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On Some Theorems Concerning the Equality of Lebesgue and Peano Areas. (*)

In § 24 of the book « L. CESARI, **Surface Area**, Princeton University Press, 1956 » a direct proof is given of the equality $L = V$, $V = P$, where L, V, P are the LEBESGUE, GEÖCZE, PEANO area of any given continuous mapping (T, A) from an admissible set $A \subset E_2$ into E_3. The proof of § 24 of the book is based on the known relations between the area of a mapping and the area of its projections (S.A.; 1.4.L_1, and 18.10, i-v), and on certain preliminary theorems (L. CESARI, S.A.; 23.1.i, 23.3.i, 23.3.ii). These theorems are used also in the discussion of the representation problem (see Chapter X of the same book). In this paper we are concerned, mainly, with the latter three theorems which, for the convenience of the reader, are restated in § **1** as Theorems I, II, III. In the present paper two new theorems (IV and V) are proved. All these theorems (I-V) concern continuous mappings (T, A) of finite GEÖCZE area $V(A, T) < +\infty$ from an admissible set $A \subset E_2$ into E_3. Theorem I and the more detailed Theorem III state that given any positive number ε, there exists an appropriate finite collection $[\pi]$ of nonoverlapping simple polygonal regions $\pi \subset A$ and a corresponding collection $[\tau]$ of projection maps $\tau: E_3 \rightarrow E_2' \subset E_3$ from the three-dimensional space E_3 onto two-dimensional spaces E_2' (one for every π) such that $V(A, T) - \sum V(\pi, \tau T) < \varepsilon$, where $\sum$ ranges over all $\pi \in [\pi]$ (cf. Note at the end of § **1**). Theorem IV of the present paper allows us to replace the above

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inequality $V(A, T) - \sum V(\pi, \tau T) < \varepsilon$ of Theorem I with $V(A, T) - \sum v(\pi, \tau T) < \varepsilon$. An analogous relation subsists between Theorem III and Theorem V proved in the present paper. With either Theorem III or V one can prove Theorem II (cf. Remark 1 of § 4) which implies the basic equality $L = V = P$. Theorem II states that there is another continuous mapping (T^*, A) with $V(A, T^*) < +\infty$ such that $d(T, T^*, A) < \varepsilon$, T^* maps a finite collection $[q]$ of disjoint squares $q \subset A$ onto squares $Q \in E_3$, and

$$V(A, T) + \varepsilon > V(A, T^*) > \sum | Q | > V(A, T) - \varepsilon,$$

where $\sum$ ranges over all $q \in [Q]$. The proof Theorems IV and V are given in § **3**. In § **2** we prove some lemmas concerning indices of finite systems of nonoverlapping simple polygonal regions $\pi \subset A$. In § **4** we correct a few misprints and imprecisions which occurred in Chapter VII of the book **Surface Area**. This book will be referred to as S.A., followed by the number of the section.

§ 1. – Restatement of known Theorems.

Let (T, A) be any continuous mapping from an admissible set A of the w-plane E_2, $w = (u, v)$, into the p-space E_3, $p = (x, y, z)$, let (T_r, A) $(r = 1, 2, 3)$ be the plane mappings which are projections of T on the coordinate planes E_{2r} of E_3 $(r = 1, 2, 3)$. We shall denote by S any finite system of nonoverlapping closed polygonal regions $\pi \subset A$, and, for each $\pi \in S$, we shall denote by τ_π an orthogonal linear transformation

$$\xi = a_{11} x + a_{12} y + a_{13} z, \quad \eta = a_{21} x + a_{22} y + a_{23} z, \quad \zeta = a_{31} x + a_{32} y + a_{33} z,$$

depending on π, from the p-space E_3, $p = (x, y, z)$, into the p'-space E_3', $p' = (\xi, \eta, \zeta)$ (a change of orthogonal coordinates depending on π). For each $\pi \in S$ and τ_π, let $T' = \tau_\pi T = (T', \pi)$ be the new mapping from π into E_3' and let (T_r', π) $(r = 1, 2, 3)$ be the plane mappings which are the projections of (T', π) on the coordinate $\eta\zeta$, $\zeta\xi$, $\xi\eta$-planes E_{2r}' of E_3' $(r = 1, 2, 3)$. We shall denote by $\sum$ any sum extended over all regions $\pi \in S$.

Theorem I (S.A.; 23.1.i):

Given any continuous mapping (T, A) from $A \subset E_2$ into E_3 with $V(A, T) < +\infty$, and any positive number $\varepsilon > 0$, there exists a $\delta > 0$, $\delta = \delta(T, A, \varepsilon) \leqslant \varepsilon$, such that any finite system S of nonoverlapping simple *polygonal regions* $\pi \subset A$ *having indices* d, m, μ *all* $< \delta$, *has the following properties: to each* $\pi \in S$ *we may associate an orthogonal linear transformation* τ_π *such that*

(a) $\sum V(\pi, T_3') > V(T, A) - \varepsilon;$

(b) $\sum V(\pi, T_i') < \varepsilon \qquad (i = 1, 2);$

(c) $\sum v(\pi, T) > V(T, A) - \varepsilon .$

Theorem II (S.A.; 23.3.i):

Given any continuous mapping (T, A) *from* $A \subset E_2$ *into* E_3 *with* $V(A, T) < +\infty$ *and any number* $\varepsilon > 0$, *there exists another continuous mapping* (T^*, A) *with the following properties:*

(1) $d(T^*, T, A) < \varepsilon;$

(2) $V(A, T^*) < V(A, T) + \varepsilon;$

(3) *there is a finite system* $[q]$ *of closed disjoint squares* $q \subset A$ *such that* T^* *is linear on each* q *and* $Q = T^*(q)$ *is also a square* $Q \subset E_3$ *of diameter* $< \varepsilon$;

(4) $\sum |Q| > V(A, T) - \varepsilon$, *where* $\sum |Q|$ *denotes the sum of the areas of all squares* Q .

