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On Generalized Darboux Lines. (**)

1. – A generalization of DARBOUX lines, recently suggested by GOROWARA [1], can be obtained by considering, together with the surface S, an arbitrary line congruence K, and determining the lines l on S at every point P of which the center of the osculating sphere of l in P lies on the ray of K through P. DARBOUX lines are included amongst these curves and correspond to the choice of the congruence of normals as congruence K.

Although a family of curves obtained in this manner will not, in general, reflect any intrinsic geometrical aspect of the surface S whatsoever (since, given a random family of ∞^1 curves on the surface S, it is sufficient to join each point of each curve with the center of the corresponding osculating sphere to obtain a line congruence with respect to which the curves are generalized DARBOUX lines), some interesting properties can be found by choosing the congruence K conveniently.

2. – Let the vector function

$$\vec{x} = \vec{x}(u,\ v) \tag{2.1}$$

represent the surface S and let S be the base surface of the congruence K.

$$\vec{k} = \vec{k}(u,\ v) \tag{2.2}$$

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(1) K. K. GOROWARA, *General and hyper Darboux lines*, Riv. Mat. Univ. Parma **6** (1955), 301-317.

being the unit vector of the ray through P, we may write

$$\vec{r} = \vec{x}(u, v) + w\vec{k}(u, v)\,. \tag{2.3}$$

Consider now a curve l, traced on S and passing through P: if dots are to denote derivation with respect to the arc length s of l, l being given as $u = u(s)$, $v = v(s)$ on S, we have

$$\vec{x}^{\cdot} = \vec{t}, \tag{2.4}$$

$$\vec{x}^{\cdot\cdot} = \varrho\vec{n}, \tag{2.5}$$

$$\vec{x}^{\cdot\cdot\cdot} = -\varrho^2\vec{t} + \varrho^{\cdot}\vec{n} + \varrho\tau\vec{b}, \tag{2.6}$$

where $\vec{t}$, $\vec{n}$, $\vec{b}$ are the unit vectors of the tangent, principal normal and binormal of the curve l and $\varrho = 1/R$, $\tau = 1/T$ are its curvature and torsion. The vector $\vec{c}$, joining P with the center of the osculating sphere, C, is then given by

$$\vec{c} = R\vec{n} + R^{\cdot}T\vec{b} \tag{2.7}$$

and consequently

$$\vec{x}^{\cdot}.\vec{c} = 0, \qquad \vec{x}^{\cdot\cdot\cdot}.\vec{c} = 0, \tag{2.8}$$

as shown elsewhere by ŞEMIN (²).

Now, if $\vec{c}$ is to lie on the ray of K passing through P, the unit vector of the ray, $\vec{k}$, must be parallel to $\vec{c}$: consequently it will be orthogonal both to $\vec{x}^{\cdot}$ to $\vec{x}^{\cdot\cdot\cdot}$, so we may write

$$\vec{k}\,.\,\vec{x}^{\cdot} = 0, \tag{2.9}$$

$$\vec{k}\,.\,\vec{x}^{\cdot\cdot\cdot} = 0. \tag{2.10}$$

(²) F. ŞEMIN, *On Darboux lines*, Rev. Fac. Sci. Univ. Istanbul (A) **17** (1952), 351-383.

This system of differential equations defines the GD*-lines* (as we shall, from now on, call generalized DARBOUX lines) *of the surface S with respect to the congruence K, straight lines and circles excepted* (3).

As for the exceptions contained in the previous statement, since the osculating sphere of both straight lines and circles are indeterminate, *straight lines and circles drawn on S are* GD*-lines, provided the rays of the congruence K through each point of any such line or circle cut it under right angles.*

Similar direct considerations can yield further results: for instance, since the osculating sphere of a plane curve (different from a circle) coincides with the plane which contains it, *a plane non-circular curve on S is a* GD*-line of S only and only if the rays of the congruence K through each of its points stay normal to the curve's plane.*

Furthermore, as the center of the osculating sphere of a MONGE curve is its center of curvature, we find that *a Monge curve (different from a circle) belonging to S is a* GD*-lines of S only and only if the rays of the congruence K through each of its points coincide with the curve's principal normals.*

