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Disturbances Due to Localized Periodic Surface Traction in a Semi Infinite Elastic Medium with Transverse Isotropy.

Introduction.

The problem of waves in a semi infinite perfectly elastic medium produced by concentrated periodic forces on the surface have been investigated by LAMB [1]. The corresponding problem in the transversely isotropic case has been attempted in this paper and solution has been obtained in an integral form, when an isolated force has been applied at the origin which is taken on the free surface.

Using the values of the elastic constants for BERYL as given by VOIGT [2] an approximation for the surface displacement has been made.

Method of solution.

Let the axis of symmetry of the transversely isotropic medium be perpendicular to the surface taken as the plane $z = 0$. The axis of symmetry is taken as the z-axis.

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Then the stress-strain relations in transversely isotropic medium can be written as [**2**, Art. 110]

$$
(1) \quad \begin{cases}
\widehat{rr} = c_{11}\, e_{rr} + c_{12}\, e_{\theta\theta} + c_{13}\, e_{zz} \\
\widehat{\theta\theta} = c_{12}\, e_{rr} + c_{11}\, e_{\theta\theta} + c_{13}\, e_{zz} \\
\widehat{zz} = c_{13}\, e_{rr} + c_{13}\, e_{\theta\theta} + c_{33}\, e_{zz} \\
\widehat{rz} = c_{44}\, e_{rz} \\
\widehat{\theta z} = c_{44}\, e_{\theta z} \\
\widehat{r\theta} = \{ (c_{11} - c_{12})/2 \}\, e_{r\theta},
\end{cases}
$$

where r, θ, z are the cylindrical coordinates and c_{11}, c_{12}, c_{13}, c_{33}, c_{44} are the elastic constants of the medium.

Further, if u, v, w are the displacement components at any point of the medium in the directions of r, θ, z respectively, the equations of motion are, in the absence of body forces [**2**],

$$
(2) \quad \frac{\partial\, \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial\, \widehat{r\theta}}{\partial \theta} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = \varrho\, \ddot{u},
$$

$$
(3) \quad \frac{\partial\, \widehat{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\, \widehat{\theta\theta}}{\partial \theta} + \frac{\partial\, \widehat{\theta z}}{\partial z} + \frac{2\, \widehat{r\theta}}{r} = \varrho\, \ddot{v},
$$

$$
(4) \quad \frac{\partial\, \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial\, \widehat{\theta z}}{\partial \theta} + \frac{\partial\, \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = \varrho\, \ddot{w}.
$$

Let the resulting motion be independent of θ, and $v = 0$. For a periodic solution, let us take

$$
(5) \quad \begin{cases}
u = U(r)\cdot \exp(-\gamma z + ipt) \\
w = W(r)\cdot \exp(-\gamma z + ipt)\,.
\end{cases}
$$

The equations of motion (2) and (4) become

$$
(6) \quad c_{11}\cdot\left(\frac{\mathrm{d}^2 U}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}r} - \frac{U}{r}\right) + (\varrho p^2 + c_{44}\gamma^2)U - \gamma\cdot(c_{13} + c_{44})\,\frac{\mathrm{d}W}{\mathrm{d}r} = 0,
$$

(7) $$c_{44}\cdot\left(\frac{\mathrm{d}^2W}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right)+(\varrho p^2+c_{33}\gamma^2)W-\gamma\cdot(c_{13}+c_{44})\left(\frac{\mathrm{d}U}{\mathrm{d}r}+\frac{U}{r}\right)=0,$$

while (3) is identically satisfied.

If we assume U, W in the forms

(8) $$U=A\cdot J_1(kr),\qquad W=B\cdot J_0(kr),$$

they satisfy the two equations (6) and (7) provided γ satisfies the equation

(9) $$[c_{44}\,\gamma^2+(\varrho p^2-c_{11}\,k^2)][c_{33}\,\gamma^2+(\varrho p^2-c_{44}\,k^2)]+\gamma^2\,k^2(c_{13}+c_{44})^2=0\,.$$

Let us define $q=\gamma/k$ and $c^2=p^2/k^2$, then equation (9) takes up the form

(10) $$q^4-q^2\left(\frac{c_{11}}{c_{44}}-\frac{2c_{44}+c_{13}}{c_{44}}\frac{c_{13}}{c_{33}}-\frac{\varrho c^2}{c_{44}}-\frac{\varrho c^2}{c_{33}}\right)+\frac{c_{11}}{c_{33}}\left(\frac{\varrho c^2}{c_{33}}-1\right)\left(\frac{\varrho c^2}{c_{11}}-1\right)=0\,.$$

Let q_1^2, q_2^2 be two roots (taken positive) of this equation. Hence the complete solution can be written in the form

(11) $$\begin{cases} u\,e^{-ipt}=(A_1\,e^{-kq_1z}+A_2\,e^{-kq_2z})\cdot J_1(kr)\\ w\,e^{-ipt}=(B_1\,e^{--kq_1z}+B_2\,e^{-kq_2z})\cdot J_0(kr),\end{cases}$$

where $A_1=m_1\,q_1\,B_1$, $A_2=m_2\,q_2\,B_2$, m_1, m_2 being given by

(12) $$m_1=\frac{c_{13}+c_{44}}{c_{11}-(\varrho c^2+c_{44}\,q_1^2)}\,,\qquad m_2=\frac{c_{13}+c_{44}}{c_{11}-(\varrho c^2+c_{44}\,q_2^2)}\,.$$