Theorem III (S.A.; 23.3.ii):

Given any continuous mapping (T, A) *from any admissible set* $A \subset E_2$ *into* E_3 *with* $V(A, T) < +\infty$, *given any number* $\varepsilon > 0$, *there exists another number* $\delta = \delta(T, A, \varepsilon) > 0$ *such that each finite system* $[\pi]$ *of nonoverlapping simple polygonal regions* $\pi \subset A$ *of indices* $d, m, \mu < \delta$ *has the following properties:*

(1) *To each* $\pi \in [\pi]$ *we may associate an orthogonal linear transformation* τ_π *from the p-space* E_3, $p = (x, y, z)$, *into the p'-space* E_3', $p' = (\xi, \eta, \zeta)$, *such that*

$$\sum_\pi V(\pi, T_3') > V(A, T) - \varepsilon; \qquad \sum_\pi V(\pi, T_i') < \varepsilon \qquad (i = 1, 2);$$

$$\sum_\pi v(\pi, T) > V(A, T) - \varepsilon,$$

where $\sum_\pi$ *denotes any sum ranging over all* $\pi \in [\pi]$, *where* $T' = \tau_\pi T$, *and* T_r, T_r' $(r = 1, 2, 3)$ *are the projections of* T, T' *on the coordinate planes* E_{2r}, E_{2r}' $(r = 1, 2, 3)$ *of* E_3 *and* E_3'.

(2) *There are two continuous mappings* (T_0, A), (T^*, A) *such that*

$$d(T_0, T, A) < \varepsilon, \quad d(T^*, T, A) < \varepsilon, \quad T = T_0 = T^* \quad on \quad A - \sum_\pi \pi .$$

(3) *In each* $\pi \in [\pi]$ *there is a finite system* $[G]_\pi$ *of disjoint simply connected open sets* $G \subset \pi$ *and a corresponding system* $[c]_\pi$ *of the same number of simple closed polygonal regions* $c \subset G$ *such that each G contains one and only one c and such that* $T = T_0 = T^*$ *on* $\pi - \sum G$, $T_0 = T^*$ *on* $\pi - \sum c$.

(4) *For each* $\pi \in [\pi]$ *and* $c \in [c]_\pi$ *the curve* $C : (T_0, c^*) = (T^*, c^*)$ *has finite length and its projection* C_3 *on the* $\xi\,\eta$*-plane* E'_{23} *is contained in the boundary* Q_0^* *of a square* Q_0 *of the same plane* E'_{23} *and of diameter* $< \varepsilon$; C_3 *is homotopic in* Q_0^* *to* Q_0^* *itself counted a certain number* $m' \geqslant 1$ *of times (all clockwise or all counterclockwise depending upon T); i.e.,* $C_3 \cong m' Q_0^*(Q_0^*)$.

(5) *In each* $c \in [c]_\pi$ *there is a system* $[q]_c$ *of m disjoint equal squares* $q \subset c^0$, $1 \leqslant m \leqslant m'$; T^* *is linear on each q and maps these m squares onto m equal squares* $Q \subset E_3$ *parallel to and projecting on the square* Q_0 *of the plane* E'_{23}.

(6) *If* $\sum |Q|$ *denotes the sum of the areas of all squares Q, then*

$$V(A, T) - \varepsilon < \sum |Q| \leqslant V(A, T^*) < V(A, T) + \varepsilon .$$

Note. In I(a), $\sum V(\pi, T'_3)$ replaces the misprint $\sum v(\pi, T'_3)$ of (S.A.; 23.1.i). In II(3), $Q = T^*(q)$ replaces $Q = T(q)$ of (S.A.; 23.3.i). In III (1), $\sum_\pi V(\pi, T'_3)$ replaces $\sum_\pi v(\pi, T'_3)$ of (S.A.; 23.3.ii); in III(4), m' replaces m; and in III(5), the specification $m' \geqslant m \geqslant 1$ is added.

§ 2. – Some preliminary considerations.

Let (T, A) be a BV mapping from an admissible set $A \subset E_2$ into E_3. Then the multiplicity functions $N(p; T_r, A)$, $p \in E_{2r}$, $(r = 1, 2, 3)$, are L-integrable in E_{2r} (S.A., 12.1). Hence, there is a function $\delta(\varepsilon) \geqslant 0$ for $\varepsilon \geqslant 0$, depending on (T, A) only, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$(h) \int N(p;\, T_r,\, A) \leqslant \delta(\varepsilon), \qquad (r = 1, 2, 3),$$

for any measurable set $h \subset E_{2r}$ with $|h| \leqslant \varepsilon$. As usual, we shall denote by $\| C, C' \|$ the Fréchet distance of two oriented closed curves C, C'.

Lemma 1:

Let (T, A) be a BV mapping from any admissible set $A \subset E_2$ into E_3, and S any finite system of nonoverlapping simple polygonal regions $\pi \subset A$ with $|T_r(\sum \pi^*)| < \varepsilon$, $(r = 1, 2, 3)$, where $\sum$ ranges over all $\pi \in S$. For each $\pi \in S$ let $[\pi_n]$ be any sequence of simple polygonal regions with $\pi_n \subset \pi_{n+1}$, $\pi_n^0 \uparrow \pi^0$, $\| \pi_n^*, \pi^* \| \to 0$ as $n \to \infty$. Then we have

$$\overline{\lim_{n\to\infty}} \sum_{\pi \in S} | u(\pi, T_r) - u(\pi_n, T_r) | \leqslant 2\delta(\varepsilon), \qquad (r = 1, 2, 3).$$

Proof:

The set $T_r(\sum\pi^*)$ is compact. Hence, there exists a ϱ-neighborhood H_r of $T_r(\sum\pi^*)$ such that $|H_r| < \varepsilon$ $(r = 1, 2, 3)$, where $\varrho > 0$ may well depend on S and so H_r. There is a $n_0 > 0$ such that $\| C_r, C_{nr} \| < \varrho$, $n \geqslant n_0$, $(r = 1, 2, 3)$, where C_r: (T_r, π^*), C_{nr}: (T_r, π_n^*). Hence, by (S.A., 8.3.i) we have $O(p; C_r) = O(p; C_{nr})$ for all $p \in E_{2r} - H_r$, $(r = 1, 2, 3)$. We deduce, for $n \geqslant n_0$,

$$\sum | u(\pi, T_r) - u(\pi_n, T_r) | = \sum | (E_{2r}) \int [O(p; C_r) - O(p; C_{nr})] | =$$

$$= \sum | (H_r) \int [O(p; C_r) - O(p; C_{nr})] | \leqslant \sum (H_r) \int [| O(p; C_r) | + | O(p; C_{nr}) |] \leqslant$$

$$\leqslant 2 (H_r) \int N(p; T_r, A) \leqslant 2\delta(\varepsilon).$$

This proves Lemma 1.