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(3) This result, obtained simply enough above, can be worked out analytically in full detail. Because of (2.8), $\vec{c}$ can be expressed in the form

$$\text{(a)} \qquad \vec{c} = g(s)(\vec{x}^{\cdot} \wedge \vec{x}^{\cdots}) \qquad [g(s) \neq 0]$$

provided that

$$\text{(b)} \qquad \vec{x}^{\cdot} \wedge \vec{x}^{\cdots} \neq 0\,.$$

Equations (2.4) and (2.6) show that this last condition is not verified by straight lines (for which $\vec{x}^{\cdots} = 0$) and circles (for which $\vec{x}^{\cdots} \neq 0$). Consequently, straight lines and circles excepted, the equation for GD-lines,

$$\vec{c} \wedge \vec{k} = 0,$$

can be rewritten by means of (a): we then get

$$(\vec{k}\,.\,\vec{x}^{\cdot})\vec{x}^{\cdots} - (\vec{k}\,.\,\vec{x}^{\cdots})\vec{x}^{\cdot} = 0,$$

and since a linear relation between $\vec{x}^{\cdot}$ and $\vec{x}^{\cdots}$ obviously contradicts (b), this last equation reduces to the two scalar equations (2.9) and (2.10).

3. – Reverting to equations (2.9) and (2.10), let us point out that for DARBOUX lines, where $\vec{k}$ is normal to S, the first equation of the above system is identically verified, whatever the nature of the curve on S, and equation

$$\vec{k}\,.\,\vec{x}^{\cdots} = 0$$

(ŞEMIN's equation) is sufficient to define such curves. It is however insufficient to solve the problem in the general case of GD-lines, hence GOROWARA's assumption (4) is incorrect.

If the curve l is referred to a parameter t different from its arc-length, equations (2.9) and (2.10) can be easily obtained in their more general form (5)

$$\vec{k}\,.\,\vec{x}' = 0, \tag{3.1}$$

$$(\vec{x}'^2)(\vec{k}\,.\,\vec{x}''') - 3(\vec{x}'\,.\,\vec{x}'')(\vec{k}\,.\,\vec{x}'') = 0, \tag{3.2}$$

where dashes indicate derivation with respect to t.

It can be easily proved (6) that the second differential equation of the system (2.9), (2.10) [or that of their more general equivalents (3.1), (3.2)] is at most of order two. Since the first is always of order one, it defines a direction (a tangent element) at each point of the surface, while usually the second equation, if considered by itself, defines a simple infinity of tangent elements at each point. Consequently, in the normal case for a given surface-congruence system, we will obtain a family of ∞^1 GD-lines. If however equation (2.10) [and consequently equation (3.2)], for one reason or another reduces to a first order equation, it will define at each point a tangent element of its own, and then a question of compatibility (coincidence of the two tangent elements thus obtained) will arise: it may happen that the surface congruence system has no GD-lines. Similarly if equation (2.9) [and similarly (3.1)] vanishes, the surface-congruence system may have ∞^2 GD-lines, as is the case with DARBOUX lines.

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(4) Loc. cit. in (1), formula (1.7), p. 302.
(5) Loc. cit. in (2), p. 357.
(6) Loc. cit. in (2), formula (4.7), p. 357.

4. – Independently from the above considerations on the solution of the differential equation system for GD-lines, let us take a DARBOUX-RIBAUCOUR trihedral associated with such a curve. Let $\vec{t}$ be the unit vector of the tangent, $\vec{N}$ that of the surface normal and $\vec{T}$ that of the tangent to the surface S, transverse to $\vec{t}$ and such that the trihedral $\vec{t}$, $\vec{T}$, $\vec{N}$ is right-handed. The unit vector $\vec{k}$ through P can be expressed in terms of these vectors, in the form

$$\vec{k} = m_1 \vec{t} + m_2 \vec{T} + m_3 \vec{N} . \tag{4.1}$$

Since the curve under consideration is admittedly a GD-line, it must verify both equation (2.9) and equation (2.10). The former implies that $m_1 = 0$, so that the ray of the congruence through P will lie in the normal plane of the curve. If φ is the angle between the normal $\vec{N}$ of the surface in P and the ray of the congruence, we nay rewrite (4.1) in the following manner:

