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The Second Extension of a Covariant Differentiation Process. (**)

Considering tensors $T^{\alpha\,\cdots}_{\beta\,\cdots}$ whose components are functions of n variables given by x and their m derivatives $x', x'', x''', \ldots, x^{(m)}$, CRAIG [1] obtained the covariant derivative

$$T^{\alpha\,\cdots}_{\beta\,\cdots\,x^{(m-1)}\gamma} - m\, T^{\alpha\,\cdots}_{\beta\,\cdots\,x^{(m)}\lambda} \left\{ \begin{matrix} \lambda \\ \gamma \end{matrix} \right\} \qquad (m \geqslant 2), \tag{1}$$

where

$$\left\{ \begin{matrix} \lambda \\ \gamma \end{matrix} \right\} \equiv x'^{\alpha}\, T^{\lambda}_{\gamma\alpha} + \frac{1}{2}\, x''^{\beta} f_{\gamma\delta\beta}\, f^{\delta\lambda}, \tag{2}$$

in which partial derivatives are indicated by subscripts and primes have been employed to denote differentiation with respect to the parameter. The curves involved in the discussions are supposed to be given in parametric form. Throughout, a repeated letter in one term denotes a sum of n terms.

The above process was extended by MARIE M. JOHNSON [2] and H. D. SINGH [3] to derive another tensor of one higher covariant order. The purpose of this paper is to extend the results of the above writers to obtain a tensor of one higher covariant order. The general process will be shown clearly by taking the tensor $T^{\alpha}_{\gamma}(x, x', x'', x''', x'''')$ into consideration.

The extended point transformation

$$x^{\alpha} = x^{\alpha}(y^1, y^2, y^3, \ldots, y^n), \qquad x'^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^i}\, y'^i, \qquad x''^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^i}\, y''^i + \frac{\partial^2 x^{\alpha}}{\partial y^i\, \partial y^j}\, y'^i y'^j,$$

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gives the following transformation in T^{α}_{γ},

$$\overline{T}^{i}_{j}(y,\ y',\ y'',\ y''',\ y'''') = T^{\alpha}_{\gamma}(x,\ x',\ x'',\ x''',\ x'''')\,\frac{\partial y^{i}}{\partial x^{\alpha}}\,\frac{\partial x^{\gamma}}{\partial y^{j}}\ , \tag{3}$$

in which y indicates n variables $y^1, y^2, y^3, \ldots, y^n$ and a similar notation is used for the derivatives y', y'', y''', y''''.

Differentiating (3) with respect to y'^k, we get

$$\overline{T}^{i}_{j\,y'^k} = \left(T^{\alpha}_{\gamma\ x'^\beta}\frac{\partial x^\beta}{\partial y^k} + T^{\alpha}_{\gamma\ x''^\beta}\frac{\partial x''^\beta}{\partial y'^k} + T^{\alpha}_{\gamma\ x'''^\beta}\frac{\partial x'''^\beta}{\partial y'^k} + T^{\alpha}_{\gamma\ x''''^\beta}\frac{\partial x''''^\beta}{\partial y'^k}\right)\cdot\frac{\partial y^{i}}{\partial x^{\alpha}}\,\frac{\partial x^{\gamma}}{\partial y^{j}} \tag{4}$$

which, by the virtue of the following general formulas

$$\left\{\begin{aligned} &\frac{\partial x^{(m-1)\beta}}{\partial y^{(m-2)k}} = (m-1)\,\frac{\partial x'^\beta}{\partial y^k}\ , \qquad \frac{\partial x^{(m)\beta}}{\partial y^{(m-2)k}} = \frac{m(m-1)}{2}\,\frac{\partial x''^\beta}{\partial y^k}\ , \\ &\frac{\partial x^{(m+1)\beta}}{\partial y^{(m-2)k}} = \frac{(m+1)m(m-1)}{3\,!}\,\frac{\partial x'''^\beta}{\partial y^k}\ , \end{aligned}\right. \tag{5}$$

