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Stresses Due to a Nucleus of Thermo-elastic Strain in an Infinite Slab of Isotropic Material with Two Fixed Plane Faces.

Introduction.

The problem of an isotropic semi-infinite solid with a free plane surface and having a nucleus of thermo-elastic strain inside it was solved by different methods by MINDLIN and CHENG (1950) and SEN (1951). The object of this paper is to find the distribution of stresses in an infinite slab of isotropic material when a nucleus of thermo-elastic strain is situated inside it while its plane faces are rigidly fixed. A simple direct method of solving the equations of equilibrium has been used to find the necessary displacements and stresses.

Method of solution.

We assume that the boundaries of the slab are given by the planes $z = 0$ and $z = b + c$ while the nucleus of thermo-elastic strain is supposed to be operative on an element $d\Omega$ at the point $(0, 0, c)$. Then the components of displacement due to this nucleus are given by

$$u = \frac{\partial \psi}{\partial x}, \qquad v = \frac{\partial \psi}{\partial y}, \qquad w = \frac{\partial \psi}{\partial z}, \tag{1.1}$$

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where

$$\psi = -P/R, \tag{1.2}$$

P standing for the constant (GOODIER, 1937)

$$-\frac{1+\sigma}{1-\sigma}\frac{\alpha T}{4\pi}$$

and R being given by the relation

$$R^2 = x^2 + y^2 + (z - c)^2. \tag{1.3}$$

In the above expression T is the temperature of the element $d\Omega$, σ is POISSON's ratio and α the coefficient of linear expansion for heat.

From relations (1.1) we get the displacement components at the surface $z = 0$ as

$$\begin{aligned} u_1 &= \frac{Px}{(x^2+y^2+c^2)^{3/2}} = P\int_0^\infty e^{-ck}\frac{\partial}{\partial x}J_0(kr)\,dk \\ v_1 &= \frac{Py}{(x^2+y^2+c^2)^{3/2}} = P\int_0^\infty e^{-ck}\frac{\partial}{\partial y}J_0(kr)\,dk \\ w_1 &= -\frac{Pc}{(x^2+y^2+c^2)^{3/2}} = P\int_0^\infty k\,e^{-ck}\;J_0(kr)\,dk\,. \end{aligned} \tag{1.4}$$

At the surface $z = b + c$, the displacement components are

$$\left\{\begin{aligned} u_2 &= \frac{Px}{(x^2+y^2+b^2)^{3/2}} = P\int_0^\infty e^{-bk}\frac{\partial}{\partial x}J_0(kr)\,dk \\ v_2 &= \frac{Py}{(x^2+y^2+b^2)^{3/2}} = P\int_0^\infty e^{-bk}\frac{\partial}{\partial y}J_0(kr)\,dk \\ w_2 &= \frac{Pb}{(x^2+y^2+b^2)^{3/2}} = -P\int_0^\infty k\,e^{-bk}\;J_0(kr)\,dk\,. \end{aligned}\right. \tag{1.5}$$

As the two-plane boundaries are kept fixed we shall make the displacement components as obtained in relations (1.4) and (1.5) vanish by the superposition of a complementary displacement system obtained on the hypothesis that there is no temperature distribution.

The equations of equilibrium in terms of displacements are

$$\left\{\begin{aligned} &\nabla^2 u + \frac{1}{1-2\sigma}\frac{\partial \Delta}{\partial x} = 0 \\ &\nabla^2 v + \frac{1}{1-2\sigma}\frac{\partial \Delta}{\partial y} = 0 \\ &\nabla^2 w + \frac{1}{1-2\sigma}\frac{\partial \Delta}{\partial z} = 0\,, \end{aligned}\right. \tag{1.6}$$

where Δ is the dilatation.

Let

$$\Delta = -2(1-2\sigma)\frac{\partial F}{\partial z} \tag{1.7}$$

where F is a harmonic function.