$$\vec{k} = \vec{T} \sin\varphi + \vec{N} \cos\varphi . \tag{4.2}$$

On the other hand

$$\vec{x}^{\cdot} = \vec{t} \tag{4.3}$$

and on derivating we have

$$\vec{x}^{\cdot\cdot} = \varrho_g \vec{T} + \varrho_n \vec{N}, \tag{4.4}$$

$$\vec{x}^{\cdot\cdot\cdot} = -\varrho^2 \vec{t} + (\dot{\varrho}_g - \varrho_n \tau_g)\vec{T} + (\dot{\varrho}_n + \varrho_g \tau_g)\vec{N}, \tag{4.5}$$

where ϱ_g, ϱ_n and τ_g are respectively the geodesic curvature, the normal curvature and the geodesic torsion of the curve. Then the second equation of the system reduces to

$$(\dot{\varrho}_g - \varrho_n \tau_g) \sin\varphi + (\dot{\varrho}_n + \varrho_g \tau_g) \cos\varphi = 0 . \tag{4.6}$$

The various remarks on plane curves and MONGE curves which are also GD-lines, formulated in paragraph **2**, can be deduced analytically from this last formula, which can easily be rewritten in the form

$$\varrho^{\cdot} \cos(\theta - \varphi) + \varrho\tau \sin(\theta - \varphi) = 0 \tag{4.7}$$

by means of the expressions

$$(4.8) \qquad \varrho_n = \varrho \cos \theta, \qquad \varrho_g = \varrho \sin \theta, \qquad \tau_g = \tau + \theta^{\cdot},$$

θ being the angle formed by the surface normal in P and the principal normal of the curve in the same point.

When the congruence K coincides with the congruence of normals, $\sin\varphi = 0$, and equation (4.6) becomes

$$(4.9) \qquad \dot{\varrho}_n + \varrho_g \tau_g = 0,$$

which is the form in which it is to be found in ȘEMIN's paper (7) on DARBOUX lines.

If, on the other hand, the rays of the congruence K are tangent in P to the surface S, we obtain a further class of special GD-lines, called by GOROWARA (8) *TD-lines*, which are simultaneously a sub-class of the family of curves whose osculating spheres cut S orthogonally in P, and which have been considered previously by S. ÖZTÜRK (9). For such curves $\cos\varphi = 0$, so equation (4.6) becomes

$$(4.10) \qquad \dot{\varrho}_g - \varrho_n \tau_g = 0 .$$

5. – Let us first rewrite the previous equation for TD-lines

$$(5.1) \qquad \dot{\varrho}_g - \varrho_n \tau_g = 0$$

and suppose that a TD-line is simultaneously an asymptotic line of the surface S: then together with (5.1), we will have

$$(5.2) \qquad \varrho_n = 0, \qquad \varrho_g = \varrho,$$

so from equation (5.1) it follows that

$$(5.3) \qquad \varrho^{\cdot} = 0$$

(7) Loc. cit. in (2), formula (4.7), p. 359.
(8) Loc. cit. in (1), heading b), p. 304.
(9) S. ÖZTÜRK, unpublished *Habilitation Thesis*, presented in 1955, and unavailable to the present writer.

and this proves that every asymptotic TD-line is a MONGE curve. Conversely, since equation (5.1) can be rewritten in the form

$$\varrho^{\cdot} \sin \theta - \varrho\tau \cos \theta = 0 \tag{5.4}$$

by means of the formulae shown in (4.8), if a TD-line is a MONGE curve we have

$$\varrho\tau \cos \theta = 0 \tag{5.5}$$

and the TD-line is a straight line ($\varrho = 0$), a circle ($\varrho = \text{const.}$, $\tau = 0$) or else an asymptotic line of S.

If a curve on S is both an asymptotic and a MONGE curve, equation (5.1) is identically satisfied but this is *not* sufficient to show that it is a TD-line, precisely because equation (5.1) contains intrinsic invariants of the surface but gives no element about the congruence. Indeed, the curve being a MONGE curve, the center of its osculating sphere lies on its osculating plane, and since it is an asymptotic line this plane is the tangent plane of the surface so that the only result achieved is that the curve has the property that the center of its osculating sphere at any point lies on the tangent plane of S at that point. For the curve to be a TD-line we must further require that the ray through the point considered lie in the tangent plane normally to the tangent of the curve.

