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A Generalization of Some Polynomials Related to the Theta Functions. (**)

1. - Introduction.

The polynomials

$$(1.1) \qquad \begin{cases} H_{n+1}(x) = (1+x)\,H_n(x) - [n]\,x\,H_{n-1}(x), \\ H_0(x) = 1, \qquad H_1(x) = 1 + x, \end{cases}$$

where $[n] = 1 - q^n$, have been studied by SZEGÖ [6] and recently by CARLITZ [3].

We recall that the ordinary HERMITE polynomials satisfy

$$(1.2) \qquad \begin{cases} He_{n+1}(x) = x\,He_n(x) - n\,He_{n-1}(x), \\ He_0(x) = 1, \qquad He_1(x) = x. \end{cases}$$

PALAMÀ [5] and TOSCANO [7] generalized (1.2) by studying the polynomials

$$(1.3) \qquad \begin{cases} G_{n,\nu}(x) = x\,G_{n-1,\nu}(x) - (n+\nu)\,G_{n-2,\nu}(x) \qquad (n > 1), \\ G_{0,\nu}(x) = 1, \qquad G_{1,\nu}(x) = x. \end{cases}$$

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The purpose of this paper is to generalize the polynomial (1.1) in a similar manner. In particular we shall obtain several q-analogs of some relations involving the polynomial $G_{n,\nu}(x)$ of TOSCANO.

Let us define $H_n^{(\nu)}(x)$ and $G_n^{(\nu)}(x)$ by means of

$$(1.4) \qquad \begin{cases} H_{n+1}^{(\nu)}(x) = (1+x)\,H_n^{(\nu)}(x) - [n+\nu]\,x\,H_{n-1}^{(\nu)}(x) & (n \geqslant 1), \\ H_0^{(\nu)}(x) = 1, \qquad H_1^{(\nu)}(x) = 1 + x, \end{cases}$$

and

$$(1.5) \qquad \begin{cases} G_{n+1}^{(\nu)}(x) = (1+x)\,G_n^{(\nu)}(x) + q^{-n-\nu}[n+\nu]\,x\,G_{n-1}^{(\nu)}(x) & (n \geqslant 1), \\ G_0^{(\nu)}(x) = 1, \qquad G_1^{(\nu)}(x) = 1 + x. \end{cases}$$

Clearly we have

$$H_n(x) = H_n^{(0)}(x).$$

WIGERT [**8**] and CARLITZ [**3**] studied the polynomial

$$G_n(x) = G_n^{(0)}(x).$$

It is obvious from (1.4) that $H_n^{(\nu)}(x)$ is actually a polynomial in the two variables x and $z = q^\nu$. For example we find below that

$$H_n^{(\nu)}(x) = \sum_s \sum_k q^{k^2} \begin{bmatrix} s \\ k \end{bmatrix} \begin{bmatrix} n-s \\ k \end{bmatrix} x^s z^k,$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1}) \dots (1-q^{m-r+1})}{(q)_r}, \qquad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1,$$

and

$$(a)_r = (1-a)(1-aq) \dots (1-aq^{r-1}), \qquad (a)_0 = 1.$$

We shall also have occasion to use the notation

$$[r]! = (q)_r.$$

Another formula which we shall prove below is, for $m \leqslant n$,

$$H_m(x)\, H_n^{(\nu)}(x) = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r\, H_{n+m-2r}^{(\nu)}(x)\,. \tag{1.6}$$

This formula is a q-analog of a generalization of NIELSEN's formula obtained by the present writer [**1**].

2. – The recurrence relation (1.4) together with the given values of $H_0^{(\nu)}(x)$ and $H_1^{(\nu)}(x)$ determine uniquely the value of $H_n^{(\nu)}(x)$. In fact from (1.4) we have

$$H_n^{(\nu)}(x) = \tag{2.1}$$

$$\begin{vmatrix} 1+x & [n+\nu-1]\,x & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1+x & [n+\nu-2]\,x & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1+x & [n+\nu-3]\,x & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1+x & [\nu+2]\,x & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1+x & [\nu+1]\,x \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1+x \end{vmatrix}$$

The corresponding expression for $G_n^{(\nu)}(x)$:

$$G_n^{(\nu)}(x) = \tag{2.2}$$

$$\begin{vmatrix} 1+x & -q^{-n-\nu+1}[n+\nu-1]\,x & 0 & \dots & 0 & 0 & 0 \\ 1 & 1+x & -q^{-n-\nu+2}[n+\nu-2]\,x & \dots & 0 & 0 & 0 \\ 0 & 1 & 1+x & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 1+x & -q^{-n-\nu}[\nu+1]\,x \\ 0 & 0 & 0 & \dots & 0 & 1 & 1+x \end{vmatrix}$$

is obtained from (1.5).