Then the relations (1.6) reduce to

$$\nabla^2 u = 2\frac{\partial^2 F}{\partial x\,\partial z}\,, \qquad \nabla^2 v = 2\frac{\partial^2 F}{\partial y\,\partial z}\,, \qquad \nabla^2 w = 2\frac{\partial^2 F}{\partial z^2}\,. \tag{1.8}$$

Since

$$\nabla^2\left(z\,\frac{\partial F}{\partial x}\right) = 2\frac{\partial^2 F}{\partial x\,\partial z}\,, \qquad \nabla^2\left(z\,\frac{\partial F}{\partial y}\right) = 2\frac{\partial^2 F}{\partial y\,\partial z}\,, \qquad \nabla^2\left(z\,\frac{\partial F}{\partial z}\right) = 2\frac{\partial^2 F}{\partial z^2}\,,$$

solutions of equation (1.8) will be

$$u = z\frac{\partial F}{\partial x} + \Phi_1, \qquad v = z\frac{\partial F}{\partial y} + \Phi_2, \qquad w = z\frac{\partial F}{\partial z} + \Phi_3, \tag{1.9}$$

where Φ_1, Φ_2 and Φ_3 are harmonic functions.

Now,

$$-2(1-2\sigma)\frac{\partial F}{\partial z} = \Delta =$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = z\,\nabla^2 F + \frac{\partial F}{\partial z} + \left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} + \frac{\partial \Phi_3}{\partial z}\right).$$

Thus

(1.10) $$\frac{\partial F}{\partial z} = -\frac{1}{3-4\sigma}\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} + \frac{\partial \Phi_3}{\partial z}\right).$$

Assuming

$$\Phi_1 = \int_0^\infty [A(k)\ \cosh kz\ +\ B(k)\ \sinh kz]\frac{\partial}{\partial x}\ J_0(kr)\ \mathrm{d}k,$$

$$\Phi_2 = \int_0^\infty [A(k)\ \cosh kz\ +\ B(k)\ \sinh kz]\ \frac{\partial}{\partial y}\ J_0(kr)\ \mathrm{d}k,$$

$$\Phi_3 = \int_0^\infty k\ [C(k)\ \cosh kz\ +\ D(k)\ \sinh kz]\ \ J_0(kr)\ \mathrm{d}k,$$

we have

(1.11) $$\frac{\partial F}{\partial z} = -\frac{1}{3-4\sigma}\int_0^\infty k^2\,[C(k)\ \sinh kz + D(k)\ \cosh kz -$$

$$- A(k)\ \cosh kz - B(k)\ \sinh kz]\,J_0(kr)\ \mathrm{d}k.$$

So,

(1.12) $$F = -\frac{1}{3-4\sigma}\int_0^\infty k\ [C(k)\ \cosh kz + D(k)\ \sinh kz -$$

$$- A(k)\ \sinh kz - B(k)\ \cosh kz]\ J_0(kr)\ \mathrm{d}k.$$

So from the relations (1.9) and (1.12) we get the displacement components as

$$(1.13)\quad \begin{cases} u = -\dfrac{1}{3-4\sigma}\displaystyle\int_0^\infty [kz\{(C(k)-B(k))\cosh kz+(D(k)-A(k))\sinh kz\}- \\ \qquad -(3-4\sigma)\{A(k)\cosh kz+B(k)\sinh kz\}]\dfrac{\partial}{\partial x}J_0(kr)\,dk\,, \\ v = -\dfrac{1}{3-4\sigma}\displaystyle\int_0^\infty [kz\{(C(k)-B(k))\cosh kz+(D(k)-A(k))\sinh kz\}- \\ \qquad -(3-4\sigma)\{A(k)\cosh kz+B(k)\sinh kz\}]\dfrac{\partial}{\partial y}J_0(kr)\,dk\,, \\ w = -\dfrac{1}{3-4\sigma}\displaystyle\int_0^\infty [k^2z\{(C(k)-B(k))\sinh kz+(D(k)-A(k))\cosh kz\}- \\ \qquad -(3-4\sigma)\{C(k)\cosh kz+D(k)\sinh kz\}]J_0(kr)\,dk\,. \end{cases}$$