We may therefore state the following theorem:

Given a surface S, a congruence K whose rays are tangent to S, and a TD-line on S, each of the following properties:

a) *to be an asymptotic line,*

b) *to be a Monge curve,*

implies for the TD-line the remaining one, straight lines and circles excepted.

6. – Let S be a surface and K the congruence formed by the tangents of one family of its lines of curvature: the orthogonal trajectories of the rays of the congruence lying on S are the lines of curvature of the remaining family, and since for lines of curvature we have

$$\tau_g = 0, \tag{6.1}$$

the condition for the lines of curvature to be TD-lines for S for this choice of K is that

$$\varrho_g = \text{const.} \tag{6.2}$$

which shows that *the condition for a line of curvature to be a TD-line with respect to the congruence formed by the tangents of the lines of curvature of the other family is that its geodesic curvature be constant* (*geodesic circle* in DARBOUX's sense).

On the other hand, the curves under consideration being TD-lines, their radii of geodesic curvature equal their radii of spherical curvature, so that condition (6.2) implies that *lines of curvature which are TD-lines with respect to the congruence formed by the tangents of the lines of curvature belonging to the other family, are spherical curves* (*or else are plane*) (10).

Indeed, curves whose radius of spherical curvature is constant either lie on a sphere, or else have constant curvature. In the latter case the line of curvature, being a TD-line, would, by the previous theorem, be also an asymptotic and consequently must necessarily be plane.

Suppose that conditions (6.2) is imposed to all the lines of curvature of the surface S: then these lines of curvature will form a double family of orthogonal curves with constant geodesic curvature, but this is possible only for surfaces whose linear element can be written in the form (11)

$$ds^2 = \frac{du^2 + dv^2}{[U(u) + V(v)]^2}\,; \tag{6.3}$$

consequently *if the lines of curvature of a surface are TD-lines with respect to the congruences formed by the tangents of the lines of curvature of the other system, the surface must be isometric.*

Furthemore, the surface will have all its lines of curvature spherical; then, taken a surface with ∞^2 plane lines of curvature (for instance a surface of revolution), a surface enjoying the property stated above will be obtained from it by inversion (12).

Consider now a minimal surface S: the asymptotic lines on S form an orthogonal system and if we are to choose as congruence K the tangents to one

(10) This remark was suggested by Prof. F. SEMIN.

(11) L. BIANCHI, **Lezioni di Geometria differenziale**, Vol. **I**, N. Zanichelli, Bologna 1927 (cf. pp. 308-310).

(12) Loc. cit. in (11), pp. 503-504 and p. 530.

family of asymptotics, the asymptotics of the second family constitute the orthogonal trajectories of the rays of K on the surfaces S. Since asymptotic lines are characterized by having

$$\varrho_n = 0, \tag{6.4}$$

the condition for these lines to be TD-lines is again that

$$\varrho = \varrho_g = \text{const.}, \tag{6.5}$$

so *the condition for an asymptotic line belonging to a minimal surface to be a TD-line with respect to the congruence formed by the tangents to the asymptotics belonging to the other family is that its curvature be constant.*

7. – We can now return to the equation

$$(\dot{\varrho_g} - \varrho_n \tau_g) \sin\varphi + (\dot{\varrho_n} + \varrho_g \tau_g) \cos\varphi = 0 \tag{7.1}$$

of the GD-lines of a surface S with respect to a congruence K.

