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L. C. YOUNG (*)

Partial Area - I. (**)

1. – The geometry of a surface is traditionally based on constructing sufficiently elementary systems of its curves. However, on a surface of finite area, it is by no means evident that there exist even rectifiable curves; and on a generalized surface there is no immediate notion of curve whatsoever. The machinery for remedying this somewhat elusive state of affairs originates in a familiar inequality of elementary analysis, connecting m-and $(m + k)$-dimensional measures, which has been much used in area theory and earlier in dimension theory and in H. A. SCHWARZ's treatment of the isoperimetric property of the sphere.

We shall sharpen this inequality into an equality, one form of which is really nothing more than a special case of FUBINI's theorem, and the equality is highly analogous to the identification of LEBESGUE area with GEÖCZE area. In fact this central identification theorem may perhaps be reached more directly by means of our equality, which is logically simpler.

We shall apply the machinery to generalized surfaces of the type needed in PLATEAU's Problem and we shall derive from it, by means of SCHWARZ sym-

(*) Address: Department of Mathematics, University of Wisconsin, Madison, Wisconsin, U.S.A. .

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metrization, a comparison theorem which furnishes a simple criterion for degeneracy of a surface of least area. Another application is an expression for area, related to well-known but unpublished work of ARONSZAJN and CHOQUET, which we use elsewhere to estimate certain geodesics and, in particular, the minimum perimeter of a disc equivalent to a given surface.

Part I. - Various expressions for the partial area.

2. – We denote by $[p+q]$ Euclidean space of dimension $p+q$, or more precisely the product space of two Euclidean spaces $[p]$ and $[q]$, termed horizontal and vertical projection of $[p+q]$. We write $x=(y, z)$ for a point of $[p+q]$, and $z \in [q]$ is termed the *level* of x; we term horizontal section, or simply *section*, of $[p+q]$ at this level the subset for which z is kept constant. We write $[k]^*$ for a coordinate subspace of $[p+q]$ of dimension k, and we term it horizontal (vertical) if it is either a subspace or a superspace of $[p]$ ($[q]$); otherwise it is non-horizontal (non-vertical).

A function of $[k]^*$ for a fixed k will be termed k-dimensional multivector in $[p+q]$; its value for each $[k]^*$ is termed the corresponding component, and the latter is said to be vertical if $[k]^*$ is so; the vertical components themselves define a multivector, termed vertical projection of the given multivector, and which is actually a k-dimensional one in $[q]$ if $k \leqslant q$ and a $(k-q)$-dimensional one in $[p]$ if $k \geqslant q$. We term norm of a multivector the square root of the sum of the squares of its components, and *partial norm* the norm of its vertical projection.

An ordered set of k vectors defines a k-dimensional multivector, termed their skew product, whose component fo any $[k]^*$ is the determinant of their projections in $[k]^*$. Multivectors occurring in the sequel will be so expressible, or will be vertical components of ones so expressible, unless termed composite. (It is desirable to avoid the overworked adjective simple).

We shall be concerned with Lipschitzian maps into $[p+q]$, from a subset W of an affine m-dimensional hyperplane $[m]$. We shall use Cartesian coordinates for the points w of W, just as for the points x of $[p+q]$; however, the quantities that we ultimately wish to study are easily seen to be unaffected by an *isomorphism* of the spaces $[m]$ and $[p+q]$ into themselves, i.e. by an affine transformation of $[m]$ combined with Euclidean movements in $[p]$ and in $[q]$. By the norm of such an isomorphism we shall mean the greater of the LIPSCHITZ constants of the affine transformation and its inverse. Maps, sets, figures, etc., transformed into one another by an isomorphism will be termed isomorphic. We shall suppose the dimensions fixed and subject to $q \leqslant m \leqslant p+q$.

Given a map $x(w)$, $w \in W$, the skew product of the m partial derivatives will be termed Jacobian and denoted by $J(w)$, the corresponding matrix termed Jacobian matrix and written $\mathfrak{J}(w)$, and the vertical projection of $J(w)$ termed partial Jacobian and written $j(w)$. These quantities will be regarded as defined only wherever $x(w)$ is differentiable (almost everywhere in W if $x(w)$ is Lipschitzian). The integrals over W of $|J(w)|$ and $|j(w)|$ will be termed *classical m-dimensional area* and *partial m-dimensional area*: we shall write them $A(T)$ and $a(T)$, where T is the map $x(w)$, $w \in W$. In the higher dimensions the terms length, area, volume are interchangeable, and the choice of one term in preference to the other two is generally dictated by considerations of analogy. Here length will be used for dimensions $< m$, volumes for dimensions $> m$, and the reference to the dimension m will be dropped.