Lemma 2:

Let (T, A) be any continuous BV mapping of the admissible set $A \subset E_2$ into E_3. Let A' be an admissible set contained in A. Given $\varepsilon > 0$, there exists a $\delta = \delta(A', \varepsilon) > 0$ such that for all systems S of nonoverlapping simple polygonal regions $\pi \subset A$ with indices d, m, μ (S.A.; 22.4) with respect to the mapping (T, A) less than δ, we have

$$U(A', T) - \sum{}' u(\pi, T) < \varepsilon,$$

$$U(A', T_r) - \sum{}' | u(\pi, T_r) | < \varepsilon, \qquad (r = 1, 2, 3),$$

where $\sum'$ is the sum over all $\pi \subset A'$, $\pi \in S$.

Proof:

Suppose the lemma is false. Then there is an $\varepsilon_0 > 0$ and a sequence of finite systems $S^{(n)}$ of nonoverlapping simple polygonal regions $\pi \subset A$ with indices $d^{(n)}$, $m^{(n)}$, $\mu^{(n)} < n^{-1}$ and one of the mappings (T, A), (T_r, A) $(r = 1, 2, 3)$, say (T_1, A), such tat $U(A', T_1) - \sum' | u(\pi, T_1) | \geqslant \varepsilon_0$. By virtue of Lemma 1 above, we may assume that each $S^{(n)}$ is a disjoint system of simple polygonal regions $\pi \subset A^0$.

Let $G_{nr} = T_r (\sum^{(n)} \pi^*)$, where $\sum^{(n)}$ ranges over all $\mu \in S^{(n)}$. Then G_{nr} is compact and $| G_{nr} | \leqslant m^{(n)} < n^{-1}$. Hence for each n, there exists a $\varrho = \varrho(n) < n^{-1}$ such that if H_{nr} is the ϱ-neighborhood of G_{nr} then $| H_{nr} | < n^{-1}$, $(r = 1, 2, 3)$.

Let F_n be a sequence of figures such that $\sum_{i=1}^{n} \sum^{(n)} \pi \subset F_n^0$ $(n = 1, 2, \ldots)$, $F_n \subset F_{n+1}$, and $F_n^0 \uparrow A^0$. Since $S^{(n)}$ is a disjoint system of simple polygonal regions $\pi \subset A^0$, $F_n' = F_n - \sum^{(n)} \pi^0$ is also a figure. Hence by (S.A.; 21.1.i) there is a subdivision $S_n = S_n' + S_n''$ of F_n', where S_n' are the simple polygonal regions q and S_n'' are the nonsimple polygonal regions R, such that the indices d_n, m_n, σ_n of the subdivision S_n are all less than $\varrho(n) < n^{-1}$, where $\varrho(n)$ is defined above. For each simple polygonal region $q \in S_n'$ and each $\pi \in S^{(n)}$ such that $q\pi^* \neq 0$, we have $T_r(q) \subset H_{nr}$. Therefore $S^{(n)} + S_n = S^{(n)} + S_n' + S_n''$ is a finite subdivision of F_n into nonoverlapping polygonal regions with indices d_n^*, m_n^*, $\sigma_n^* < 2n^{-1}$.

By (S.A.; 21.3 Note) we have that $\sum^{(n)'} u(\pi, T) + \sum_n' u(q, T) \to U(A', T)$ and $\sum^{(n)'} | u(\pi, T_r) | + \sum_n' | u(q, T_r) | \to U(A', T_r)$ $(r = 1, 2, 3)$, as $n \to \infty$, where $\sum^{(n)'}$ ranges over $\pi \in S^{(n)}$, $\pi \subset A'$ and $\sum_n'$ ranges over $q \in S_n'$, $q \subset A'$. Since $\mu^{(n)} \to 0$ as $n \to \infty$ implies that $U(A, T) - \sum^{(n)} u(\pi, T) \to 0$, we have $\sum_n' | u(q, T_r) | \leqslant \sum_n' u(q, T) \leqslant U(A, T) - \sum^{(n)} u(\pi, T) \to 0$ as $n \to \infty$ $(r = 1, 2, 3)$. Hence $\sum^{(n)'} u(\pi, T) \to U(A', T)$ and $\sum^{(n)'} | u(\pi, T_r) | \to U(A', T_r)$, $(r = 1, 2, 3)$ as $n \to \infty$. But this is a contradiction since it was assumed that $U(A', T_1) - \sum^{(n)'} | u(\pi, T_1) | \geqslant \varepsilon_0 > 0$ for all n. Hence Lemma 2 is established.

Since $U(A, T) = V(A, T)$ and $v(\pi, T)$, (S.A.; 21.3), we have the following lemma as an immediate consequence of Lemma 2.

Lemma 3:

Let (T, A) be any continuous BV mapping of the admissible set $A \subset E_2$ into E_3. Let A' be an admissible set contained in A. Given $\varepsilon > 0$, there exists a $\delta = \delta(A', \varepsilon) > 0$ such thtat for all systems S of nonoverlapping simple polygonal regions $\pi \subset A$ with indices d, m, μ (S.A.; 22.4) with respect to the mapping (T, A) less than δ, we have

$$V(A', T) - \sum' v(\pi, T) < \varepsilon,$$

$$V(A', T_r) - \sum' v(\pi, T_r) < \varepsilon, \qquad (r = 1, 2, 3),$$

where $\sum'$ is the sum over all $\pi \subset A'$, $\pi \in S$.

§ 3. - Some new theorems.

We are now in a position to prove the following Theorems IV and V analogous to the Theorems I, III restated in § 1.

Theorem IV:

Given any continuous mapping (T, A) *from* $A \subset E_2$ *into* E_3 *with* $V(A, T) < +\infty$ *and any positive number* $\varepsilon > 0$, *there exists*

(a) *a finite collection* $\mathfrak{T} = \{\tau\}$ *of orthogonal linear transformations* τ *of* E_3, *and*

(b) *a number* $\delta = \delta(A, T, \mathfrak{T}, \varepsilon) > 0$, $\delta \leqslant \varepsilon$,

such that any finite system S *of nonoverlapping simple polygonal regions* $\pi \subset A$, *having indices* d_τ, m_τ, μ_τ *with respect to the mappings* $(\tau T, A)$, $\tau \in \mathfrak{T}$, *all less than* δ *has the following property: To each* $\pi \in S$ *we may associate a orthogonal linear transformation* τ_π, $\tau_\pi \in \mathfrak{T}$, *such that*

(i) $$\sum v(\pi, T'_3) > V(A, T) - \varepsilon,$$

(ii) $$\sum V(\pi, T'_i) < \varepsilon \quad (i = 1, 2),$$

(iii) $$\sum v(\pi, T) > V(A, T) - \varepsilon,$$

where $\sum$ *is the sum extended over all* $\pi \in S$ *and* $T' = \tau_\pi T$.