reduces to

$$\overline{T}^{i}_{j\,y'^k} = \left(T^{\alpha}_{\gamma\ x'^\beta}\,\frac{\partial x^\beta}{\partial y^k} + 2\,T^{\alpha}_{\gamma\ x''^\beta}\,\frac{\partial x'^\beta}{\partial y^k} + 3\,T^{\alpha}_{\gamma\ x'''^\beta}\,\frac{\partial x''^\beta}{\partial y^k} + 4\,T^{\alpha}_{\gamma\ x''''^\beta}\,\frac{\partial x'''^\beta}{\partial y^k}\right)\cdot\frac{\partial y^{i}}{\partial x^{\alpha}}\,\frac{\partial x^{\gamma}}{\partial y^{j}}\ , \tag{6}$$

in which $\partial x'^\beta/\partial y^k$ are eliminated by

$$\left\{\begin{matrix}\overline{l}\\ k\end{matrix}\right\}\frac{\partial x^\beta}{\partial y^l} = \frac{\partial x'^\beta}{\partial y^k} + \left\{\begin{matrix}\beta\\ \delta\end{matrix}\right\}\frac{\partial x^\delta}{\partial y^k} \tag{7}$$

which is due to J. H. TAYLOR [4].

To eliminate $\partial x''^\beta/\partial y^k$, we first write x''^β in the form

$$x''^\beta = \frac{\partial x^\beta}{\partial y^j}\,y''^j + \overline{T}^{r}_{jk}\,y'^j\,y'^k\,\frac{\partial x^\beta}{\partial y^r} - T^{\beta}_{\alpha\delta}\,x'^\alpha\,x'^\delta, \tag{8}$$

with the help of (2), (7) and [4, p. 248] $f_{\alpha\beta\gamma}\, x'^{\beta} = 0$, and so by means of formula (7) and the tensor

$$T^{*\beta}(x,\ x',\ x'') = x''^{\beta} + T^{\beta}_{\alpha\delta}\, x'^{\alpha}\, x'^{\delta}, \tag{9}$$

the partial derivative of (8) have the form

$$\frac{\partial x''^{\beta}}{\partial y^{k}} = T^{*\alpha} \frac{\partial y^{j}}{\partial x^{\alpha}} \frac{\partial^{2} x^{\beta}}{\partial y^{j}\, \partial y^{k}} - \left(T^{*\beta}_{x^{\gamma}} - T^{*\beta}_{x'^{\alpha}} \left\{ {\alpha \atop \gamma} \right\} \right) \frac{\partial x^{\gamma}}{\partial y^{k}} + \tag{10}$$

$$+ \overline{T}^{*r}_{y^{k}} \frac{\partial x^{\beta}}{\partial y^{r}} - \overline{T}^{*r}_{y'^{l}} \left\{ {\overline{l} \atop k} \right\} \frac{\partial x^{\beta}}{\partial y^{r}} + 2 \left\{ {\overline{l} \atop k} \right\} \left(\left\{ {\overline{r} \atop l} \right\} \frac{\partial x^{\beta}}{\partial y^{r}} - \left\{ {\beta \atop \alpha} \right\} \frac{\partial x^{\alpha}}{\partial y^{l}} \right).$$

To obtain the value of $\partial^{2} x^{\beta}/(\partial y^{j}\, \partial y^{k})$, we multiply the TAYLOR's formula [4, p. 254, formula (19)] by

$$\overline{f}^{kl} \frac{\partial x^{\varrho}}{\partial y^{l}} = f^{\varrho\varepsilon} \frac{\partial y^{k}}{\partial x^{\varepsilon}}$$

and substituting for $\partial x'^{\gamma}/\partial y^{j}$ from (7) we have:

$$\overline{f}^{kl}\, \overline{[ij,\ k]}\, \frac{\partial x^{\varrho}}{\partial y^{l}} =$$

$$= f_{\alpha\beta}\, f^{\varrho\varepsilon} \frac{\partial^{2} x^{\alpha}}{\partial y^{i}\, \partial y^{j}} \frac{\partial x^{\beta}}{\partial y^{k}} \frac{\partial y^{k}}{\partial x^{\varepsilon}} + f^{\varrho\varepsilon}[\alpha\beta,\ \gamma] \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{j}} \frac{\partial x^{\gamma}}{\partial y^{k}} \frac{\partial y^{k}}{\partial x^{\varepsilon}} +$$