Also from (1.4) and (1.5) we see that

$$x^n H_n^{(\nu)}(1/x) = H_n^{(\nu)}(x) \qquad \text{and} \qquad x^n G_n^{(\nu)}(1/x) = G_n^{(\nu)}(x) .$$

It is also obvious from (1.4) that $G_n^{(\nu)}(x)$ can be obtained from $H_n^{(\nu)}(x)$ by replacing q by q^{-1}. This shows that for any finite formula for $H_n^{(\nu)}(x)$ one can obtain a corresponding one involving $G_n^{(\nu)}(x)$.

We note here that, when $\nu = 1$,

$$H_{n-1}^{(1)}(x) = u_n(x)$$

where $u_n(x)$ is a second solution of (1.1) for which $u_0 = 0$, $u_1(x) = 1$ (see [**3**]).

We remark further that

$$\eta_n^{(\nu)}(x) = H_{n-1}^{(\nu+1)}(x), \qquad \eta_0^{(\nu)}(x) = 0$$

is a second solution of (1.4).

3. – We prove here by induction the formula

$$H_m(x)\, H_n^{(\nu)}(x) = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r H_{n+m-2r}^{(\nu)}(x) \qquad (m \leqslant n). \tag{3.1}$$

For $m = 1$, (3.1) obviously holds. Assume it holds for $m = k$. Then

$$H_{k+1}(x)\, H_n^{(\nu)}(x) = (1+x)\, H_k(x)\, H_n^{(\nu)}(x) - [k]\, x\, H_{k-1}(x)\, H_n^{(\nu)}(x) =$$

$$= (1+x) \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r H_{k+n-2r}^{(\nu)}(x) -$$

$$- [k]\, x \sum_{r=0}^{k-1} \begin{bmatrix} k-1 \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r H_{n+k-1-2r}^{(\nu)}(x) =$$

$$= \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r \left\{ H_{k+n+1-2r}^{(\nu)}(x) + [k+n+\nu-2r]\, x\, H_{k+n-1-2r}^{(\nu)}(x) \right\} -$$

$$- [k]\, x \sum_{r=0}^{k-1} \begin{bmatrix} k-1 \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r H_{n+k-1-2r}^{(\nu)}(x) =$$

$$= \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r H^{(\nu)}_{k+n+1-2r}(x) +$$

$$+ \sum_{r=1}^{k+1} \begin{bmatrix} k \\ r-1 \end{bmatrix} \begin{bmatrix} n+\nu \\ r-1 \end{bmatrix} [r-1]!\,[n+k+\nu+2-2r]\, x^r H^{(\nu)}_{n+k+1-2r}(x) -$$

$$- \sum_{r=1}^{k} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r-1 \end{bmatrix} [r]!\; x^r\; H^{(\nu)}_{n+k+1-2r}(x)\,.$$

But $[k+n+\nu+2-2r] = [k-r+1] + q^{k-r+1}[n+\nu-r+1]$. Hence

$$H_{k+1}(x)\, H_n^{(\nu)}(x) = \sum_r \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]! \left\{ \begin{bmatrix} k \\ r \end{bmatrix} + q^{k-r+1} \begin{bmatrix} k \\ r-1 \end{bmatrix} \right\} x^r H^{(\nu)}_{k+n+1-2r}(x) =$$

$$= \sum_r \begin{bmatrix} k+1 \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r H^{(\nu)}_{k+n+1-2r}(x)\,.$$

Thus the proof is complete.