Proof:

By Theorem I, given $\varepsilon > 0$, there exists a finite system γ of nonoverlapping simple closed polygonal regions $\nu \subset A^0$ and a corresponding system $\mathfrak{T}' = \{\tau_\nu\}$ of orthogonal linear transformations τ_ν of E_3 such that

$$(\alpha) \qquad \begin{cases} \sum_\nu V(\nu, T'_3) > V(A, T) - \varepsilon/4, \\ \sum_\nu V(\nu, T'_i) < \varepsilon/4 \quad (i = 1, 2), \\ \sum_\nu v(\nu, T) \quad > V(A, T) - \varepsilon/4, \end{cases}$$

where $T' = \tau_\nu T$ and $\sum_\nu$ is the sum extended over all $\nu \in \gamma$. In the remaining, we will drop the subscript ν of τ_ν.

Let N be the number of simple polygonal regions $\nu \in \gamma$. For each ν and corresponding τ, determine a $\delta_\tau > 0$, $\delta_\tau < \varepsilon/(4N)$, such that, for any finite system S of nonoverlapping simple polygonal regions $\pi \subset A$, with indices d_τ, m_τ, μ_τ with respect to the mapping (T', A), $T' = \tau T$, less than δ_τ, we have

$$(\beta) \qquad \begin{cases} V(\nu, T') - \sum' v(\pi, T') < \varepsilon/(4N), \\ V(\nu, T'_r) - \sum' v(\pi, T'_r) < \varepsilon/(4N) \quad (r = 1, 2, 3), \end{cases}$$

where $\sum'$ is the sum over all $\pi \in S$, $\pi \subset \nu$ (Lemma 3).

Let $\mathfrak{T}$ be the system $\mathfrak{T}'$ with the identity transformation added if it is not in $\mathfrak{T}'$ already, and let $\delta = \min[\delta_\tau, \tau \in \mathfrak{T}']$. Consider any finite system S of nonoverlapping simple polygonal regions $\pi \subset A$ with indices d_τ, m_τ, μ_τ with respect to the mappings (T', A), $T' = \tau T$, $\tau \in \mathfrak{T}$, all less than δ. Then for each $\pi \in S$, let τ_π be determined as follows: If $\pi \subset \nu$ for some $\nu \in \gamma$, then let $\tau_\pi = \tau$. If $\pi \not\subset \nu$ for any $\nu \in \gamma$, then let τ_π be the identity.

We have $\sum v(\pi, T) \geqslant V(A, T) - \mu \geqslant V(A, T) - \varepsilon$, since the identity transformation belongs to $\mathfrak{T}$. Hence (iii) is verified.

From (β) we have

$$V(\nu, T') - \sum' V(\pi, T') < V(\nu, T') - \sum' v(\pi, T') < \varepsilon/(4N).$$

Since $V(\nu, T') = V(\nu, T)$ and $V(\pi, T) = V(\pi, T')$ (S.A.; 22.3.i), we have

$$V(\nu, T) - \sum' V(\pi, T) < \varepsilon/(4N),$$

$$V(A, T) \geqslant \sum V(\pi, T) = \sum_\nu \sum' V(\pi, T) + \sum'' V(\pi, T) >$$

$$> \sum_\nu V(\nu, T) - \varepsilon/4 + \sum'' V(\pi, T) > V(A, T) - \varepsilon/4 - \varepsilon/4 + \sum'' V(\pi, T) =$$

$$= V(A, T) - \varepsilon/2 + \sum'' V(\pi, T'),$$

where $\sum''$ is extended over all $\pi \in S$ with $\pi \not\subset \nu$ for any $\nu \in \gamma$. Therefore, $\sum'' V(\pi, T') < \varepsilon/2$ and $\sum'' V(\pi, T'_r) < \varepsilon/2$ $(r = 1, 2)$. Hence

$$\sum V(\pi, T'_r) = \sum_\nu \sum' V(\pi, T'_r) + \sum'' V(\pi, T'_r) \leqslant \sum_\nu V(\nu, T'_r) + \varepsilon/2 \leqslant \varepsilon/4 + \varepsilon/2 < \varepsilon$$

$(r = 1, 2)$, and (ii) is verified.

$$\sum v(\pi, T'_3) = \sum_\nu \sum' v(\pi, T'_3) + \sum'' v(\pi, T'_3) \geqslant$$

$$\geqslant \sum_\nu \sum' v(\pi, T'_3) \geqslant \sum_\nu V(\nu, T'_3) - \varepsilon/4 \geqslant V(A, T) - \varepsilon/4 - \varepsilon/4 > V(A, T) - \varepsilon;$$

and hence (i) is verified. Thus Theorem IV is proved.

Theorem V:

Given any continuous mapping (T, A) *from any admissible set* $A \subset E_2$ *into* E_3 *with* $V(A, T) < +\infty$, *and any positive number* $\varepsilon > 0$, *there exists a finite collection* $\mathfrak{T} = \{\tau\}$ *of orthogonal linear transformations* τ *of* E_3 *and a number* $\delta = \delta(A, T, \mathfrak{T}, \varepsilon) > 0$, $\delta \leqslant \varepsilon$, *such that any finite system* $\mathfrak{S}$ *of nonoverlapping simple polygonal regions* $\pi \subset A$, *having indices* d_τ, m_τ, μ_τ *with respect to the mappings* $(\tau T, A)$, $\tau \in \mathfrak{T}$, *all less than* δ *has the following properties:*

(1) *To each* $\pi \in \mathfrak{S}$ *we may associate an orthogonal linear transformation* $\tau_\pi \in \mathfrak{T}$ *from the p-space* E_3, $p = (x, y, z)$, *into the* p'*-space* E'_3, $p' = (\xi, \eta, \zeta)$, *such that*

$$\sum_\pi v(\pi, T'_3) > V(A, T) - \varepsilon, \qquad \sum_\pi V(\pi, T'_i) < \varepsilon \quad (i = 1, 2),$$

$$\sum_\pi v(\pi, T) > V(A, T) - \varepsilon,$$

where $\sum_\pi$ *denotes any sum ranging over all* $\pi \in \mathfrak{S}$, *where* $T' = \tau_\pi T$, *and* T_r, T'_r, $(r = 1, 2, 3)$ *are the projections of* T, T' *on the coordinate planes* E_{2r}, E'_{2r}, $(r = 1, 2, 3)$ *of* E_3 *and* E'_3.