$$+ \frac{1}{2} \left(f^{\varrho\varepsilon} f_{\alpha\beta\gamma} \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{k}} \frac{\partial y^{k}}{\partial x^{\varepsilon}} \left[\left\{ {\overline{r} \atop j} \right\} \frac{\partial x^{\gamma}}{\partial y^{r}} - \left\{ {\gamma \atop \tau} \right\} \frac{\partial x^{\tau}}{\partial y^{j}} \right] + \right.$$

$$+ f^{\varrho\varepsilon} f_{\alpha\beta\gamma} \frac{\partial x^{\alpha}}{\partial y^{j}} \frac{\partial x^{\beta}}{\partial y^{k}} \frac{\partial y^{k}}{\partial x^{\varepsilon}} \left[\left\{ {\overline{r} \atop i} \right\} \frac{\partial x^{\gamma}}{\partial y^{r}} - \left\{ {\gamma \atop \tau} \right\} \frac{\partial x^{\tau}}{\partial y^{i}} \right] -$$

$$\left. - f^{\varrho\varepsilon} f_{\alpha\beta\gamma} \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{j}} \frac{\partial y^{k}}{\partial x^{\varepsilon}} \left[\left\{ {\overline{r} \atop k} \right\} \frac{\partial x^{\gamma}}{\partial y^{r}} - \left\{ {\gamma \atop \tau} \right\} \frac{\partial x^{\tau}}{\partial y^{k}} \right] \right),$$

which reduces to

$$\overline{T}^{l}_{ij}\frac{\partial x^{\varrho}}{\partial y^{l}}=\frac{\partial^2 x^{\varrho}}{\partial y^i\,\partial y^j}+T^{\varrho}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial y^i}\frac{\partial x^{\beta}}{\partial y^j}+$$

$$+\frac{1}{2}f^{\varrho\varepsilon}\left(f_{\alpha\varepsilon\gamma}\left\{\begin{matrix}\bar{r}\\ j\end{matrix}\right\}\frac{\partial x^{\gamma}}{\partial y^r}\frac{\partial x^{\alpha}}{\partial y^i}+f_{\alpha\varepsilon\gamma}\left\{\begin{matrix}\bar{r}\\ i\end{matrix}\right\}\frac{\partial x^{\gamma}}{\partial y^r}\frac{\partial x^{\alpha}}{\partial y^j}-f_{\alpha\beta\gamma}\left\{\begin{matrix}\bar{r}\\ k\end{matrix}\right\}\frac{\partial x^{\gamma}}{\partial y^{\tau}}\frac{\partial x^{\alpha}}{\partial y^i}\frac{\partial x^{\beta}}{\partial y^j}\frac{\partial y^k}{\partial x^{\varepsilon}}\right)-$$

$$-\frac{1}{2}f^{\varrho\varepsilon}\left(f_{\alpha\varepsilon\gamma}\left\{\begin{matrix}\gamma\\ \beta\end{matrix}\right\}\frac{\partial x^{\beta}}{\partial y^j}\frac{\partial x^{\alpha}}{\partial y^i}+f_{\beta\varepsilon\gamma}\left\{\begin{matrix}\gamma\\ \alpha\end{matrix}\right\}\frac{\partial x^{\alpha}}{\partial y^i}\frac{\partial x^{\beta}}{\partial y^j}-f_{\alpha\beta\gamma}\left\{\begin{matrix}\gamma\\ \varepsilon\end{matrix}\right\}\frac{\partial x^{\alpha}}{\partial y^i}\frac{\partial x^{\beta}}{\partial y^j}\right).$$

Therefore,

$$\frac{\partial^2 x^{\beta}}{\partial y^j\,\partial y^k}=\overline{\Lambda}^{t}_{jk}\frac{\partial x^{\beta}}{\partial y^t}-\Lambda^{\beta}_{\alpha\delta}\frac{\partial x^{\alpha}}{\partial y^j}\frac{\partial x^{\delta}}{\partial y^k},\tag{11}$$

where

$$\Lambda^{\beta}_{\alpha\delta}=T^{\beta}_{\alpha\delta}-\frac{1}{2}f^{\beta\gamma}\left(f_{\delta\gamma\tau}\left\{\begin{matrix}\tau\\ \alpha\end{matrix}\right\}+f_{\gamma\alpha\tau}\left\{\begin{matrix}\tau\\ \delta\end{matrix}\right\}-f_{\alpha\delta\tau}\left\{\begin{matrix}\tau\\ \gamma\end{matrix}\right\}\right);$$