Let a force act normal to the surface. The force is supposed to be given by $Pe^{ipt}\delta(r)/(2\pi r)$, where $\delta(r)$ is DIRAC delta function [3, p. 32] defined by

$$\delta(x)\neq 0\quad\text{for}\quad x\neq 0,\qquad \int_{-\infty}^{+\infty}\delta(x)\,\mathrm{d}x=1\,.$$

Also

(13) $$\delta(r)/(2\pi r)=\int_0^{\infty}k\cdot J_0(kr)\,\mathrm{d}k.$$

We can therefore write down the boundary conditions as

$$(\widehat{rz})_{z=0} = 0, \tag{14}$$

$$(\widehat{zz})_{z=0} = -P\int_0^\infty k\cdot J_0(kr)\, e^{ipt}\,\mathrm{d}k\,. \tag{15}$$

A suitable solution for the case therefore is taken in the form

$$u\, e^{-ipt} = \int_0^\infty (A_1\, e^{-kq_1 z} + A_2\, e^{-kq_2 z})\, J_1(kr)\,\mathrm{d}k, \tag{16}$$

$$w\, e^{-ipt} = \int_0^\infty (B_1\, e^{-kq_1 z} + B_2\, e^{-kq_2 z})\, J_0(kr)\,\mathrm{d}k\,. \tag{17}$$

The condition (14) gives

$$B_1 = -\frac{1 + q_2^2\, m_2}{1 + q_1^2\, m_1}\, B_2, \tag{18}$$

while the condition (15) gives

$$B_2 = -\frac{P\cdot(1 + q_1^2\, m_1)}{(1 + q_2^2\, m_2)(c_{13}\, m_1 - c_{33})q_1 - q_2(c_{13}\, m_2 - c_{33})(1 + q_1^2\, m_1)}\,. \tag{19}$$

Hence the solution is fully obtained as given by equations (16), (17) together with (18) and (19).

The evaluation of the integrals in (16) and (17) become untractable owing to the complicated equation (10) to determine q_1, q_2.

An approximate calculation of the surface displacement can be made using the value of c_{11}, c_{12}, c_{13}, c_{33} and c_{44} for BERYL as given by VOIGT [2, Art. 113]. Thus, for BERYL,

$$\begin{aligned}
c_{11} &= 2746 \times 10^6 \text{ gr wt. per cm}^2,\\
c_{12} &= 980 \times 10^6 \quad » \quad » \quad » \quad » \;,\\
c_{13} &= 674 \times 10^6 \quad » \quad » \quad » \quad » \;,\\
c_{33} &= 2409 \times 10^6 \quad » \quad » \quad » \quad » \;,\\
c_{44} &= 666 \times 10^6 \quad » \quad » \quad » \quad » \;,
\end{aligned}$$

q_1, q_2 are determined from the equation (10) on the assumption that frequency of the applied force is small compared to the elastic constants so that in the equation the terms $\varrho c^2/c_{11}$, $\varrho c^2/c_{33}$ can be neglected. Thus

$$q_1 = 0{,}28 \qquad q_2 = -1{,}7 \qquad B_1 \cong -23{,}8 \times 10^{-8} P \qquad B_2 \cong 3{,}4 \times 10^{-8} P.$$

Hence

$$(20)\quad \begin{cases} u\, e^{-ipt} \cong P \times 10^{-8} \int_0^\infty (-3{,}3 \cdot e^{-0{,}28 \cdot kz} + 11{,}5 \cdot e^{-1{,}7 \cdot kz}) J_1(kz)\, dk \\ w\, e^{-ipt} \cong P \times 10^{-8} \int_0^\infty (23{,}8 \cdot e^{-0{,}28 \cdot kz} + 3{,}4 \cdot e^{-1{,}7 \cdot kz}) J_0(kr)\, dk = \\ \qquad = \dfrac{P \times 10^{-8}(-23{,}8)}{\sqrt{r^2 + (0{,}28^2 z^2}} + \dfrac{3{,}4 \times 10^{-8} P}{\sqrt{r^2 + (1{,}7)^2 z^2}}. \end{cases}$$

Then, on the surface $z = 0$,

$$(21)\qquad u\, e^{-ipt} \cong \frac{8{,}2 \times 10^{-8} P}{r}, \qquad w\, e^{-ipt} \cong -\frac{20{,}4 \times 10^{-8} P}{r}.$$

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References.

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2. Love A. E. H.: **A Treatise on the Mathematical Theory of Elastic.** 4. ed., Cambridge University Press, Cambridge 1927.

3. Sneddon, Ivan N.: **Fourier Transforms.** McGraw-Hill Book Co., New York 1951 (cf. p. 32).

Summary.

Propagation of disturbances in a semi infinite elastic medium possessing transverse isotropy about a line perpendicular to the boundary plane surface has been investigated when the source of disturbance is a concentrated periodic force on the plane surface.