In a similar manner one can prove the inverse formula

$$\text{(3.2)} \qquad H^{(\nu)}_{n+m}(x) = \sum_{r=0}^{m} (-1)^r q^{r(r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r H_{m-r}(x)\, H^{(\nu)}_{n-r}(x)\,.$$

We also remark that in (3.1) and (3.2) we can replace $H_n^{(\nu)}(x)$ by $U_\mu^{(\nu)}(x)$, where $U_\mu^{(\nu)}(x)$ is any solution of

$$U^{(\nu)}_{\mu+1}(x) = (1+x)\, U^{(\nu)}_\mu(x) - [\mu+\nu]\, x\, U^{(\nu)}_{\mu-1}(x),$$

μ and ν being arbitrary complex numbers in this case. In the right hand side of (3.1) and (3.2)

$$\begin{bmatrix} n+\nu \\ r \end{bmatrix} \quad \text{is to replaced by} \quad \begin{bmatrix} \mu+\nu \\ r \end{bmatrix}.$$

In the next place if $V_\mu^{(\nu)}(x)$ denotes an arbitrary solution of

$$V^{(\nu)}_{\mu+1}(x) = (1+x)\, V^{(\nu)}_\mu(x) + q^{-\mu-\nu}[\mu+\nu]\, x\, V^{(\nu)}_{\mu-1}(x),$$

then as above we can prove

$$\text{(3.3)} \quad G_m(x)\, V^{(\nu)}_\mu(x) = \sum_{r=0}^{m} (-1)^r q^{r\cdot(r-m-\mu-\nu)+r\cdot(r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} \mu+\nu \\ r \end{bmatrix} [r]!\, x^r V^{(\nu)}_{m+\mu-2r}(x)$$

and

$$(3.4)\qquad V^{(\nu)}_{m+\mu}(x) = \sum_{r=0}^{m} q^{r\cdot(r-m-\mu-\nu)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} \mu+\nu \\ r \end{bmatrix} [r]!\, x^r\, G_{m-r}(x)\, V^{(\nu)}_{\mu-r}(x)\,.$$

In particular (3.3) and (3.4) imply, respectively,

$$(3.5)\qquad G_m(x)\, G^{(\nu)}_n(x) = \sum_{r=0}^{m} (-1)^r q^{r\cdot(r-m-n-\nu)+r\cdot(r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r\, G^{(\nu)}_{n+m-2r}(x),$$

$$(3.6)\qquad G^{(\nu)}_{n+m}(x) = \sum_{r=0}^{m} q^{r\cdot(r-m-n-\nu)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} [r]!\, x^r\, G_{m-r}(x)\, G^{(\nu)}_{n-r}(x)\,.$$

4. – From (1.1) we see that

$$H^2_n(x) - H_{n+1}(x)\, H_{n-1}(x) =$$

$$= x\,[n-1]\, H^2_{n-1}(x) - H_n(x)\, H_{n-2}(x) + x\, q^{n-1}\,(1-q)\, H^2_{n-1}(x)\,.$$

Thus we get

$$(4.1)\qquad H^2_n(x) - H_{n+1}(x)\, H_{n-1}(x) = (1-q)\,[n-1]!\sum_{r=1}^{n} x^r\, q^{n-r}\, H^2_{n-r}(x)/[n-r]!\,.$$

This formula has indeed a striking resemblence to the formula of DEMIR [4] for the HERMITE polynomials

$$He^2_n(x) - He_{n+1}(x)\, He_{n-1}(x) = (n-1)!\sum_{r=1}^{n} He^2_{n-r}(x)/(n-r)!\,.$$

Formula (4.1) can be easily generalized by means of (1.4). We get

$$(4.2)\qquad \left\{ \begin{aligned} &\{ H^{(\nu)}_n(x) \}^2 - H^{(\nu)}_{n+1}(x)\, H^{(\nu)}_{n-1}(x) = \\ &= z\,(1-q)\,(zq)_{n-1} \sum_{r=1}^{n} x^r\, q^{n-r}\, \{ H^{(\nu)}_{n-r}(x) \}^2/(zq)_{n-r} \end{aligned} \right.$$

which in turn resembles the formula obtained by TOSCANO for his generalized HERMITE polynomials.