3. – The terms length, area, partial area will also be used in another context. By a multiple system M in a metric space X, we shall mean a function $M(x)$ termed counting function, or multiplicity function, whose values are cardinal numbers; and for our purposes the infinite cardinals need not be distinguished in this respect. A set of points in X is regarded as a special case, the counting function being then identified with the characteristic function: however, the notation will disagree mildly with standard set theory notations as a result, and to avoid confusion a multiple system arising from one or more sets of points by our operations will be distinguished from a set arising from similar operations in set theory notation, by stating that it is a multiple system.

Union and intersection of multiple systems are defined by addition or multiplication of the counting functions, with the convention that zero takes precedence, i.e. $0 \cdot \infty = \infty \cdot 0 = 0$; and if $\xi(x)$ is a map of X into another metric space Ξ, we denote by $\xi(M)$ the multiple system in Ξ whose counting function at $\xi \in \Xi$ is the sum of the values of $M(x)$ at the distinct points $x \in X$ for which $\xi(x)$ takes this value ξ. We observe associativity: if $\eta[\xi]$ maps further Ξ into H and $\eta(x)$ denotes the combined map $\eta[\xi(x)]$, we have $\eta[\xi(M)] = \eta(M)$.

A multiple system M will be termed measurable (relative to a measure or to an additive class of sets) if each set M_ν is measurable, where M_ν denotes the set of $x \in X$ for which $M(x) \geqslant \nu$, and ν is a positive integer. Any measure μ is extended to such multiple systems by writing $\mu(M) = \sum \mu(M_\nu)$. The set M_1, in particular, will be termed *carrier* of the multiple system M.

Given, for subsets E of X, a non-negative function $g(E)$, termed gauge, we associate with it an outer measure γ given by

$$\gamma(E) = \lim_{\varepsilon \to 0} \gamma(E, \varepsilon), \qquad \gamma(E, \varepsilon) = \operatorname{Inf} \sum g(E_\sigma),$$

where the infimum is for all sequences of sets E_σ of diameters $< \varepsilon$ such that $\bigcup E_\sigma \supset E$.

In Euclidean space, if c_k denotes for any positive integer k the usual measure of the k-dimensional ball of unit diameter, we shall use particularly the gauges

$$g(E) = c_k\, d^k \qquad \text{and} \qquad g(E) = c_q\, c_{m-q}\, (d')^{m-q}\, (d'')^q,$$

where d, d', d'' are the diameters of E and of its horizontal and vertical projections, i.e. the projections in $[p]$ and $[q]$. The corresponding outer measures will be termed, respectively, k-dimensional *length* (or area or volume) and m-dimensional *partial area*; usually k will here be $m - q$ and reference to it will be omitted.

We shall write $A(M)$ and $a(M)$ for the area and for the partial area of a multiple system M, just as for a map. We write further M_z for the section of M at the level z, i.e. for its intersection with the corresponding section of $[p+q]$, and $s(M_z)$ for its length (of dimension $m - q$).

4. – As an immediate corollary of a recent version of FUBINI's theorem [**1**]:

(4.1) *For any multiple system M, measurable with respect to a in $[p+q]$, we have*

$$a(M) = \int s(M_z)\, \mathrm{d}z\,.$$

We shall establish further:

(4.2) *If T is a Lipschitzian map into $[p+q]$ given by $x(w)$, $w \in W$, where W is a Borel set in $[m]$, and if M denotes the multiple system $x(W)$, we have*

$$a(M) = a(T)\,.$$

Theorem (4.2) supplements a theorem of FEDERER [**2**, Th. 4.5, p. 448] which states

(4.3) *With the hypotheses of* (4.2), *we have* $A(M) = A(T)$.

Before proving (4.2), we interpose some preparatory material.