If the GD-line is an asymptotic line on S, we also have

$$\varrho_n = 0, \qquad \cos\theta = 0 \tag{7.2}$$

so

$$\varrho_g = \varrho, \qquad \tau_g = \tau \tag{7.3}$$

and the above equation becomes

$$\varrho^{\cdot} \sin\varphi + \varrho\tau \cos\varphi = 0\,. \tag{7.4}$$

If furthermore the GD-line is a MONGE curve, equation (7.4) becomes

$$\varrho\tau \cos\varphi = 0 \tag{7.5}$$

so that the GD-line is a straight line ($\varrho = 0$), or a circle ($\tau = 0$, $\varrho = $ const.) or else $\cos\varphi = 0$, i.e. $\varphi = \pi/2$. Now, by the BELTRAMI-ENNEPER theorem, the torsion of the two asymptotic lines through a point of a surface is expressed as the function $\pm\sqrt{-K}$ of the Gaussian curvature K, hence *every non-rectilinear asymptotic Mongean GD-line on a surface of constant curvature is a TD-line.*

Similarly, suppose a GD-line is a geodesic of S: then

$$\varrho_g = 0, \qquad \sin\theta = 0 \tag{7.6}$$

so

$$\varrho_n = \varrho, \qquad \tau_g = \tau \tag{7.7}$$

and equation (7.1) becomes

$$\varrho^{\cdot}\cos\varphi - \varrho\tau\sin\varphi = 0\,. \tag{7.8}$$

If furthermore the curve is of constant curvature, we find

$$\varrho\tau\sin\varphi = 0 \tag{7.9}$$

so that the GD-line must reduce to a straight line ($\varrho = 0$), a circle ($\tau = 0$, $\varrho = \text{const.}$) or else $\varphi = 0$. Then *every non-planar geodesic Mongean GD-line is necessarily a Darboux line.*

The results obtained in this paragraph can be considered partial extensions to GD-lines of the theorem on page 6 for TD-lines and of a theorem of ŞEMIN's on DARBOUX lines ([13]).

8. – Some of the results obtained above can be applied to ruled surfaces, to which it is possibile to associate easily various special ray sistems.

Let $\vec{d} = \vec{d}(u)$ be the striction curve of the ruled surface S, referred to its arc-length u, and $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ be the unit vectors of the generator, the surface normal and the surface tangent transverse to the generator (geodesic normal of the generator) in the central point. We have

$$\vec{x}(u,\, v) = \vec{d}(u) + v\,\vec{a}_1(u), \tag{8.1}$$

([13]) Loc. cit. in ([2]), theorem (5.4) on page 362.

so, using BLASCHKE's notations [14], we get

$$\vec{x}^{\cdot} = (\bar{q}u^{\cdot} + v^{\cdot})\vec{a}_1 + u^{\cdot}vp\,\vec{a}_2 + u^{\cdot}\,\bar{p}\,\vec{a}_3\,. \tag{8.2}$$

Consider now the congruence K defined by means of the unit vector $\vec{a}_3$, which is tangent to S in the central point of the generator: the GD-curves on S corresponding to this choice of the line congruence will be given by equations (2.13) and (2.14), where $\vec{k}$ is to be substituted with $\vec{a}_3$. The first of these equations reduces to

$$u^{\cdot}\bar{p} = 0, \tag{8.3}$$

consequently *the GD-lines of an arbitrary ruled surface for the congruence defined by* $\vec{a}_3$ *are the surface's generators.* Indeed, if $u^{\cdot} = 0$ we have

$$\vec{x}^{\cdot} = v^{\cdot}\,\vec{a}_1, \qquad \vec{x}^{\cdot\cdot} = v^{\cdot\cdot}\,\vec{a}_1, \qquad \vec{x}^{\cdot\cdot\cdot} = v^{\cdot\cdot\cdot}\,\vec{a}_1 \tag{8.4}$$

and both equations of the system are identically verified. The above result can also be reached directly, since the congruence K defined by $\vec{a}_3$ is such that its rays cut the generators under right angles and the theorem on straight GD-lines given on page 2 can therefore be applied.