In a similar fashion we find

$$(4.3)\quad \begin{cases} \{ G_n^{(\nu)}(x) \}^2 - G_{n+1}^{(\nu)}(x)\, G_{n-1}^{(\nu)}(x) = \\ = (1-q)\,(zq)_{n-1} \sum_{r=1}^{n} (-x/z)^r\, q^{-r\cdot(2n-r-1)/2} \{ G_{n-r}^{(\nu)}(x) \}^2/(zq)_{n-r} \end{cases}$$

where as before $z = q^\nu$.

5. – From (1.4) we have

$$(1-q^{n+\nu+1})\, H_{n+1}^{(\nu)}(x)/(zq)_{n+1} = (1+x)\, H_n^{(\nu)}(x)/(zq)_n - x\, H_{n-1}^{(\nu)}(x)/(zq)_{n-1}\,.$$

Now let

$$(5.1)\qquad F(t) = \sum_{n=0}^{\infty} t^n\, H_n^{(\nu)}(x)/(zq)_n\,,$$

where ν is not a negative integer. Hence we find from above

$$(5.2)\qquad z\,F(t) = (1-t)\,(1-xt)\,F(t) + z - 1\,.$$

Consequently

$$(5.3)\qquad F(t) = \delta_\nu\, H(t) + (1-z) \sum_{k=1}^{\infty} z^{k-1}/\{ (t)_k\, (tx)_k \},$$

where

$$\delta_\nu = \begin{cases} 0 & \text{if} \quad \nu \neq 0, \\ 1 & \text{if} \quad \nu = 0 \end{cases}$$

and

$$(5.4)\qquad H(t) = \prod_{0}^{\infty} (1-t\,q^j)^{-1}\,(1-tx\,q^j)^{-1} = \sum_{n=0}^{\infty} t^n\, H_n(x)/(q)_n\,.$$

Thus, if $\nu \neq 0, -1, -2, \ldots,$

$$F(t) = (1-z) \sum_{k=1}^{\infty} z^{k-1}/\{ (t)_k\, (tx)_k \}.$$

By means of the identity

$$\frac{1}{(t)_r} = \sum_{n=0}^{\infty} \begin{bmatrix} n+r-1 \\ r-1 \end{bmatrix} t^n$$

we obtain

$$F(t) = (1-z) \sum_{k=1}^{\infty} \sum_{m,n=0}^{\infty} \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix} \begin{bmatrix} m+k-1 \\ k-1 \end{bmatrix} t^{m+n} x^m z^{k-1} =$$

$$= (1-z) \sum_{j=0}^{\infty} t^j \sum_{m=0}^{j} x^m \sum_{k=0}^{\infty} \begin{bmatrix} j+k-m \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix} z^k.$$

From this we find, using the notation of basic hypergeometric functions,

$$H_n^{(\nu)}(x) = (z)_{n+1} \sum_{s=0}^{n} x^s \;{}_2\Phi_1\begin{bmatrix} q^{s+1},\ q^{n-s+1};\ z \\ q \end{bmatrix}. \tag{5.5}$$

The campanion formula

$$G_n^{(\nu)}(x) = (-1)^{n+1} q^{(n+1)(2\nu+1)/2} \sum_{s=0}^{n} x^s \;{}_2\Phi_1\begin{bmatrix} q^{s+1},\ q^{n-s+1};\ z \\ q \end{bmatrix} \tag{5.6}$$

can be easily obtained .

Now since

$$(z)_{n+1} = \sum_{r=0}^{n+1} (-1)^r q^{r\cdot(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} z^r,$$

then

$$(z)_{n+1} \;{}_2\Phi_1\begin{bmatrix} q^{s+1},\ q^{n-s+1};\ z \\ q \end{bmatrix} =$$

$$= \sum_{k=0}^{\infty} z^k \sum_{r+j=k} (-1)^r q^{r\cdot(r-1)/2} (q^{s+1})_j (q^{n-s+1})_j \begin{bmatrix} n+1 \\ k-j \end{bmatrix} / \{ (q)_j (q)_j \} =$$

$$= \sum_{k=0}^{\infty} (-z)^k q^{k\cdot(k-1)} \begin{bmatrix} n+1 \\ k \end{bmatrix} \;{}_3\Phi_2\begin{bmatrix} q^{-k},\ q^{s+1},\ q^{n-s+1};\ q \\ q,\quad q^{n-k+2} \end{bmatrix}.$$