and thus (10) becomes

$$\frac{\partial x''^{\beta}}{\partial y^k}=-\left|\begin{matrix}\beta\\ \gamma\end{matrix}\right|\frac{\partial x^{\gamma}}{\partial y^k}+\left|\begin{matrix}\bar{r}\\ k\end{matrix}\right|\frac{\partial x^{\beta}}{\partial y^r}-2\left\{\begin{matrix}\beta\\ \alpha\end{matrix}\right\}\left\{\begin{matrix}\bar{l}\\ k\end{matrix}\right\}\frac{\partial x^{\alpha}}{\partial y^l}+2\left\{\begin{matrix}\bar{r}\\ i\end{matrix}\right\}\left\{\begin{matrix}\bar{i}\\ k\end{matrix}\right\}\frac{\partial x^{\beta}}{\partial y^{\tau}},\tag{12}$$

where we have the non-tensor form

$$\left|\begin{matrix}\beta\\ \gamma\end{matrix}\right|=T^{*\beta}_{x^{\gamma}}-T^{*\beta}_{x'^{\alpha}}\left\{\begin{matrix}\alpha\\ \gamma\end{matrix}\right\}+T^{*\alpha}\Lambda^{\beta}_{\alpha\gamma}.\tag{13}$$

The derivatives $\partial x'''^{\beta}/\partial y^k$ are simplified by first writing

$$x'''^{\beta}=\left(y'''^{r}+\overline{T}^{*j}\left\{\begin{matrix}\bar{r}\\ j\end{matrix}\right\}+\overline{T}^{*r}_{y^i}\,y'^{i}+\overline{T}^{*r}_{y'^i}\,y''^{i}\right)\frac{\partial x^{\beta}}{\partial y^r}-\tag{14}$$

$$-\left(T^{*\alpha}\left\{\begin{matrix}\beta\\ \alpha\end{matrix}\right\}+T^{*\beta}_{x^{\gamma}}\,x'^{\gamma}+T^{*\beta}_{x'^{\gamma}}\,x''^{\gamma}\right)$$

by differentiating (8) with respect to the parameter and using the tensor (9).

By means of formulas (7), (11) and (12) and using the tensor

$$Q^{\beta}(x,\, x',\, x'',\, x''') = x'''^{\beta} + T^{*\alpha} \left\{ \begin{matrix} \beta \\ \alpha \end{matrix} \right\} + T^{*\beta}_{x\delta}\, x'^{\delta} + T^{*\beta}_{x'\delta}\, x''^{\delta},$$

the partial derivatives of (14) have the form

$$\frac{\partial x'''^{\beta}}{\partial y^k} = -\left(Q^{\beta}_{x\gamma} - Q^{\beta}_{x'\alpha} \left\{ \begin{matrix} \alpha \\ \gamma \end{matrix} \right\} - Q^{\beta}_{x''\alpha} \left| \begin{matrix} \alpha \\ \gamma \end{matrix} \right| + Q^{\alpha} \Lambda^{\beta}_{\alpha\gamma}\right) \frac{\partial x^{\gamma}}{\partial y^k} +$$

$$+ \left(\bar{Q}^{r}_{yk} - \bar{Q}^{r}_{y'l} \left\{ \begin{matrix} \bar{l} \\ k \end{matrix} \right\} - \bar{Q}^{r}_{y''l} \left| \begin{matrix} \bar{l} \\ k \end{matrix} \right| + \bar{Q}^{j} \bar{\Lambda}^{r}_{jk}\right) \frac{\partial x^{\beta}}{\partial y^r} +$$

$$+ 3 \left[\left(\left| \begin{matrix} \bar{l} \\ k \end{matrix} \right| \left\{ \begin{matrix} \bar{r} \\ l \end{matrix} \right\} + \left\{ \begin{matrix} \bar{l} \\ k \end{matrix} \right\} \left| \begin{matrix} \bar{r} \\ l \end{matrix} \right| + 2 \left\{ \begin{matrix} \bar{r} \\ i \end{matrix} \right\} \left\{ \begin{matrix} \bar{i} \\ l \end{matrix} \right\} \left\{ \begin{matrix} \bar{l} \\ k \end{matrix} \right\} \right) \frac{\partial x^{\beta}}{\partial y^r} - \right.$$