If, however, the ruled surface under consideration is developable, we have $\bar{p} = 0$, and consequently the first equation of the differential system for GD-lines is verified identically: the GD-lines will therefore be given by the remaining second order differential equation

$$u^{\cdot 3}\,pq + 3u^{\cdot 2}\,v^{\cdot}pq + 3vu^{\cdot}\,u^{\cdot\cdot}\,pq + vu^{\cdot 3}\,\frac{d(pq)}{du} + vu^{\cdot 3}\,\frac{dp}{du} = 0, \tag{8.5}$$

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[14] W. BLASCHKE, **Vorlesungen über Differential Geometrie**, Bd **1**, Dover Publications, New York 1945 (cf. pp. 260-277). We have

$$d\vec{a}_1/du = p\,\vec{a}_2, \qquad d\vec{a}_2/du = -p\vec{a}_1 + q\vec{a}_3, \qquad d\vec{a}_3/du = -q\vec{a}_2$$

and the tangent of the striction curve is

$$\vec{t} = \bar{q}\,\vec{a}_1 + \bar{p}\,\vec{a}_3$$

so that $\bar{q} = \cos(\vec{t}, \vec{a}_1)$ and $\bar{p} = \sin(\vec{t}, \vec{a}_1)$.

which, provided $q \neq 0$, can be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}v}\left[\left(1+2\,\frac{v}{p}\,\frac{\mathrm{d}p}{\mathrm{d}u}+\frac{v}{q}\,\frac{\mathrm{d}q}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2+3\,\frac{\mathrm{d}u}{\mathrm{d}v}+3v\,\frac{\mathrm{d}^2u}{\mathrm{d}v^2}\right]=0 \tag{8.6}$$

or again

$$\frac{\mathrm{d}u}{\mathrm{d}v}\left\{[1+v\,A(u)]\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2+3\,\frac{\mathrm{d}u}{\mathrm{d}v}+3v\,\frac{\mathrm{d}^2u}{\mathrm{d}v^2}\right\}=0 \tag{8.7}$$

where

$$A(u)=\frac{2}{p}\,\frac{\mathrm{d}p}{\mathrm{d}u}+\frac{1}{q}\,\frac{\mathrm{d}q}{\mathrm{d}u}\,. \tag{8.8}$$

Equation (8.7) gives the generators of S as one set of the GD-lines of the developable, as was to be excepted by virtue of the theorem quoted above, but the complete family is given by the general solution of equation

$$[1+v\,A(u)]\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2+3\,\frac{\mathrm{d}u}{\mathrm{d}v}+3v\,\frac{\mathrm{d}^2u}{\mathrm{d}v^2}=0: \tag{8.9}$$

therefore *given a developable surface S and the congruence defined by $\vec{a}_3$, there are ∞^1 GD-lines through each point of S.* Of course, the surface being developable, $\vec{a}_3$ defines a congruence whose rays are tangent to the surface in all the points of the generator and consequently these GD-lines are TD-lines.

If $\bar{p}=0$ and $q=0$, then $\vec{a}_3$ is constant, so the surface S is a developable whose generators keep parallel to a plane and is therefore itself a plane ruled by the tangents of one of its curves: both equations (2.13) and (2.14) are verified identically, hence *all* the curves contained in the plane are GD-lines with respect to $\vec{a}_3$.

Let us now take for $\vec{k}$ the unit vector of the central normal of the surface, $\vec{a}_2$, and examine the nature of the GD-lines corresponding to this particular choice of the congruence K. As first equation of our differential system we obtain

$$vu'p=0. \tag{8.10}$$

It is easily seen, both by direct reasoning and from this last equation that again *the GD-lines of an arbitrary ruled surface for the congruence defined by $\vec{a}_2$ are its generators.*

$v = 0$ gives the line of striction of S, which will also be a GD-line if the second equation of the system is verified. Since in this case

$$\vec{x}^{\cdot} = \bar{q}\,\vec{a}_1 + \bar{p}\,\vec{a}_3, \tag{8.11}$$

$$\vec{x}^{\cdot\cdot} = \bar{q}^{\cdot}\,\vec{a}_1 + (p\bar{q} - q\bar{p})\,\vec{a}_2 + \bar{p}^{\cdot}\,\vec{a}_3, \tag{8.12}$$

$$\vec{x}^{\cdot\cdot\cdot} = \dots + [p\bar{q}^{\cdot} + (p\bar{q} - q\bar{p})^{\cdot} - q\bar{p}^{\cdot}]\vec{a}_2 + \dots \tag{8.13}$$

the condition for the line of striction of S to be a GD-*line for the congruence K defined by $\vec{a}_2$ is that*

$$p\bar{q}^{\cdot} + (p\bar{q} - q\bar{p})^{\cdot} - q\bar{p}^{\cdot} = 0\,. \tag{8.14}$$