5. – Evidently for a map T we have $A(T) \geqslant a(T)$. The corresponding relation for a multiple system M is, however, far from obvious, and perhaps even false, unless we weaken it by a constant factor $c = c_q\, c_{m-q}/c_m$. It then reads

$$a(M) \leqslant c\, A(M) \tag{5.1}$$

and is clearly valid for BOREL measurable multiple systems M in $[p+q]$, since the gauge used for a does not exceed c times that used for A.

Similarly, since the gauge used for a is not increased when the sets concerned are subjected to a projection P of $[p+q]$ into any flat subspace, we have

$$a(PM) \leqslant a(M)\,. \tag{5.2}$$

Since (5.1) holds in particular for a set of area 0, it follows that

(5.3) A-measurability implies a-measurability.

Similarly, a-measurability of a set, or multiple system, implies that of its projection, and we can extend appropriately the class of multiple systems for which (5.1) or (5.2) is asserted.

Of course in an m-dimensional flat subspace, $A(E)$ agrees with the LEBESGUE measure $|E|$ of dimension m. We observe that this special case of (4.3) implies:

(5.4) An elementary property of the m-dimensional ball: its measure is not less than that of any set in $[m]$ with the same diameter.

For otherwise, with arbitrarily small disjoint replicas of such a set, obtained by change of scale and translation, we could cover, by VITALI's theorem, almost the whole of a simple figure E, and deduce that $A(E) < |E|$.

From (5.4) with m replaced successively by $m-q$ and by q, and A replaced by appropriate lengths, it follows that a similar elementary property holds for the *dicylinder*, i.e. the Cartesian product of a horizontal and a vertical ball of dimensions $m-q$ and q:

(5.5) The measure of the dicylinder is not less than that of any set whose horizontal projection has the same diameter and whose vertical projection has the same diameter.

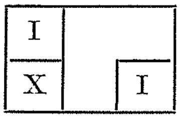
From this last remark we easily derive by VITALI's theorem the equality $a(M) = |M| = A(M)$ for a multiple system M situated in a vertical $[m]^*$. And by applying this equality to a projection PM of an a-measurable multiple system M into a *vertical* $[m]^*$, we find that for such a projection, by (5.2),

$$A(PM) \leqslant a(M)\,. \tag{5.6}$$

It is important ot observe further that with the notation of (4.2),

(5.7) The quantities $a(T)$ and $a(M)$ are unaltered by isomorphism.

A linear map whose Jacobian does not vanish will be termed non-singular, and we then define its vertical rank r to be the rank of the Jaocobian matrix of its vertical projection. The map and its Jacobian matrix $\mathfrak{J}$ will be said to be in *standard form* if the matrix of the first $m-r$ rows and columns of $\mathfrak{J}$ and that of the last r rows and columns are identity matrices I, and the remaining elements of $\mathfrak{J}$ vanish except those of the rectangular matrix X shown in the figure. X is formed of the first $m-r$ columns and the last r rows of $\mathfrak{J}$ and is not subject to any special condition. We term standard isomorphism of our map, one which transforms it into a linear map of the same vertical rank, in standard form: it need not be unique, but we easily verify that, by elementary constructions,



(5.8) We can attach to each non-singular linear map from $[m]$ to $[p+q]$, a corresponding standard isomorphism.

We shall require one more elementary result, in which N, r denote fixed integers $\geqslant 0$, η is an arbitrarily small positive real, and δ a suitable positive real. We denote by Q the dicylinder in $[m]$ defined by the inequalities $|u| \leqslant 1$, $|v| \leqslant \delta$, where u, v consist of the first $m-r$ and the last r coordinates in $[m]$, respectively. We term two maps $x(w)$, $x'(w)$ differentially close in W if the difference $x(w) - x^*(w)$ exists in W and has a Lipschitz constant $\leqslant \delta$. An elementary calculation establishes for a suitable δ the following statement:

(5.9) Let $x(w)$ be differentially close to a non-singular linear map in standard form, whose Lipschitz constant is $\leqslant N$ and whose vertical rank is r, and let E denote the carrier of $x(WQ)$. Then

$$g(E) \leqslant \eta \, |Q| \quad \text{if} \quad r < q, \qquad g(E) \leqslant (1+\eta)\, |Q| \quad \text{if} \quad r = q,$$

where g is the gauge used in defining partial area.

6. – The proof of (4.2) will be in three parts.