But [**2**, p. 68]

$$ {}_3\Phi_2 \begin{bmatrix} q^{-k}, & q^{s+1}, & q^{n-s+1}, & q \\ & q, & q^{n-k+2} & \end{bmatrix} = \frac{(q^{-s})_k\,(q^{-n+s})_k}{(q)_k\,(q^{-n-1})_k}. $$

Then (5.5) and (5.6) become

$$ (5.7) \qquad H_n^{(\nu)}(x) = \sum_{s=0}^{n} \sum_{k} (q^{-s})_k\,(q^{-n+s})_k\, x^s\,(zq^{n+1})^k / \{ (q)_k\,(q^{-n-1})_k \} = $$

$$ = \sum_s \sum_k q^{k^2} \begin{bmatrix} n-s \\ k \end{bmatrix} \begin{bmatrix} s \\ k \end{bmatrix} x^s z^k, $$

$$ (5.8) \qquad G_n^{(\nu)}(x) = \sum_{s=0}^{n} \sum_{h} (q^{-s})_k\,(q^{-n+s})_k\, q^k x^s z^{-k} / \{ (q)_k\,(q)_k \} = $$

$$ = \sum_{s=0}^{n} \sum_{k} \begin{bmatrix} s \\ k \end{bmatrix} \begin{bmatrix} n-s \\ k \end{bmatrix} q^{k\cdot(k-n)}\, x^s z^{-k}. $$

We observe that although (5.5) and (5.6) do not hold for $\nu = 0$ and certain negative integers, nonetheless (5.7) and (5.8) hold for all values of ν.

An interesting functional relation can be obtained from (5.7). Indeed it easy to show that

$$ (5.9) \qquad H_n^{(\nu)}(x) - H_n^{(\nu-1)}(xq) = x\,H_{n-1}^{(\nu)}(x) - x\,q\,H_{n-1}^{(\nu)}(qx). $$

This formula can be use to characterize our polynomials. In fact we prove the following theorem.

Let $W_n(x, z)$ be a polynomial in the two variable x and $z = q$ of total degree n. We have:

Theorem. *Let the sequence* $\{ W_n(x, z)$ *satisfy the functional equation*

$$ (5.10) \quad W_n(x, z) - W_n(xq, zq^{-1}) = x\,W_{n-1}(x, z) - x\,q\,W_{n-1}(xq, z) \quad (n = 1, 2, \ldots) $$

such that

$$ (5.11) \qquad W_n(0, z) = 1, \qquad W_n(x, 1) = H_n(x), \qquad (n = 0, 1, 2, \ldots). $$

Then $W_n(x, z) = H_n^{(\nu)}(x)$.

Proof.

Assume $W_n(x, z) = H_n^{(\nu)}(x) + g_n(x, z)$ where $g_n(x, z)$ is a polynomial of total degree $\leqslant n$. Hence

$$g_n(x, z) - g_n(xq, zq^{-1}) = x\, g_{n-1}(x, z) - x\, q\, g_{n-1}(xq, z), \tag{5.12}$$

where

$$g_n(x, 1) = 0, \qquad g_n(0, z) = 0, \qquad (n = 0, 1, 2, \ldots). \tag{5.13}$$

Now (5.13) and (5.11) imply that $g_0(x, z) = 0$. Similarly, if we put $n = 1$ and $z = 1$ in (5.12), we get $g_1(x, q^{-1}) = 0$. Hence

$$g_1(x, z) = (1 - z)(1 - zq)\, f(x, z),$$

where $f(x, z)$ is a polynomial in x and z. This contradicts the assumption that g_1 is of total degree $\leqslant 1$. Thus $g_1(x, z) \equiv 0$.

Now assume

$$g_n(x, z) \equiv 0 \qquad (n = 0, 1, 2, \ldots, k).$$

Hence (5.12) gives

$$g_{k+1}(x, z) = g_{k+1}(xq, zq^{-1}) = g_{k+1}(sq^2, zq^{-2}) = \ldots .$$

Thus by (5.13)

$$g_n(x, z) = G(x, z) \prod_{k=0}^{\infty} (1 - zq^k).$$

This is obviously a contradiction and completes the proof of the Theorem.