$$\left. - \left(\left| \begin{matrix} \bar{l} \\ k \end{matrix} \right| \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} + \left\{ \begin{matrix} \bar{l} \\ k \end{matrix} \right\} \left| \begin{matrix} \beta \\ \delta \end{matrix} \right| + 2 \left\{ \begin{matrix} \bar{l} \\ i \end{matrix} \right\} \left\{ \begin{matrix} \bar{i} \\ k \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} \right) \frac{\partial x^{\delta}}{\partial y^l} \right],$$

which can be put in the form

$$\frac{\partial x'''^{\beta}}{\partial y^k} = - \left\| \begin{matrix} \beta \\ \gamma \end{matrix} \right\| \frac{\partial x^{\gamma}}{\partial y^k} + \left\| \begin{matrix} \bar{r} \\ k \end{matrix} \right\| \frac{\partial x^{\beta}}{\partial y^r} + \tag{15}$$

$$+ 3 \left[\left(\left| \begin{matrix} \bar{l} \\ k \end{matrix} \right| \left\{ \begin{matrix} \bar{r} \\ l \end{matrix} \right\} + \left\{ \begin{matrix} \bar{l} \\ k \end{matrix} \right\} \left| \begin{matrix} \bar{r} \\ l \end{matrix} \right| + 2 \left\{ \begin{matrix} \bar{r} \\ i \end{matrix} \right\} \left\{ \begin{matrix} \bar{i} \\ l \end{matrix} \right\} \left\{ \begin{matrix} \bar{l} \\ k \end{matrix} \right\} \right) \frac{\partial x^{\beta}}{\partial y^r} - \right.$$

$$\left. - \left(\left| \begin{matrix} \bar{l} \\ k \end{matrix} \right| \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} + \left\{ \begin{matrix} \bar{l} \\ k \end{matrix} \right\} \left| \begin{matrix} \beta \\ \delta \end{matrix} \right| + 2 \left\{ \begin{matrix} \bar{l} \\ i \end{matrix} \right\} \left\{ \begin{matrix} \bar{i} \\ k \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} \right) \frac{\partial x^{\delta}}{\partial y^l} \right],$$

in which we have the non-tensor form

$$\left\| \begin{matrix} \beta \\ \gamma \end{matrix} \right\| = Q^{\beta}_{x\gamma} - Q^{\beta}_{x'\alpha} \left\{ \begin{matrix} \alpha \\ \gamma \end{matrix} \right\} - Q^{\beta}_{x''\alpha} \left| \begin{matrix} \alpha \\ \gamma \end{matrix} \right| + Q^{\alpha} \Lambda^{\beta}_{\alpha\gamma}\,. \tag{16}$$

Substituting the values given by (7), (12) and (15) in the equation (6) we have

$$\overline{T}^{i}_{jy'k} = \left(T^{\alpha}_{\gamma x'\beta} \frac{\partial x^{\beta}}{\partial y^{k}} - 2 T^{\alpha}_{\gamma x''\beta} \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} \frac{\partial x^{\delta}}{\partial y^{k}} - \right.$$

$$\left. - 3 T^{\alpha}_{\gamma x'''\beta} \left| \begin{matrix} \beta \\ \delta \end{matrix} \right| \frac{\partial x^{\delta}}{\partial y^{k}} - 4 T^{\alpha}_{\gamma x''''\beta} \left\| \begin{matrix} \beta \\ \delta \end{matrix} \right\| \frac{\partial x^{\delta}}{\partial y^{k}} \right) \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial y^{j}} +$$

$$+ 2 \left\{ \begin{matrix} \overline{l} \\ k \end{matrix} \right\} \left[T^{\alpha}_{\gamma x''\beta} \frac{\partial x^{\beta}}{\partial y^{l}} + 3 T^{\alpha}_{\gamma x'''\beta} \left(\left\{ \begin{matrix} \overline{r} \\ l \end{matrix} \right\} \frac{\partial x^{\beta}}{\partial y^{r}} - \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} \frac{\partial x^{\delta}}{\partial y^{l}} \right) + \right.$$

