (4.1) and (4.2).

Cor. For $r > r_0$,

$$I(r) \cdot r^{(\lambda-\varepsilon-1)s} < I^{(s)}(r) < I(r) \cdot r^{(\varrho+\varepsilon-1)s} . \tag{4.3}$$

The following are direct consequence of (4.3) and the inequality

$$I^{(s-1)}(r) \cdot r^{\lambda-\varepsilon-1} < I^{(s)}(r) < I^{(s-1)}(r) \cdot r^{\varrho+\varepsilon-1},$$

for $r > r_0$.

(i) If $0 \leqslant \varrho < \infty$,

$$\overline{\lim}_{r \to \infty} \frac{\log I^{(s)}(r)}{N(r) \cdot \log r} \leqslant 1 .$$

(ii) If $\lambda > 1$, $I(r)$, $I^{(1)}(r)$, ... form an increasing sequence for $r > r_0$.

(iii) If $\varrho < 1$, $I(r)$, $I^{(1)}(r)$, ... form a decreasing sequence for $r > r_0$.

(iv) If $\lambda > 1$,

$$\lim_{r \to \infty} \frac{I^{(s)}(r)}{r^p \cdot I(r)} = \infty,$$

provided $p < s \cdot (\lambda - 1)$.

(v) If $\varrho < 1$,

$$\lim_{r\to\infty} \left\{ r^p \frac{I^{(s)}(r)}{I(r)} \right\} = 0,$$

provided $p < s\cdot(1-\varrho)$.

Results (ii) and (iii) can be improved upon by a change of argument.

Theorem 4. *If $\lambda = 1$ and $\varrho > 1$, $I(r)$, $I^{(1)}(r)$, ... form an increasing sequence for all values of $r > r_0 > 1$.*

Proof: From (4.3) we can write

$$I^{(s)}(r) = I^{(s-1)}(r)\cdot r^{\lambda-1+\theta\cdot(\varrho-\lambda)+\varepsilon'} = I^{(s-1)}(r)\cdot r^{\theta\cdot(\varrho-1)+\varepsilon'} > I^{(s-1)}(r),$$

where $0 < \theta < 1$. Hence the result follows by giving s the values 1, 2, 3, ...

Theorem 5. *If $\varrho = 1$ and $\lambda < 1$, $I(r)$, $I^{(1)}(r)$, ... form a decreasing sequence for all values of $r > r_0 > 1$.*

Proof: If we proceed as in the proof of Theorem 4, we have

$$I^{(s)}(r) = I^{(s-1)}(r)\cdot r^{(\lambda-1)(1-\theta)+\varepsilon'}$$

and from this the result easily follows.

Theorem 6. *If $\lambda \geqslant 1$ and $\varrho \neq 1$, for $r > r_0$,*

$$I^{(s)}(r) > I(r)\cdot[\{\log I(r)\}/(r\cdot\log r)]^s.$$

Proof is similar to ([1], p. 79).

Section II.

We define *logarithmic mean value* of an integral function $f(z)$ as

$$I_l(r) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta\,.$$

Obviously $I_l(r)$ is a non-decreasing function of r.

Let

$$\overline{\underline{\lim}}_{r\to\infty} \frac{I_l(r)}{r^\varrho} = \begin{cases} T \\ t \end{cases}, \qquad \overline{\underline{\lim}}_{r\to\infty} \frac{n(r)}{r^\varrho} = \begin{cases} \gamma \\ \delta \end{cases},$$

where $n(r)$ is the number of zeros of $f(z)$ within and on the circle $|z| = r$.

We prove the following

(i) $$\varrho \leqslant \frac{\gamma}{e} e^{\delta/\gamma} \leqslant \varrho T \leqslant \gamma,$$

(ii) $$\delta \leqslant \varrho t \leqslant \delta \cdot \left\{ 1 + \log (\gamma/\delta) \right\} \leqslant \gamma .$$

Proof [1].

From JENSEN's formula, we can write,

$$I_l(r) = I_l(r_0) + \int_{r_0}^{r} \frac{n(x)}{x} \, \mathrm{d}x .$$

Hence, we have

$$I_l(rK^{1/\varrho}) = A_1 + \left(\int_{r_0}^{r} + \int_{r}^{rK^{1/\varrho}} \right) \frac{n(x)}{x} \, \mathrm{d}x > A_2 + \frac{(\delta - \varepsilon) r^\varrho}{\varrho} + \frac{n(r) \cdot \log K}{\varrho},$$

therefore,

$$\frac{K \cdot I_l(rK^{1/\varrho})}{K r^\varrho} > A_2 r^{-\varrho} + \frac{\delta - \varepsilon}{\varrho} + \frac{n(r) \cdot \log K}{\varrho \cdot r^\varrho}$$

and hence

$$Kt \geqslant \frac{\delta + \delta \cdot \log K}{\varrho}, \qquad KT \geqslant \frac{\delta + \gamma \cdot \log K}{\varrho} .$$

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[1] The method of proof is similar to that of SHAH ([**2**], p. 221).

Taking $K=1$ in the first inequality and $\log K = (\gamma - \delta)/\gamma$ in the second, we get

$$t \geqslant \delta/\varrho, \qquad \varrho T e \geqslant \gamma e^{\delta/\gamma} \geqslant e\delta .$$

Further,

$$I_t(rK^{1/\varrho}) \leqslant A_3 + \frac{\gamma+\varepsilon}{\varrho} r^\varrho + \frac{n(rK^{1/\varrho})}{\varrho} \log K .$$

Hence

$$KT \leqslant \frac{\gamma + \gamma K \cdot \log K}{\varrho}, \qquad Kt \leqslant \frac{\gamma + \delta K \cdot \log K}{\varrho} .$$

Putting $K=1$ in the first inequality and $K=\gamma/\delta$ in the second, we get

$$T \leqslant \gamma/\varrho, \qquad \varrho t \leqslant \delta \left\{ 1 + \log(\gamma/\delta) \right\} \leqslant \gamma .$$

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