(2) *There are two continuous mappings* (T_0, A), (T^*, A) *such that*

$$d(T_0, T, A) < \varepsilon, \quad d(T^*, T, A) < \varepsilon, \qquad T = T_0 = T^* \quad on \quad A - \sum_\pi \pi.$$

(3) *In each* $\pi \in \mathfrak{S}$ *there is a finite system* $[G]_\pi$ *of disjoint simply connected open sets* $G \subset \pi$ *and a corresponding system* $[c]_\pi$ *of the same number of simple closed polygonal regions* $c \subset G$ *such that each* G *contains one and only one* c *and such that* $T = T_0 = T^*$ *on* $\pi - \sum G$, $T_0 = T^*$ *on* $\pi - \sum c$.

(4) *For each* $\pi \in \mathfrak{S}$ *and* $c \in [c]_\pi$ *the curve* C: $(T_0, c^*) = (T^*, c^*)$ *has finite length and its projection* C_3 *on the* $\xi\eta$*-plane* E'_{23} *is contained in the boundary* Q^*_0 *of a square* Q_0 *of the same plane* E'_{23} *and of diameter* $< \varepsilon$; C_3 *is homotopic in* Q^*_0 *to* Q^*_0 *itself counted a certain number* $m' \geqslant 1$ *of times (all clockwise or all counterclockwise, depending upon* T*); i.e.,* $C_3 \cong m' Q^*_0(Q^*_0)$.

(5) *In each* $c \in [c]_\pi$ *there is a system* $[q]_c$ *of* m *disjoint equal squares* $q \subset c^0$, $1 \leqslant m \leqslant m'$; T^* *is linear on each* q *and maps these* m *squares onto* m *equal squares* $Q \subset E_3$ *parallel to and projecting on the square* Q_0 *of the plane* E'_{23}.

(6) *If* $\sum |Q|$ *denotes the sum of the areas of all squares* Q, *then* $V(A, T) - \varepsilon < \sum |Q| \leqslant V(A, T^*) < V(A, T) + \varepsilon$.

Proof:

The major portion of the proof of this Theorem is exactly the same as that of (S.A.; 23.3, ii.). The changes which are necessary occur in the first two paragraphs of the proof and are given below.

If $V(A, T)=0$, let $T^*=T_0=T$, $\mathfrak{T}$ consist of the identity transformation and $\delta>0$ be arbitrary. Then for each system $\mathfrak{S}=[\pi]$ and for every region $\pi\in\mathfrak{S}$ let τ_π be the only transformation in $\mathfrak{T}$ and let all collections in the theorem be the empty collection.

Suppose now $V(A, T)>0$ and let $M_0=\max[1, V(A, T)]$, $\sigma=$ $=\min[1, (138\, M_0)^{-1}\varepsilon, 3^{-1} M_0, (10\, M_0)^{-1}\varepsilon^4]$; hence $0<\sigma\leqslant 1\leqslant M_0<+\infty$. By Theorem IV (where we replace ε by σ^4), there exists a finite collection $\mathfrak{T}=\{\tau\}$ of linear orthogonal transformations τ of E_3 and a number $\delta=\delta(A, T, \mathfrak{T}, \sigma^4)>0$, $\delta\leqslant\sigma^4<\varepsilon$, such that any finite system $\mathfrak{S}=[\pi]$ of non-overlapping simple polygonal regions $\pi\subset A$, having indices d_τ, m_τ, μ_τ with respect to the mapping $(\tau T, A)$, $\tau\in\mathfrak{T}$ all less than δ has the following property:

To each $\pi\in\mathfrak{S}$ we may associate an orthogonal linear transformation $\tau_\pi\in\mathfrak{T}$ from the p-space E_3, $p=(x, y, z)$, onto the p'-space E_3', $p'=(\xi, \eta, \zeta)$, such that

(C 12) $$V(A, T)\geqslant\sum_\pi V(\pi, T)\geqslant\sum_\pi v(\pi, T)>V(A, T)-\sigma^4,$$

$$\sum_\pi V(\pi, T_3')\geqslant\sum_\pi v(\pi, T_3')\geqslant V(A, T)-\sigma^4, \qquad \sum_\pi V(\pi, T_i')<\sigma^4$$

$(i=1, 2)$, where $T'=\tau_\pi T$ and T_r' $(r=1, 2, 3)$ are the projections of T' on the $\eta\zeta$, $\zeta\xi$, $\xi\eta$-planes.

§ 4. – Some remarks.

Remark 1:

Theorem II is included both in Theorem III as well as in Theorem V. Thus the present Theorem V gives a new proof of Theorem II. The short Theorem II was given in (S.A.; 23.1.i, 23.3.i, 23.3.ii) since the proof of the basic equality $L=V=P$ (S.A.; § 24) is based only on the very simple statement of Theorem II.

Remark 2:

In (S.A.; 24.2) the first displayed formula on page 393 should read

$$\sum V(\pi, T_3')\geqslant V(A, T)-\varepsilon.$$

Then by the upper additivity of P, and by (S.A.; 9.11) we have

$$P(A,\ T) \geqslant \sum P(\pi,\ T) \geqslant \sum V(\pi,\ T_3') \geqslant V(A,\ T) - \varepsilon$$

which replaces the second displayed formula on page 393 .

Remark 3:

In (S.A.; 23.3), formula (12) on page 372, should be replaced by

$$\text{(C12)} \quad \begin{cases} U(A,\ T_r) \geqslant \sum_\pi |\, u(\pi,\ T_r) \,| > U(A,\ T_r) - \sigma^4 & (r = 1,\ 2,\ 3), \\ V(A,\ T) \geqslant \sum_\pi V_\pi(\pi,\ T) \geqslant \sum_\pi v(\pi,\ T) > V(A,\ T) - \sigma^4, & \\ \sum_\pi V(\pi,\ T_3') > V(A,\ T) - \sigma^4, \qquad \sum_\pi V(\pi,\ T_i') < \sigma^4 & (i = 1,\ 2), \end{cases}$$

where $T' = \tau_\pi T$, and T_r, T_r' $(r = 1, 2, 3)$ are the projections of T and T' on the yz, zx, xy, $\eta\zeta$, $\zeta\xi$, $\xi\eta$-planes respectively .