1) We observe that, by (4.3) and (5.1), the formula to be established is certainly valid in the following two particular cases: (I) when the Jacobian $J(w)$ of $x(w)$ vanishes almost everywhere in W, and (II) when the counting function $M(x)$ of M is infinite everywhere on its carrier. As a result, we can make some preliminary reductions. We may suppose in the first place that W is bounded, that the counting function $M(x)$ is never infinite, and that $j(w)$ exists and is $\neq 0$ for all $w \in W$. Each subset of W in which $x(w)$ remains constant is then a finite set, and by ordering in familiar fashion the points of $[m]$ we can select from each such subset a first element, and therefore locate in W a Borel subset W' on which $x(w)$ is one-to-one, and for which $x(W')$ coincides

with the carrier of M. Since this implies that W is a countable union of disjoint BOREL sets on each of which $x(w)$ is one-to-one, we need only treat the case in which $x(w)$ is one-to-one on W itself.

Again by the theorems of STEPANOV, LUSIN and EGOROV, we may suppose that W is a closed set on which $x(w)$ is continuously differentiable and the sequence of functions

$$f_\nu(w) = \operatorname*{Sup}_{\substack{|w'-w|<1/\nu \\ w' \neq w,\, w' \in W}} \frac{|x(w') - x(w) - \mathfrak{J}(w)(w'-w)|}{|w'-w|}$$

converges uniformly to zero. Finally we may suppose that there is a fixed positive integer N and an integer r, where $0 \leqslant r \leqslant q$, such that for each $w \in W$ the linear function, of w', $x(w) + \mathfrak{J}(w)(w'-w)$ has an isomorph in accordance with (5.8) in standard form, for which the vertical rank is r, and the LIPSCHITZ constant and the norm of the isomorphism are $\leqslant N$. Since the general case follows by addition, all these additional hypotheses are immaterial.

2) Suppose $r = q$. It will suffice to prove, for an arbitrary small $\eta > 0$, the two inequalities

$$(6.1) \qquad (1+\eta)\, a[x(W)] \geqslant \int_W |j(w)|\, dw \geqslant (1-\eta)\, a[x(W)]\,.$$

We may suppose W contained in a sufficiently small ball $|w - w_0| < \varrho$, and $\mathfrak{J}(w_0)$ in standard form. Since the norm of the isomoprphism which secures this is $\leqslant N$, all the additional hypotheses listed in 1) remain in force, and moreover $|j(w_0)| = 1$.

Let P denote projection of $[p+q]$ onto the vertical $[m]^*$ comprising $[q]$ and the first $m-q$ coordinates in $[p]$. The map $P\,x(w)$ has a Jacobian $J_P(w)$ for which $|J_P(w_0)| = |j(w_0)| = 1$. Hence if ϱ is small $|j(w)| < (1+\eta)\,|J_P(w)|$ throughout W, so that by applying (4.3) to the map $P\,x(w)$ and using (5.6), we find that

$$\int_W |j(w)|\, dw \leqslant (1+\eta) \int_W |J_P(w)|\, dw = (1+\eta)\, A(PM) \leqslant (1+\eta)\, a(M),$$

which establishes the first inequality in (6.1).

We now determine $\delta > 0$ so as to secure, for our values of η, r, N, the validity of (5.9) and ν so that $f_\nu(w) < \delta/2$ for all $w \in W$, and we then choose $\varrho < 1/\nu$ small enough to make

$$(6.2) \qquad |\mathfrak{J}(w) - \mathfrak{J}(w_0)| > \delta/2 \quad \text{and} \quad |j(w) - j(w_0)| > \eta^2 \quad \text{for} \quad -w \in W\,.$$

From the former of these two inequalities, it follows, since $f_\nu(w) < \delta/2$, that

$$| x(w') - x(w) - \mathfrak{J}(w_0)(w' - w) | < \delta \, | w' - w | \quad \text{for} \quad w' \in W, \; w \in W,$$

and therefore that $x(w)$ is differentially close to the map in sta-dard form $x(w_0) + \mathfrak{J}(w_0)(w - w_0)$.