$$\left. + 6 T^{\alpha}_{\gamma x''''\beta} \left(- \left| \begin{matrix} \beta \\ \delta \end{matrix} \right| \frac{\partial x^{\delta}}{\partial y^{l}} + \left| \begin{matrix} \overline{r} \\ l \end{matrix} \right| \frac{\partial x^{\beta}}{\partial y^{r}} - 2 \left\{ \begin{matrix} \overline{t} \\ l \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} \frac{\partial x^{\delta}}{\partial y^{t}} + 2 \left\{ \begin{matrix} \overline{r} \\ t \end{matrix} \right\} \left\{ \begin{matrix} \overline{t} \\ l \end{matrix} \right\} \frac{\partial x^{\beta}}{\partial y^{r}} \right) \right] \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial y^{j}} +$$

$$+ 3 \left| \begin{matrix} \overline{l} \\ k \end{matrix} \right| \left[T^{\alpha}_{\gamma x'''\beta} \frac{\partial x^{\beta}}{\partial y^{l}} + 4 T^{\alpha}_{\gamma x''''\beta} \left(\left\{ \begin{matrix} \overline{r} \\ l \end{matrix} \right\} \frac{\partial x^{\beta}}{\partial y^{r}} - \left\{ \begin{matrix} \beta \\ \delta \end{matrix} \right\} \frac{\partial x^{\delta}}{\partial y^{l}} \right) \right] \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial y^{j}} +$$

$$+ 4 \left\| \begin{matrix} \overline{l} \\ k \end{matrix} \right\| T^{\alpha}_{\gamma x''''\beta} \frac{\partial x^{\beta}}{\partial y^{l}} \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial y^{j}} ;$$

which reduces to

$$\overline{T}^{i}_{jy'k} = \left(T^{\alpha}_{\gamma x'\beta} - 2 T^{\alpha}_{\gamma x''\delta} \left\{ \begin{matrix} \delta \\ \beta \end{matrix} \right\} - 3 T^{\alpha}_{\gamma x'''\delta} \left| \begin{matrix} \delta \\ \beta \end{matrix} \right| - 4 T^{\alpha}_{\gamma x''''\delta} \left\| \begin{matrix} \delta \\ \beta \end{matrix} \right\| \right) \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial y^{j}} \frac{\partial x^{\beta}}{\partial y^{k}} +$$

$$+ 2 \overline{T}^{i}_{jy''l} \left\{ \begin{matrix} \overline{l} \\ k \end{matrix} \right\} + 3 \overline{T}^{i}_{jy'''l} \left| \begin{matrix} \overline{l} \\ k \end{matrix} \right| + 4 \overline{T}^{i}_{jy''''l} \left\| \begin{matrix} \overline{l} \\ k \end{matrix} \right\| .$$

Hence the new tensor is

$$T^{\alpha}_{\gamma x'\beta} - 2 T^{\alpha}_{\gamma x''\delta} \left\{ \begin{matrix} \delta \\ \beta \end{matrix} \right\} - 3 T^{\alpha}_{\gamma x'''\delta} \left| \begin{matrix} \delta \\ \beta \end{matrix} \right| - 4 T^{\alpha}_{\gamma x''''\delta} \left\| \begin{matrix} \delta \\ \beta \end{matrix} \right\| , \tag{17}$$

where $\left\{ \begin{matrix} \delta \\ \beta \end{matrix} \right\}$, $\left| \begin{matrix} \delta \\ \beta \end{matrix} \right|$ and $\left\| \begin{matrix} \delta \\ \beta \end{matrix} \right\|$ are defined by (2), (13) and (16).