In (S.A.; 23.11) the lines from page 381, line 11 to page 383, line 8, contain a recourse to the previous formula (12) and should be replaced by the lines given below . No other major change is needed in (S.A.; §§ 23, 24) .

We shall prove that

$$\sum_\pi \sum' V(\alpha,\ T_3') < 54\ \sigma^4,$$

where $\sum_\pi$, $\sum'$ range over all $\pi \in [\pi]$ and $\alpha \in [\alpha]_\pi'$. This relation replaces farmula (31) of pages 381 (S.A.; 23.11). For each $\alpha \in [\alpha]_\pi^0$ let us consider the finite collection $[k]_\alpha$ of those components k of α^* which separate $g \subset \alpha$ from either U^*, or some set $\alpha' \in [\alpha]_\pi'$ (if any) with $\alpha' \subset \beta' \subset \beta' + \beta'^0 \subset \beta$, where β is the set β relative to α. Thus $[k]_\alpha$ contains the continuum $k_0 = k = \beta^*$ (S.A.; 23.9) which separates g from U^* and may contain some more continua .

For each $k \neq k_0$, $k \in [k]_\alpha$ we may consider the relative collections of ends η, of prime ends ω of k in α, and the ordered collection $[E_\omega]$ of the continua $E_\omega \subset k$. The mapping T is constant on each $E_\omega \subset k$, and the curve Γ: $(T,\ [E]_\omega)$ is rectifiable . We shall denote by Γ_r, Γ_r' $(r = 1, 2, 3)$ the projections of Γ on the planes E_{2r}, E_{2r}', $(r = 1, 2, 3)$, and by Ω_r, Ω_r', the numbers

$$\Omega_r = (E_{2r}) \int O(p;\ \Gamma_r), \quad \Omega_r' = (E_{2r}') \int O(p;\ \Gamma_r'), \qquad (r = 1,\ 2,\ 3)\,.$$

By (S.A.; 8.11.i) we have

$$\Omega_r' = \sum_{s=1}^{3} \alpha_{rs}\, \Omega_s, \qquad \Omega_r = \sum_{r=1}^{3} \alpha_{rs}\, \Omega_r',$$

where (α_{rs}) is an orthogonal matrix. Hence

$$(C1) \qquad |\Omega_r| \leqslant |\Omega_1'| + |\Omega_2'| + |\Omega_3'| \qquad (r = 1, 2, 3).$$

Now $\Gamma_3' \subset Q^*$ and we must have $O(p; \Gamma_3') = 0$, where p is the center of Q, $p \in [p]_\pi$, since otherwise we could include k in a simple polygonal region r with $\|C, \Gamma\|$ as small as we want, C: (T, r^*), and hence also $O(p; C_3') \neq 0$ and finally, by (S.A.; 14.3.ii), there would be another set $g \in [g]_p$ in U relative to the same point p, a contradiction. This implies that $O(p; \Gamma_3') = 0$ for all $p \in E_{23}' - Q^*$, and finally $O(p; \Gamma_3') = 0$ for all $p \in E_{23}'$ and $\Omega_3' = 0$. Thus, by (C1),

$$(C2) \qquad |\Omega_r| \leqslant |\Omega_1'| + |\Omega_2'| \qquad (r = 1, 2, 3).$$

Let us prove that

$$(C3) \qquad \sum{}^* |\Omega_i'| \leqslant V(\pi, T_i') \qquad (i = 1, 2),$$

where $\sum^*$ is extended over all $k \neq k_0$, $k \in [k]_\alpha$, $\alpha \in [\alpha]_\pi^0$. Indeed we may consider, as above, disjoint regions r with $r \subset \pi$, $r^0 \supset k$, and $\|C, \Gamma\| \leqslant \tau$ with τ as small as we want. Then

$$O(p; C_i') = O(p; \Gamma_i')$$

for all points $p \in E_{2i}' - [\Gamma_i']_\tau$, where $[\Gamma_i']_\tau$ is the τ-neighborhood of the set $[\Gamma_i']$. Since Γ_i' is rectifiable, the set $[\Gamma_i']$ is compact and of measure zero (S.A.; 8.8.i). Then $|[\Gamma_i']_\tau| \to 0$ as $\tau \to 0$ and thus we may assume τ so small that $|[\Gamma_i']_\tau|$ is as small as we wish. Now we have

$$|\Omega_i' - u(r, T_i')| = |(E_{2r}') \int O(p; \Gamma_i') - O(p; C_i')]| \leqslant$$

$$\leqslant [\Gamma_i']_\tau \int |O(p; \Gamma_i')| + [\Gamma_i']_\tau \int N(p; T_i', \pi).$$

Since both $|O(p; \Gamma_i')|$, $N(p; T_i', \pi)$ are L-integrable in E_{2i}', we may take τ so small that both integrals above and therefore $|\Omega_i' - u(r, T_i')|$ is less than any given number. Since

$$\sum{}^* |u(r, T_i')| \leqslant \sum{}^* v(r, T_i') \leqslant V(\pi, T_i'),$$

we conclude that we have also

$$\sum{}^* |\Omega_i'| \leqslant V(\pi, T_i') \qquad (i = 1, 2),$$

and (C 3) is proved.

By the last relation of (C12) and (C2), (C3), we have

$$\sum_\pi \sum^* |\Omega_r| \leqslant \sum_\pi \sum^* |\Omega_1'| + \sum_\pi \sum^* |\Omega_2'| \leqslant \sum_\pi V(\pi, T_1') + \sum_\pi V(\pi, T_2') \leqslant 2\sigma^4,$$

that is

$$\text{(C4)} \qquad \sum_\pi \sum^* |\Omega_r| \leqslant 2\sigma^4 \qquad (r = 1, 2, 3).$$

We consider now the planes E_{2r} $(r = 1, 2, 3)$. The functions $|O(p;\ \Gamma_r)|$, $N(p;\ T_r, \pi)$, $p \in E_{2r}$, $(r = 1, 2, 3)$ are L-integrable. Hence there exists a number $\tau > 0$ such that, for every measurable set $h \subset E_{2r}$, with $|h| < \tau$, we have

$$(h) \int N(p;\ T_r,\ \pi) < M_1^{-1} \sigma^4, \qquad (h) \int |O(p;\ \Gamma_r)| < M_1^{-1} \sigma^4, \qquad (r = 1, 2, 3),$$

where M_1 is the total number of k, $k \neq k^0$, $k \in [k]_\alpha$, $\alpha \in [\alpha]_\pi^0$, $\pi \in [\pi]$.