Applying (5.9), we now cover W by a sequence of parallel similar dicylinders Q of diameters $dQ < \varepsilon/N$, such that $\sum | Q | < | W + \varepsilon |$. The corresponding sets $E = x(WQ)$ have diameters $< \varepsilon$ and cover $x(W)$, and the gauges satisfy $g(E) \leqslant (1 + \eta) | Q |$. Hence by addition $a[x(W), \varepsilon] \leqslant (1 + \eta)(| W | + \varepsilon)$, and by making $\varepsilon \to 0$, $a[x(W)] \leqslant (1 + \eta) | W |$. Multiplying by $1 - \eta$, we derive the second inequality (6.1), since it clearly follows from the second inequality (6.2) and from $| j(w_0) | = 1$ that

$$(1 - \eta^2) | W | \leqslant \int_W | j(w) | \, dw \,.$$

3) Finally suppose $r < q$. Then $j(w) = 0$ and it suffices to prove that

$$a[x(W)] \leqslant \eta \, | W | \,. \tag{6.3}$$

We may again suppose that W is contained in a sufficiently small ball $| w - w_0 | < \varrho$ and that $\mathfrak{J}(w_0)$ is in standard form-this last amounts to proving (6.3) for the original quantities only with η replaced by ηN, but this is immaterial since N is fixed.

Arguing as in 2), and using the case $r < q$ of (5.9), where the relevant gauges now satisfy $g(E) \leqslant \eta | Q |$, we find that $a[x(W), \varepsilon] < \eta(| W | + \varepsilon)$ which leads at once to (6.3).

7. – There is a further important expression for the partial area in a special case. We now suppose $q \leqslant m \leqslant p$, and we denote by $z(y)$ a fixed map from $[p]$ to $[p + q]$ whose value for any particular $y \in [p]$ is the associated *level*. To any map $y(w)$, $w \in W$ into $[p]$ we associate the *relief map* $x(w) = (y(w), z[y(w)])$ into $[p + q]$. We write P for the operation of projecting from $[p + q)$ into $[p]$, so that $P\,x(w)$ is simply $y(w)$, and we denote by $J_P(w)$, $\mathfrak{J}_P(w)$ the corresponding Jacobian and Jacobian matrix of $y(w)$.

Let $\mathfrak{X}$ be the Jacobian matrix of y, $z(y)$ with respect to y and let α and β denote respectively a vertical and a horizontal $[m]^*$. We write $\mathfrak{X}_\alpha$ for the matrix derived from $\mathfrak{X}$ by restriction to the columns of α, $X_{\alpha\beta}$ for the determinant of the square matrix derived from $\mathfrak{X}_\alpha$ by restriction to the rows of β. By the elementary rule for composite differentiation we find that the component $j_\alpha(w)$ of $j(w)$ is the determinant of the matrix $\mathfrak{X}_\alpha \mathfrak{J}_P(w)$, so that, by the LAGRANGE identity

$$j_\alpha(w) = \sum_\beta X_{\alpha\beta}(J_P(w))_\beta .$$

Evidently $X_{\alpha\beta}$ vanishes unless $\beta \supset P\alpha$ in which case it is the value of the matrix of the restriction to the rows of $\beta/P\alpha$ in the Jacobian matrix $\mathfrak{Z}$ of $z(y)$ with respect to y. Denoting this determinant by $Z_{\beta/P\alpha}$ we have thus

$$(7.1) \qquad j_\alpha(w) = \sum_{\beta \supset P\alpha} Z_{\beta/P\alpha}(J_P(w))_\beta ,$$

and this formula, which gives $j(w)$ in terms of $J_P(w)$, is valid wherever all the functions concerned are differentiable. We have to extend it to the case of Lipschitzian maps $z(y)$ and $y(w)$.

Let γ denote a horizontal $[q]^*$. We have to define a (q)-dimensional multivector Z with components Z_γ such that (7.1) is valid for almost all w; in the differentiable case, Z is the Jacobian multivector corresponding to $\mathfrak{Z}$ and is a function only of the point $y \in [p]$ concerned. In general, however, we shall use instead a *reduced* Jacobian, which d pends on y and J_P. We shall denote it by Λ, so that (7.1) will now read

$$(7.2) \qquad j_\alpha(w) = \sum_{\beta \supset P\alpha} \Lambda_{\beta/P\alpha}(J_P(w))_\beta .$$