Considering the m times extended point transformation we extend the process to tensors whose components contain derivatives of any order. We can easily verify by virtue of the general relations in (5) that the covariant rank of the tensor

$$T^{\alpha\,\cdots}_{\gamma\,\cdots\,x^{(m-3)}\beta} - (m-2)\,T^{\alpha\,\cdots}_{\gamma\,\cdots\,x^{(m-2)}\delta}\left\{\begin{matrix}\delta\\ \beta\end{matrix}\right\} -$$

$$-\frac{(m-1)(m-2)}{2}\,T^{\alpha\,\cdots}_{\gamma\,\cdots\,x^{(m-1)}\delta}\left|\begin{matrix}\delta\\ \beta\end{matrix}\right| - \frac{m(m-1)(m-2)}{3\,!}\,T^{\alpha\,\cdots}_{\gamma\,\cdots\,x^{(m)}\delta}\left\|\begin{matrix}\delta\\ \beta\end{matrix}\right\| \qquad (m \geqslant 4)$$

is one greater than that of the original tensor $T^{\alpha\,\cdots}_{\gamma\,\cdots}$ whose components are functions of $(x,\ x',\ x'',\ \ldots,\ x^{(m)})$.

Some of the obvious properties of the above process are:

(a) If the tensor equations for $\overline{T}^i_j(y,\ y',\ y'',\ y''',\ y'''')$ are differentiated with respect to y''^k, then (17) reduces to the result obtained by H. D. Singh [3] by putting $m = 5$.

(b) If the components of the tensor do not contain x'''', then (17) reduces to the result obtained by Marie M. Johnson [2].

(c) If the components of the tensor do not contain x''' and x'''' derivatives, then the result is Craig's covariant derivative [1].

(d) If there are no x'', x''' and x'''', then the result is a partial differentiation with respect to x'.

(e) The usual rules for the derivative of a sum of tensors of the same rank and kind and for the product of any tensors are conserved.

(f) If $m = 3$, a scalar $T(x,\ x',\ x'',\ x''')$ will give a covariant tensor which is similar to that of (17), when the tensor equations for $\overline{T}(y,\ y',\ y'',\ y''')$ are differentiated with respect to y^k instead of y'^k. The tensor so obtained is

$$T_{x\beta} - T_{x'\delta}\left\{\begin{matrix}\delta\\ \beta\end{matrix}\right\} - T_{x''\delta}\left|\begin{matrix}\delta\\ \beta\end{matrix}\right| - T_{x'''\delta}\left\|\begin{matrix}\delta\\ \beta\end{matrix}\right\|.$$

(g) If $m = 3$, a tensor $T^{\alpha}(x,\ x',\ x'',\ x''')$ will give

$$T^{\alpha}_{x\beta} - T^{\alpha}_{x'\delta}\left\{\begin{matrix}\delta\\ \beta\end{matrix}\right\} - T^{\alpha}_{x''\delta}\left|\begin{matrix}\delta\\ \beta\end{matrix}\right| - T^{\alpha}_{x'''\delta}\left\|\begin{matrix}\delta\\ \beta\end{matrix}\right\| + T^{\delta}\,\Lambda^{\alpha}_{\delta\beta}\,,$$

when the tensor equations for $\overline{T}^i(y,\ y',\ y'',\ y''')$ are differentiated with respect to y^k.

(h) However, if $m=3$, and a tensor $T^{\varkappa}_{\gamma}(x, x', x'', x''')$ is used under the process (g), the new tensor of one higher covariant rank is

$$T^{\varkappa}_{\gamma;\beta} - T^{\varkappa}_{\gamma x'\delta}\left\{ \begin{matrix} \delta \\ \beta \end{matrix} \right\} - T^{\varkappa}_{\gamma x''\delta}\left| \begin{matrix} \delta \\ \beta \end{matrix} \right| - T^{\varkappa}_{\gamma x'''\delta}\left\| \begin{matrix} \delta \\ \beta \end{matrix} \right\| + T^{\delta}_{\gamma} A^{\varkappa}_{\delta\beta} - T^{\varkappa}_{\delta} A^{\delta}_{\gamma\beta}$$

and so, a tensor $T_{\gamma}(x, x', x'', x''')$ will give the new tensor

$$T_{\gamma;\beta} - T_{\gamma x'\delta}\left\{ \begin{matrix} \delta \\ \beta \end{matrix} \right\} - T_{\gamma x''\delta}\left| \begin{matrix} \delta \\ \beta \end{matrix} \right| - T_{\gamma x'''\delta}\left\| \begin{matrix} \delta \\ \beta \end{matrix} \right\| - T_{\delta} A^{\delta}_{\gamma\beta}.$$

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