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A Note on a Lemma of Fullerton. (**)

In his paper « Prime Ends for Open Subsets of Two Dimensional Manifolds-I » [1], R. E. FULLERTON proves the following lemma concerning the components of the frontier of an open connected set in a finitely triangulable two-manifold M with or without boundary. If S is any subset of M let S^* denote its frontier and $\bar{S}$ its closure.

Lemma 1.

Let U be any coordinate neighborhood on M which intersects at most one boundary component of M. Let Q be a connected non-void open subset of M and let V be any component of $U - Q^*$ which contains no points of Q. Then V^* contains points of n components of Q^*, where $0 \leqslant n < \infty$.

Also a further conjecture is made concerning the upper bound of the number n of the above lemma for oriented finitely triangulable two-manifolds without boundary, namely, n has a least upper bound of the genus of the manifold increased by one.

This Note gives a shorter proof of the above lemma. The least upper bound of the number n is also derived for each finitely triangulable two-manifold (orientable or non-orientable, and with or without boundary).

In the following, the degree of multicoherence of a continuum will be used [2, p. 83].

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Lemma 2.

Let P be a PEANO space whose degree of multicoherence $r(P) = n$ is finite. If U is a non-void connected open set in P and V is a non-void component of $P - \overline{U}$, then V^* meets at most $n + 1$ components of U^*.

Proof.

$\overline{V}$ and $P - V$ are non-void subcontinua of P such that $P = (P - V) \cup \overline{V}$. Hence by the definition of the degree of multicoherence, $(P - V) \cap \overline{V} = V^*$ has at most $n + 1$ components. Since V is a component of $P - \overline{U}$, we have $V^* \subset U^*$. Hence V^* meets at most $n + 1$ components of U^*. This completes the proof of Lemma 2.

It is clear that a finitely triangulable two-manifold M with or without boundary is a PEANO space. It can be characterized by the number of handles $h(M)$, the number of cross-caps $c(M)$, and the number of boundary curves $b(M)$ [3, p. 144]. The following theorem gives the degree of multicoherence of M.

Theorem.

Let M be a finitely triangulable two-manifold with or without boundary. Then $r(M)$ is determined as follows:

i) *If $b(M) = 0$ and $c(M)$ is even, then $r(M) = h(M) + c(M)/2$.*

ii) *If $b(M) = 0$ and $c(M)$ is odd, then $r(M) = h(M) + [c(M) - 1]/2$.*

iii) *If $b(M) > 0$, then $r(M) = 2\,h(M) + c(M) + b(M) - 1$.*

Proof.

i) and ii) have been proved in [4].

To prove iii) we first show that M is the image under a quasi-monotone map of a finitely triangulable two-manifold M' with $h(M') = 2\,h(M) + b(M) - 1$ and $c(M') = 2\,c(M)$. This shows by i) and [2, p. 153] that $r(M) \leqslant 2\,h(M) + c(M) + b(M) - 1$.

Let B denote the boundary of M. Since $b(M) > 0$, we have $B \neq \Phi$. Let $f^+ : M \to M \times R$ be the homeomorphism defined by $f^+(x) = (x, \varrho(x, B))$, where $\varrho(x, B)$ is the distance from x to B. Similarly, define $f^- : M \to M \times R$ as $f^-(x) = (x, -\varrho(x, B))$. It is easily seen that $M' = f^+(M) \cup f^-(M)$ is a finitely triangulable two-manifold with $h(M') = 2\,h(M) + b(M) - 1$ and $c(M') = 2\,c(M)$ [3, p. 144].

Let $g : M' \to M$ be defined by $g(x, y) = x$. g is a continuous open light mapping of M' onto M. By [2, p. 152], g is quasi-monotone. Hence by i) above and [2, p. 153], we have

$$r(M) \leqslant r(M') = h(M') + c(M')/2 = 2\,h(M) + b(M) - 1 + c(M).$$

Next we show $r(M) \geqslant 2\,h(M) + c(M) + b(M) - 1$. Let B_1 be one component of B, the boundary of M. Then there are $N = h(M) + c(M)$ mutually disjoint arcs l_i in M so that $l_i \cap B_1$ consists of the end-point of l_i, $\bigcup_{i=1}^{N} l_i$ separates M into precisely $N+1$ components, the closure of one of the components is a JORDAN region of connectivity $b(M) - 1$ and th? closure of the remaining components of $M - \bigcup_{i=1}^{N} l_i$ is the union of $h(M)$ finitely triangulable two-manifolds with 1 boundary and 1 handle and $c(M)$ MÖBIUS bands. The quotient space formed by identifying each of the arcs l_i to a point is a PEANO space M'' and the projection map m of M onto M'' is monotone. It is easily computed that $r(M'') = 2\,h(M) + c(M) + b(M) - 1$. Hence by [2, p.154], $r(M) \geqslant \geqslant r(M'') = 2\,h(M) + c(M) + b(M) - 1$. This completes the proof of the theorem.

Lemma 1 is readily proved from Lemma 2 and the theorem above. Also, the least upper bound of the number n in Lemma 1 is established by the above theorem. In particular, for oriented finitely triangulable two-manifolds without boundary the least upper bound for n is the genus increased by one.

Bibliography.

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