6. – We next prove by induction the formula

$$H_{n+1}^{(\nu-1)}(x) = (1 + x)\, H_n^{(\nu)}(x) - (1 - q^\nu)\, x\, H_{n-1}^{(\nu+1)}(x) \qquad (n \geqslant 1). \tag{6.1}$$

Indeed this formula can be seen to hold for $n = 1$ and all values of ν.

Assume (6.1) hold for $n = k$. Then by (1.4)

$$H_{k+1}^{(\nu-1)}(x) = (1+x)\, H_k^{(\nu-1)}(x) - (1-q^{k+\nu-1})\, x\, H_{k-1}^{(\nu-1)}(x) =$$

$$= (1+x)\{(1+x)\, H_{k-1}^{(\nu)}(x) - (1-q^{\nu})\, x\, H_{n-2}^{(\nu-1)}(x)\} -$$

$$-(1-q^{n+\nu-1})\, x\{(1+x)\, H_{k-2}^{(\nu)}(x) - (1-q^{\nu})\, x\, H_{n-3}^{(\nu+1)}(x)\} =$$

$$= (1+x)\{(1+x)\, H_{n-1}^{(\nu)}(x) - (1-q^{n+\nu-1})\, x\, H_{n-2}^{(\nu)}(x)\} -$$

$$-(1-q)\, x\{(1+x)\, H_{n-2}^{(\nu+1)}(x) - (1-q^{n+\nu-1})\, x\, H_{n-3}^{(\nu+1)}(x)\} =$$

$$= (1+x)\, H_n^{(\nu)}(x) - (1-q^{\nu})\, x\, H_{n-1}^{(\nu+1)}(x)$$

which complete the proof.

The corresponding formula for $G_n^{(\nu)}(x)$ is

$$G_{n+1}^{(\nu-1)}(x) = (1+x)\, G_n^{(\nu)}(x) + q^{-\nu}(1-q^{\nu})\, x\, G_{n-1}^{(\nu+1)}(x)\,. \tag{6.2}$$

Formulas (6.1) and (6.2) are essentially relations between three solutions of the difference equation. The following generalizations are easily proved:

$$H_{n+k}^{(\nu-k)}(x) = H_k^{(\nu-k)}(x)\, H_n^{(\nu)}(x) - (1-q^{\nu})\, x\, H_{k-1}^{(\nu-k)}(x)\, H_{n-1}^{(\nu+1)}(x), \tag{6.3}$$

$$G_{n+k}^{(\nu-k)}(x) = G_k^{(\nu-k)}(x)\, G_n^{(\nu)}(x) + q^{-\nu}(1-q^{\nu})\, x\, G_{k-1}^{(\nu-k)}(x)\, G_{n-1}^{(\nu+1)}(x)\,. \tag{6.4}$$

7. – We prove here an analog of TOSCANO's formula expressing his generalized HERMITE polynomial in terms of HERMITE polynomial, namely, we prove that

$$H_n^{(\nu)}(x) = \sum_{2r\leq n} (-1)^r x^r q^{r\cdot(r+1)/2} (z)_r \begin{bmatrix} n-r \\ r \end{bmatrix} H_{n-2r}(x)\,. \tag{7.1}$$

This formula is obviously is true for $n = 0$ and $n = 1$. Assume its truth for $n = k$. Then, by (1.4) and the induction hypothesis,

$$H_{n+1}^{(\nu)}(x) = (1+x) \sum_r (-1)^r x^r q^{r(r+1)/2} (z)_r \begin{bmatrix} n-r \\ r \end{bmatrix} H_{n-2r}(x) -$$

$$-(1-q^{n+\nu})\, x \sum_r (-1)^r x^r q^{r(r+1)/2} (z)_r \begin{bmatrix} n-1-r \\ r \end{bmatrix} H_{n-1-2r}(x) =$$

$$= \sum_r (-1)^r x^r q^{r(r+1)/2} (z)_r \begin{bmatrix} n-r \\ r \end{bmatrix} \{ H_{n+1-2r}(x) + [n-2r]\, x\, H_{n-1-2r}(x) \} -$$

$$- [n+\nu]\, x \sum_r (-1)^r x^r q^{r(r+1)/2} (z)_r \begin{bmatrix} n-r-1 \\ r \end{bmatrix} H_{n-1-2r}(x) =$$