If this condition is verified, the line of striction of S is a DARBOUX-line, and not only of S, but also of the surface S^* generated by $\vec{a}_3$, which has same line of striction and central normal as S. Since the invariants of S^* are

$$p^* = q, \qquad q^* = p, \qquad \bar{p}^* = \bar{q}, \qquad \bar{q}^* = \bar{p}, \tag{8.15}$$

this last result is confirmed analytically by the symmetry of equation (8.15), which can be rewritten in the form

$$2p\,\bar{q}^{\cdot} + p^{\cdot}\,\bar{q} = 2q\,\bar{p}^{\cdot} + q^{\cdot}\,\bar{p}\,. \tag{8.16}$$

Finally if in equation (8.10) we have $p = 0$, the ruled surface S reduces to a cylinder, the vector $\vec{a}_1$ is constant and is the binormal of any normal section of the surface, while the principal normal $\vec{a}_2$ of this curve is also the surface normal. The GD-lines are then the DARBOUX lines of the cylinder.

As last case, let us assume $\vec{a}_1$ as defining the congruence K: the first equation of the system of differential equations of the GD-lines is then

$$\bar{q}(u)\,\frac{du}{ds} + \frac{dv}{ds} = 0, \tag{8.17}$$

which defines the orthogonal trajectories of the generators. For these to be GD-lines the second equation must be verified: it is given by

$$3v\,p(u)\,\frac{d^2u}{ds^2} + 3v\,\frac{dp}{du} + p(u)\,q(u)\left(\frac{du}{ds}\right)^2 + 2p(u)\,\frac{du}{ds}\,\frac{dv}{ds} = 0 \tag{8.18}$$

and it is readily seen that for (8.17) and (8.18) to be compatible the ruled surface must be specialized. Indeed, since (8.17) and (8.18) can be rewritten as

$$\bar{q}\,\frac{\mathrm{d}u}{\mathrm{d}v} + 1 = 0, \tag{8.19}$$

$$2p\,\frac{\mathrm{d}u}{\mathrm{d}v} + \left(3v\,\frac{\mathrm{d}p}{\mathrm{d}u} + q\bar{p}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 + 3vp\,\frac{\mathrm{d}^2u}{\mathrm{d}v^2} = 0, \tag{8.20}$$

we can deduce from the first of these equations that

$$v = -\int \bar{q}(u)\,\mathrm{d}u, \tag{8.21}$$

$$\mathrm{d}u/\mathrm{d}v = -1/\,\bar{q}, \tag{8.22}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}v^2} = \frac{1}{\bar{q}^2}\,\frac{\mathrm{d}\bar{q}}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}v} = -\,\frac{1}{\bar{q}^3}\,\frac{\mathrm{d}\bar{q}}{\mathrm{d}u} \tag{8.23}$$

and consequently, by substituting these expressions in the second equation we obtain the condition

$$q\,\bar{p} - 2p\,\bar{q}^2 + 3\left(\bar{q}\,.\frac{\mathrm{d}p}{\mathrm{d}u} - p\right)\int \bar{q}\,\mathrm{d}u = 0\,. \tag{8.24}$$

Condition (8.24) shows that *every ruled surface does not possess GD-lines corresponding to the choice of* $\vec{a}_1$ *as the unit vector defining the congruence* K. However, if this condition is satisfied the GD-lines are TD-lines, since the congruence defined by $\vec{a}_1$ is formed of tangents to the surface.

If the ruled surface S is such that its generators cut the line of striction under right angles, $\bar{q} = 0$ and the condition given above reduces to

$$q\bar{p} = 0, \tag{8.25}$$

so that q or $\bar{p}$ must be zero for TD-lines to exist on the surface. In the first case ($q = 0,\ \bar{q} = 0$) the surface is a *right conoid*, in the second ($\bar{p} = 0,\ \bar{q} = 0$) a *cone*, hence *the orthogonal trajectories of the generators of cones and right conoids are TD-lines for the congruence* $\vec{k} = \vec{a}_1$.