We define $\Lambda = \Lambda(y, J_P)$ to be 0 if $J_P = 0$, while if $J_P \neq 0$ we define it for almost every y of each m-dimensional hyperplane in $[p]$ parallel to J_P, by means of an isomorphism, or change of axes, which transforms this hyperplane into that of the first m coordinates of $[p]$. In that case $(J_P)_\beta$ vanishes unless β is this hyperplane β_0 and Z_γ exists for $\gamma = [q]^* \subset \beta_0$ for almost all $y \in \beta_0$. We set $\Lambda_\gamma = Z_\gamma$ for $\gamma \subset \beta_0$ and $\Lambda_\gamma = 0$ otherwise. The q-dimensional multivector Λ is then defined in the original setting by reversing the isomorphism. Evidently the formula (7.2) is then valid for every *linear* map $y(w)$ from $[m]$ to $[p]$. We have to exstend it to arbitrary *Lipschitzian* maps.

The function

$$f(y, J_P) = f_\alpha(y, J_P) = \sum_{\beta \supset P\alpha} \Lambda_{\beta/P\alpha}(y, J_P)\ (J_P)_\beta$$

is now defined for almost every y of each m-dimensional hyperplane parallel to J_P and satisfies obvious conditions of boundedness and measurability. Moreover, if S is any closed oriented m-dimensional polytope in $[p]$ the integral of $f(y, J_P)$ on S coincides with that of j_α on a corresponding closed Lipschitzian parametric surface in $[p+q]$, defined by a corresponding relief map. Hence this integral is zero. We express this by saying that $f(y, J_P)$ is *homologous to zero.* We shall see that such functions are always defined almost everywhere on any Lipschitzian m-dimensional hypersurface and that the formula (7.2) holds for any Lipschitzian map. In consequence, *the partial area of the relief map corresponding to a Lipschitzian map $y(w)$ from $W \subset [m]$ into $[p]$ may be written in the form*

$$(7.3) \qquad \int_W \Big[\sum_\alpha \Big\{ \sum_{\beta \supset P\alpha} A_{\beta/P\alpha}\,[y(w),\ J_P(w)]\ (J_P(w))_\alpha \Big\}^2\Big]^{1/2}\ dw\,.$$

This last expression may be written more shortly

$$(7.4) \qquad \int_W |\,A\,[y\,(w),\ J_P\,(w)]\,|\ |\,J_P\,(w)\,|\ dw\,.$$

To see this it is sufficient to observe that in (7.2) we may suppose by a rotation of the axes that there is only one β for wich the quantity $(J_P\,(w))_\beta \neq 0$. Squaring and summing with respect to α we deduce that the square of the integrand in (7.3) is

$$|\,j(w)\,|^2 = \{\,(J_P\,(w))_\beta\,\}^2\,|\,A\,|^2 = |\,J_P\,|^2\ |\,A\,|^2$$

and therefore that the integral in question reduces to (7.4).

Note added.

At the 1958 Summer Institute of the American Mathematical Society sponsored by the National Science Foundation and devoted to Surface Area and related topics, several mathematicians have reported on researches closely connected with the notion of Partial Area. H. FEDERER has anounced a theorem virtually equivalent to the equality

$$a(T) = \int s\,(M_z)\,dz;$$

he states it in a less general form (for a RIEMANN metric) however the general case can be deduced almost at once. Also W. H. FLEMING has used Partial Area in his work on hypersurfaces in the non-parametric form and on functional completion. Both FEDERER and FLEMING refer, in this connection, to DE GIORGI. Finally L. CESARI has staded that he has a proof of the Partial Area identity for parametric surfaces.

References.

[1] L. C. Young, *Remarks on a chapter of the integral*, Rend. Circ. Mat. Palermo (2) **7** (1958), 48-54.

[2] H. Federer, *Surface area* II, Trans. Amer. Math. Soc. **55** (1944), 438-456.

[3] H. Federer, Notices Amer. Math. Soc. (Abstracts of papers presented by title at Cincinnaty meeting Jan. 1958).

[4] W. H. Fleming, *Functions wit generalized gradient and generalized surfaces*, Ann. Mat. Pura Appl. (4) **44** (1957), 93-103.

[5] E. de Giorgi, *Su una teoria generale della misura* $(r-1)$*-dimensionale in uno spazio ad* r *dimensioni*, Ann. Mat. Pura Appl. (4) **36** (1954), 191-213.

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