As the two plane boundaries are fixed, the displacement components vanish at $z=0$ and $z=b+c$. So we obtain from relations (1.4), (1.5) and (1.13):

$$(1.14)\qquad A(k) = -Pe^{-ck}, \qquad C(k) = -Pe^{-ck},$$

$$(1.15)\quad \begin{cases} [k(b+c)\{C(k)-B(k)\}-(3-4\sigma)A(k)]\cosh k(b+c) + \\ \qquad +[k(b+c)\{D(k)-A(k)\}-(3-4\sigma)B(k)]\sinh k(b+c) = \\ \qquad\qquad = (3-4\sigma)Pe^{-bk}\,, \\ [k(b+c)\{D(k)-A(k)\}-(3-4\sigma)C(k)]\cosh k(b+c) + \\ \qquad +[k(b+c)\{C(k)-B(k)\}-(3-4\sigma)D(k)]\sinh k(b+c) = \\ \qquad\qquad = -(3-4\sigma)Pe^{-bk}. \end{cases}$$

The relations (1.15) are satisfied if we write

$$(1.16)\quad \begin{cases} B(k) = \big[P\{k^2(b+c)^2e^{-ck} + 2(3-4\sigma)k(b+c)\sinh ck - \\ \qquad -(3-4\sigma)^2\big(e^{-bk} - e^{-ck}\cosh k(b+c)\big)\sinh k(b+c)\}\big]/ \\ \qquad /\{(3-4\sigma)^2\sinh^2 k(b+c) - k^2(b+c)^2\}, \\ D(k) = \big[P\{k^2(b+c)^2e^{-ck} + 2(3-4\sigma)k(b+c)\cosh ck + \\ \qquad +(3-4\sigma)^2\big(e^{-bk} + e^{-ck}\cosh k(b+c)\big)\sinh k(b+c)\}\big]/ \\ \qquad /\{(3-4\sigma)^2\sinh^2 k(b+c) - k^2(b+c)^2\}. \end{cases}$$

So the normal stress due to the superimposed system is given by

$$(1.17)\quad \begin{cases} \widehat{zz_1} = \dfrac{E}{(1+\sigma)(1-2\sigma)}[(1-2\sigma)e_{zz} + \Delta\sigma] = \\ = -\dfrac{E}{1+\sigma}\displaystyle\int_0^\infty \big[\{k^2(1-2\sigma)/(3-4\sigma)\}\{\big(C(k)-B(k)\big)\sinh kz + \\ \qquad + \big(D(k)-A(k)\big)\cosh kz\} + \\ +\{k^3z/(3-4\sigma)\}\{\big(C(k)-B(k)\big)\cosh kz + \big(D(k)-A(k)\big)\sinh kz\} - \\ \qquad - k^2\{C(k)\sinh kz + D(k)\cosh kz\}\big]\, J_0(kr)\, dk. \end{cases}$$

From relations (1.4) we can deduce the normal stress due to the nucleus of strain:

$$(1.18)\quad \widehat{zz_2} = \frac{E}{(1+\sigma)(1-2\sigma)}[(1-2\sigma)e_{zz} + \Delta\sigma] = \frac{PE}{1+\sigma}\,\frac{r^2 - 2(z-c)^2}{[r^2+(z-c)^2]^{5/2}}.$$

The total normal stress is therefore given by the relations (1.17) and (1.18):

$$(1.19)\quad \widehat{zz} = \widehat{zz_1} + \widehat{zz_2}.$$

To have an idea how the stress $(\widehat{zz})$ at the surface $z = 0$ and $z = b + c$ varies with different values of r we calculate the integral (1.19) numerically and suppose

$$b = c = 1; \qquad E = 0{,}001; \qquad \sigma = 0{,}3 .$$

The results of calculation for the stress $(\widehat{zz})$ at the surface $z = 0$ are given in Table I.

Table I

$r =$	0	0,25	0,5	0,75	1
$(\widehat{zz}/P)_{z=0} =$	$-0{,}00024$	$-0{,}0002$	0,00013	0,00034	0,00026 .

From symmetry we shall get similar results on the other plane face.

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Summary: **In this paper stresses in an isotropic solid with two fixed plane boundaries due to a thermo-elastic strain have been obtained by a simple method.**