Let us consider the sets $H_r = \sum^* [\Gamma_r] \subset E_{2r}$. These sets are compact and of measure zero, hence there is a $\delta = \delta_0(\pi) > 0$ such that the δ-neighborhood $H_{r\delta} = [H_r]_\delta$ of H_r has measure $< \tau$ $(r = 1, 2, 3)$. Finally, let $\delta_1 = \delta_1(\pi)$ be a number such that $|T(w) - T(w')| < \delta$ for all $w, w' \in \pi$, $|w - w'| < \delta_1(\pi)$.

By (S.A.; 19.7) for each $\alpha \in [\alpha]_\pi^0$ we can now define a closed polygonal region R_α with $g \subset R_\alpha^0$, $R_\alpha^* \subset \alpha$, such that, if $R_\alpha = (r_0, r_1, \ldots, r_\nu)$, then r_0^* separates g from U^* and each $r_1^*, \ldots, r_\nu^*$ separates g from one continuum $k \in [k]_\alpha - k_0$ and hence from the regions $\alpha' \in [\alpha]_\pi'$ considered above. We may also suppose that each point $w \in r^*$ is at a distance $< \delta_1(\pi)$ from the corresponding set $k \in [k]_\alpha$, and that $\| c,\ \Gamma \| < \delta = \delta_0(\pi)$, where c: (T, r^*), and Γ: $(T, [E_w])$ is the curve Γ relative to k. Then we have $O(p;\Gamma_r) = O(p;c_r)$ for all $p \in E_{2r} - H_{r\delta}$, and then

$$|\Omega_r - u(r,\ T_r)| = |(E_{2r}) \int [O(p;\ \Gamma_r) - O(p;\ c_r)]| \leqslant$$

$$\leqslant (H_{r\delta}) \int |O(p;\ \Gamma_r)| + (H_{r\delta}) \int N(p;\ T_r,\ \pi) < 2 M_1^{-1} \sigma^4 .$$

Thus

$$|u(r,\ T_r)| \leqslant |\Omega_r| + 2 M_1^{-1} \sigma^4 \qquad (r = 1, 2, 3),$$

and, by (C 4), we have also

$$\text{(C 5)} \qquad \sum_\pi \sum^* |u(r,\ T_r)| \leqslant 2\sigma^4 + 2\sigma^4 = 4\sigma^4 \qquad (r = 1, 2, 3).$$

Let us denote by $[r]_\pi$ the total collection of all regions $r = r_i$ $(i = 1, 2, \ldots, \nu)$ relative to the polygonal regions R_α of the sets $[\alpha]^0_\pi$. Then the regions $r \in [r]_\pi$ are all disjoint and hence $R = \pi - \sum r$ is a polygonal region, where $\sum$ ranges over all $r \in [r]_\pi$.

We shall now consider an arbitrary finite subdivision $S = S' + S''$ of R into simple polygonal regions $q' \in S'$ and nonsimple polygonal regions $q'' \in S''$ whose indices d, m, σ with respect to T are $< \delta = \delta_0(\pi)$ (S.A.; 21.1.i).

If we denote by B_r the sets $B_r = \sum' T_r(q'^*) - T_r(R^*)$, where $\sum'$ ranges over all $q' \in S'$, we can suppose $|B_r| < \delta$ (S.A.; 21.1, and 12.6, Note 2). Finally, if B_{0r} are the sets corresponding to the set B_0 of (S.A.; 12.6), we have also $|B_{0r}| < 4\delta$. Here B_{0r} is the finite sum of circles $\gamma \subset E_{2r}$, each containing at least one curve, say c''_r, image under T_r, of a boundary curve of the regions $q'' \in S''$. Thus $O(p;\, c''_r) = 0$ for every $p \in E_{2r} - B_{0r}$ and for every curve c''_r.

For each $q' \in S'$, $q'' = (q_0, q_1, \ldots, q_\nu) \in S''$ and $r \in [r]_\pi$ let us define the following oriented curves: c'_r: $(T_r,\, q'^*)$, $q' \in S'$; c''_{ri}: $(T_r,\, q_i^*)$, $q_i \in q'' = (q_0, q_1, \ldots, q_\nu) \in S''$; c'''_r: $(T_r,\, r)$ $r \in [r]_\pi$. Then we have $[c'_r] \subset B_r + H_{r\delta} + [C_r]$, $[c''_{ri}] \subset B_{0r}$, $[c'''_r] \subset H_{r\delta}$. By using (S.A.; 8.6.i) we can show that

$$O(p;\, C_r) = \sum' O(p;\, c'_r) + \sum'' [O(p;\, c''_{r0}) - O(p;\, c''_{r1}) - \\ - O(p;\, c''_{r2}) - \ldots - O(p;\, c''_{r\nu})] + \sum''' O(p;\, c'''_r)$$

for all $p \notin [C_r] + B_r + B_{0r} + H_{r\delta}$ $(r = 1, 2, 3)$, where $\sum'$, $\sum''$, $\sum'''$ range over all $q' \in S'$, $q'' \in S''$ and $r \in [r]_\pi$. But $\sum'' [O(p;\, c''_{r0}) - O(p;\, c''_{r1}) - \ldots - O(p;\, c''_{r\nu})] = 0$ for the same p since each term of the sum is zero. Hence $O(p;\, C_r) = \sum' (p;\, c'_r) + \sum''' O(p;\, c'''_r)$ for all $p \notin [C_r] + B_r + B_{0r} + H_{r\delta}$ $(r = 1, 2, 3)$. By integration we have now

$$u(\pi,\, T_r) = (E_{2r}) \int O(p;\, C_r) = (B_r + B_{0r} + H_{r\delta} + [C_r]) \int O(p;\, C_r) + \\ + (E_{2r} - B_r - B_{0r} - H_{r\delta} - [C_r]) \int O(p;\, C_r) =$$

$$= (B_r + B_{0r} + H_{r\delta} + [C_r]) \int O(p;\, C_r) + \\ + (E_{2r} - B_r - B_{0r} - H_{r\delta} - [C_r]) \int [\sum' O(p;\, c'_r) + \sum''' O(p;\, c'''_r)] =$$

$$= (B_r + B_{0r} + H_{r\delta} + [C_r]) \int O(p;\, C_r) + \\ + (E_{2r} - B_r - B_{0r} - H_{r\delta} - [C_r]) \int \sum' O(p;\, c'_r) + (E_{2r}) \int \sum''' O(p;\, c'''_r) - \\ - (B_r + B_{0r} + H_{r\delta} + [C_r]) \int \sum''' O(p;\, c'''_r).$$