$$= \sum_r (-1)^r x^r q^{r(r+1)/2} (z)_r \begin{bmatrix} n-r \\ r-1 \end{bmatrix} \{ [n-2r+1]/[r] \} H_{n+1-2r}(x) -$$

$$- \sum_r (-1)^r x^r q^{r(r-1)/2} (z)_r \begin{bmatrix} n+1-r \\ r-1 \end{bmatrix} [n+2-2r]\, H_{n+1-2r}(x) +$$

$$+ [n+\nu] \sum_r (-1)^r x^r q^{r(r-1)/2} (z)_{r-1} \begin{bmatrix} n-r \\ r-1 \end{bmatrix} H_{n+1-2r}(x) =$$

$$= \sum_r (-1)^r x^r \begin{bmatrix} n-r \\ r-1 \end{bmatrix} (z)_{r-1}\, q^{r(r-1)/2} [\nu + r]\, H_{n+1-2r}(x) \cdot$$

$$\cdot \{ q^r [n+1-2r]/[r] + q^{n+1-r} \} =$$

$$= \sum_r (-1)^r x^r (z)_r\, q^{r(r+1)/2} \begin{bmatrix} n+1-r \\ r \end{bmatrix} H_{n+1-2r}(x) .$$

This completes the proof.

Similarly we have the formula

$$G_n^{(\nu)}(x) = \sum_{2r \leq n} x^r z^{-r} (z)_r\, q^{r(2r-n)} \begin{bmatrix} n-r \\ r \end{bmatrix} G_{n-2r}(x) . \tag{7.2}$$

Now since (see [**3**])

$$\Delta^r H_n(x) = \begin{bmatrix} n \\ r \end{bmatrix} [r]!\, x^r H_{n-r}(x),$$

then

$$H_n^{(\nu)}(x) = \sum_r (-1)^r q^{r(r+1)/2} \{ (z)_r/[r]! \}\, \Delta^r H_{n-r}(x), \tag{7.3}$$

where

$$\Delta f(x) = f(x) - f(qx), \qquad \Delta^{r+1} f(x) = q^r \Delta^r f(x) - \Delta^r f(qx) .$$

Therefore

$$H_n(x) = \sum_r c_r(\nu)\, \Delta^r H^{(\nu)}_{n-2}(x)\,, \tag{7.4}$$

where $c_r(\nu)$ is the coefficient in the formula

$$\Big\{ \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)/2} (z)_r\, t^r/[r]!\, \Big\}^{-1} = \sum_{r=0}^{\infty} c_r(\nu)\, t^r.$$

For $\nu = 1$, (7.4) reduces to formula (4.12) of [**3**].

8. – We now consider the polynomial $H^{(\nu)}_n(x)$ for ν a negative integer. Let r a positive integer such that $0 \leqslant r \leqslant n$. Then we prove first

$$\begin{cases} H^{(-r)}_n(x) = G_r(x)\, H_{n-r}(x) \\[2ex] G^{(-r)}_n(x) = H_r(x)\, G_{n-r}(x)\,. \end{cases} \tag{8.1}$$

For $r = 0$ (8.1) is obvious. For $r = 1$, put $\nu = 0$ in (6.1) and (6.2). We see then that (8.1) follows immediately.

Now assume that the first of (8.1) is true for $r = k$. Then employing (6.1) we see that

$$H^{(-r-1)}_n(x) = (1 + x)\, H^{(-r)}_{n-1}(x) - (1 - q^{-r})\, x\, H^{(-r+1)}_{n-2}(x) =$$

$$= (1 + x)\, G_r(x)\, H_{n-1-r}(x) - (1 - q^{-r})\, x\, G_{r-1}(x)\, H_{n-r-1}(x) =$$

$$= H_{n-1-r}(x) \{ (1 + x)\, G_r(x) - (1 - q^{-r})\, x\, G_{r-1}(x) \} = H_{n-1-r}(x)\, G_{r+1}(x)\,.$$

This completes the proof. The proof of the second part is omitted.