Finally, we have

$$| u(\pi, T_r) - (E_{2r}) \int \sum''' O(p; c_r''') | \leqslant$$

$$\leqslant (B_r + B_{0r} + H_{r\delta} + [C_r]) \int | O(p; C_r) | + (E_{2r}) \int N(p; T_r; R) +$$

$$+ (B_r + B_{0r} + H_{r\delta} + [C_r]) \int N(p; T_r, \pi) .$$

Since $| B_r + B_{0r} + H_{r\delta} | < \delta + 4\delta + \delta = 6\delta$, $\delta = \delta_0(\pi)$, and $W(R, T_r) = V(R, T_r)$, we have

$$| u(\pi, T_r) - \sum''' u(r, T_r) | < 12 M^{-1} \sigma^4 + V(R, T_r) + ([C_r]) \int N(p; T_r, \pi)$$

$$(r = 1, 2, 3) .$$

Hence, we ha·e

$$| u(\pi, T_r) | \leqslant \sum''' | u(r, T_r) | + 12 M^{-1} \sigma^4 + V(R, T_r) + ([C_r]) \int N(p; T_r, \pi)$$

$$(r = 1, 2, 3) .$$

Finally by (C5) we have also:

$$\sum_\pi | u(\pi, T_r) | \leqslant$$

$$\leqslant \sum_\pi \sum''' | u(\pi, T_r) | + 12 \sigma^4 + \sum_\pi V(R, T_r) + \sum_\pi ([C_r]) \int N(p; T_r, \pi) \leqslant$$

$$\leqslant 16 \sigma^4 + \sum_\pi V(R, T_r) + \sum_\pi ([C_r]) \int N(p; T_r, \pi) \leqslant$$

$$\leqslant 16 \sigma^4 + \sum_\pi V(R, T_r) + \sum_\pi (\sum_\pi [C_r]) \int N(p; T_r, \pi) \leqslant$$

$$\leqslant 16 \sigma^4 + \sum_\pi V(R, T_r) + (\sum_\pi [C_r]) \int \sum_\pi N(p; T_r, \pi) \leqslant$$

$$\leqslant 16 \sigma^4 + \sum_\pi V(R, T_r) + (\sum_\pi [C_r]) \int N(p; T_r, A) .$$

Observe here that $| \sum_\pi [C_r] | \leqslant m_r \leqslant m$ and that this index is $< \delta$, where $\delta = \delta(A, T, \sigma^4)$ of (S.A.; 23.1.i). In the proof of (S.A.; 23.1.i), this δ was chosen in such a manner that for all measurable sets $h \subset E_{2r}$ with $| h | < \delta$ we have $(h) \int N(p; T_r, A) < \sigma^4$ $(r = 1, 2, 3)$. Hence we have that

(C6) $$\sum_\pi | u(\pi, T_r) | < 17 \sigma^4 + \sum_\pi V(R, T_r) \quad (r = 1, 2, 3) .$$

By (S.A.; 12.14) we have now

$$\text{(C7)} \qquad V(R,\ T_r) + \sum{}' V(\alpha,\ T_r) \leqslant V(\pi,\ T_r) \qquad (r = 1,\ 2,\ 3),$$

where $\sum{}'$ denotes any sum ranging over all sets $\alpha \in [\alpha]'_{\pi}$.

Finally by comparing (C6) and (C7), and by force of (C12), we have

$$\sum_{\pi} \sum{}' V(\alpha,\ T'_3) \leqslant \sum_{\pi} \sum{}' V(\alpha,\ T') = \sum_{\pi} \sum{}' V(\alpha,\ T) \leqslant$$

$$\leqslant \sum_r \sum_{\pi} \sum{}' V(\alpha,\ T_r) \leqslant \sum_r \sum_{\pi} [V(\pi,\ T_r) - V(R,\ T_r)] =$$

$$= \sum_r [\sum_{\pi} V(\pi,\ T_r) - \sum_{\pi} V(R,\ T_r)] \leqslant \sum_r [\sum_{\pi} V(\pi,\ T_r) + 17\ \sigma^4 - \sum_{\pi} |\ u(\pi,\ T_r)\ |] \leqslant$$

$$\leqslant \sum_r [V(A,\ T_r) - \sum_{\pi} |\ u(\pi,\ T_r)\ |] + 51\ \sigma^4 =$$

$$= \sum_r [U(A,\ T_r) - \sum_{\pi} |\ u(\pi,\ T_r)\ |] + 51\ \sigma^4 < 3\ \sigma^4 + 51\ \sigma^4 = 54\ \sigma^4.$$

Thus we have proved that

$$\sum_{\pi} \sum{}' V(\alpha,\ T'_3) < 54\ \sigma^4.$$

Remark 4:

In (S.A.; 23.7) formula (21) is referred to (S.A.; 20.5.i) (CAVALIERI inequality). As a matter of fact the latter is proved in (S.A.; 20.5) for a simple polygonal region π, while formula (21) concerns an open set $\alpha_0 \subset \pi$. Nevertheless, for every t, the set $\alpha_0\ \mathfrak{F}[D^-(t)]$ is closed, where

$$D^-(t) = [w \in \pi, \quad f[T(w)] < t] \quad \text{and} \quad \mathfrak{F}[D^-(t)] = \overline{D^-(t)} - D^-(t),$$

and formula (21) can be proved by the same process used in (S.A.; 20.5), where the generalized length of the contour $C(t,\ \alpha_0)$ relative to α_0 is defined to be the generalized length of the contour $C(t,\ \pi)$ relative to π restricted to α_0. For an extension of the CAVALIERI inequality for mappings from any admissible set, in particular from any open set, see « T. NISHIURA, *On the Cesari-Cavalieri inequality* », to appear.