We next prove, for arbitrary μ,

$$\begin{cases} H^{(-n-\mu)}_n(x) = G^{(\mu)}_n(x) \\[2ex] G^{(-n-\mu)}_n(x) = H^{(\mu)}_n(x)\,. \end{cases} \tag{8.2}$$

We can verify easily that (8.2) holds for $n = 0, 1, 2$. Assume the first member is true for $n = 0, 1, \ldots, k$. Then by (1.4)

$$H_{n+1}^{(-n-\mu-1)}(x) = (1 + x)\, H_n^{(-n-\mu-1)}(x) - (1 - q^{-\mu-1})\, x\, H_{n-1}^{(-n-\mu-1)}(x) =$$

$$= (1 + x)\, G_n^{(\mu+1)}(x) - (1 - q^{-\mu-1})\, x\, G_{n-1}^{(\mu+2)}(x)\,.$$

The right hand side the above formula is, by (6.2), $G_{n-1}^{(\mu)}(x)$. Hence (8.2) is true.

9. – Let $F(t) = H(t)\, W(t)$, where $F(t)$ and $H(t)$ are as in (5.1) and (5.4) respectively. Thus by (5.2) we have

$$W(t) - z\, W(tq) = (1 - z)/H(tq)\,.$$

But [**3**, formula (2.3)]

$$1/H(tq) = \sum_{r=1}^{\infty} (-1)^r\, q^{r(r+1)/2}\, t^r\, G_n(x)/(q)_r\,.$$

We therefore have

$$W(t) = (1 - z) \sum_{n=1}^{\infty} (-1)^n\, q^{n(n+1)/2}\, t^n\, G_n(x)/\{\, (q)_n\, (1 - zq^n)\, \}. \tag{9.2}$$

Substituting in (9.1) from (5.1), (5.2) and (9.2), we get

$$H_n^{(\nu)}(x) = \{\, (z)_{n+1}/(q)_n \,\} \sum_{r=0}^{n} (-1)^r\, q^{r(r+1)/2} \begin{bmatrix} n \\ r \end{bmatrix} H_{n-r}(x)\, G_r(x)/(1 - zq^r). \tag{9.3}$$

By means of (8.1), this formula becomes

$$H_n^{(\nu)}(x) = \{\, (z)_{n+1}/(q)_n \,\} \sum_{r=0}^{n} (-1)^r\, q^{r(r+1)/2} \begin{bmatrix} n \\ r \end{bmatrix} H_n^{(-r)}(x)/(1 - zq^r) \tag{9.4}$$

or

$$H_n^{(\nu)}(x) = \{\, (z)_{n+1}/(q)_n \,\} \sum_{r=0}^{n} (-1)^{n-r}\, q^{(n-r)(n-r+1)/2} \begin{bmatrix} n \\ r \end{bmatrix} G_n^{(-r)}(x)/(1 - zq^r)\,. \tag{9.5}$$

Both of (9.4) and (9.5) can also derived from LAGRANGE interpolation formula.

Two more formulas can be written immediately if we change q into q^{-1} in (9.4) and (9.5).

References.

1. W. A. AL-SALAM, *On generalized Hermite polynomial*, Boll. Un. Mat. Ital. (3) **12** (1957), 241-246.

2. W. N. BAILEY, **Generalized hypergeometric series**. Cambridge University Press, London 1935.

3. L. CARLITZ, *Some polynomials related to the Theta functions*, Ann. Mat. Pura Appl. (4) **41** (1956), 359-373.

4. H. DEMIR, *Problem* 4215, Amer. Math. Monthly **53** (1946), p. 470.

5. G. PALAMÀ, *Polinomi più generali di altri classici e dei loro associati, e relazioni tra essi. Funzioni di seconda specie*, Riv. Mat. Univ. Parma **4** (1953), 363-386.

6. G. SZEGÖ, *Ein Beitrag zur Theorie der Thetafunktionen*, Sitzungber. Preuss. Akad. Wiss., Phys.-Math. Klasse (1926), 242-252.

7. L. TOSCANO, *Formule di riduzione tra funzioni e polinomi classici*, Riv. Mat. Univ. Parma **6** (1955), 117-140.

8. S. WIGERT, *Sur les polynomes orthogonaux et l'approximation des fonctions continues*, Ark. Math. Fys. **27** (1923), no. 18.