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ROOP NARAIN (*)

Some Properties of Generalized Laplace Transform - II. (**)

1. – The purpose of this paper (¹) is to extend certain results of LAPLACE transform to the generalized LAPLACE transform introduced by VARMA [1, p. 209) in the form

$$\Phi(s:k,\ m) = s\int_0^\infty (st)^{m-1/2}\, e^{-st/2}\, W_{k,m}(st)\, f(t)\, dt, \tag{1}$$

which has been symbolically denoted [2] by

$$W[f(t);\ k,\ m] = \Phi\,(s:k,\ m).$$

2. – Theorem 8. *If*

$$W[f(t);\ k,\ m] = \Phi\,(s:k,\ m)$$

and

$$W[f(\sqrt{t})/\sqrt{t};\ k,\ m] = h\,(s),$$

(*) Address: Department of Mathematics and Astronomy, Lucknow University, Lucknow, India.

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(¹) This paper is in continuation to my paper [2].

then

$$W[\sqrt{t}\, h(1/(4t));\ k,\ m] = \sqrt{\pi}\, 2^{-2(k+m)}\, \Phi(\sqrt{s} : 2k - 1/2,\ 2m),$$

provided the generalized *Laplace* *transforms of* $|f(t)|$, $|f(\sqrt{t})/\sqrt{t}|$ *and* $|\sqrt{t}\, h(1/(4t))|$ *exist*.

Proof. We have

$$\Phi(\sqrt{s} : 2k - 1/2,\ 2m) = \int_0^\infty y^{2m-1/2}\, s^{m+1/4}\, e^{-y\sqrt{s}/2}\, W_{2k-1/2,2m}(y\sqrt{s})\, f(y)\, dy .$$

Interpreting the integrand by the known generalized Laplace transform [3, p. 383]

$$s^{m+1/4}\, e^{-\sqrt{as}}\, W_{2k-1/2,2m}(2\sqrt{as}) = W[2^{2k-1/2}\, a^{-1/4}\, \pi^{-1/2}\, t^{-m}\, e^{-a/(2t)}\, W_{k,m}(a/t);\ k,\ m],$$

we get

$$\Phi(\sqrt{s} : 2k - 1/2,\ 2m) = W[2^{2k}\pi^{-1/2}\int_0^\infty y^{2m-1}\, t^{-m}\, e^{-y^2/(8t)}\, W_{k,m}(y^2/(4t))\, f(y)\, dy;\ k,\ m] =$$

$$= W[2^{2k+2m-2}\,(\pi t)^{-1/2}\int_0^\infty (y/(4t))^{m-1/2}\, e^{-y/(8t)}\, W_{k,m}(y/(4t))\, y^{-1/2} f(\sqrt{y})\, dy;\ k,\ m) =$$

$$= W[2^{2k+2m}\,\pi^{-1/2}\sqrt{t}\, h(1/(4t));\ k,\ m].$$

The intepretation of the integrand is justified by de la Vallée Poussin's Theorem [4, p. 504] since the generalized Laplace transforms of $|f(t)|$, $|f(\sqrt{t})/\sqrt{t}|$ and $|\sqrt{t}\, h(1/(4t))|$ exist.

2.1. Corollary. Putting $k + m = 1/2$, we get a known result of Laplace transform [5, p. 19]:

If $f(t) \doteqdot \Phi(s)$ *and* $f(\sqrt{t})/\sqrt{t} \doteqdot h(s)$, *then*

$$2\sqrt{t/\pi}\; h(1/(4t)) \doteqdot \Phi(\sqrt{s}),$$

provided the *Laplace* *transforms of* $|f(t)|$, $|f(\sqrt{t})/\sqrt{t}|$ *and* $|\sqrt{t}\, h(1/(4t))|$ *exist*.

2.2. As a verification of Theorem 8 let us consider the correspondences [6, p. 31]:

$$(2.1)\quad \begin{cases} W[t^{\mu} J_{\nu}(t);\ k,\ m] = \\ = \dfrac{\Gamma_*(\mu+\nu+1+m\pm m)}{2^{\nu} s^{\mu+\nu}\,\Gamma(\nu+1)\,\Gamma(\mu+\nu+(3/2)+m-k)}\ {}_4F_3\left\{ \begin{matrix} (\mu+\nu+(3/2)+m\pm m\pm 1/2)/2 \\ \nu+1,\ (\mu+\nu+2+m-k\pm 1/2)/2 \end{matrix};-\dfrac{1}{s^2}\right\}, \\ \Re(\mu+\nu+1+m\pm m)>0, \qquad \Re(s)>0, \quad |s|>1, \end{cases}$$

and

$$(2.2)\quad \begin{cases} W[t^{\mu} J_{\nu}(\sqrt{t});\ k,\ m] = \\ = \dfrac{\Gamma_*(\mu+(\nu/2)+1+m\pm m)}{2^{\nu} s^{\mu+\nu/2}\,\Gamma(\nu+1)\,\Gamma(\mu+(\nu/2)+(3/2)+m-k)}\ {}_2F_2\left\{ \begin{matrix} \mu+(\nu/2)+1+m\pm m \\ \nu+1,\ \mu+(\nu/2)+(3/2)+m-k \end{matrix};-\dfrac{1}{4s}\right\}, \\ \Re(\mu+(\nu/2)+1+m\pm m)>0, \qquad \Re(s)>0\,. \end{cases}$$

Let in Theorem 8, $f(t) = t^{\mu} J_{\nu}(t)$, then by (2.1),

$$\Phi(\sqrt{s}:\ 2k-1/2,\ 2m) =$$

$$= \frac{\Gamma_*(\mu+\nu+1+2m\pm 2m)}{2^{\nu} s^{(\mu+\nu)/2}\,\Gamma(\nu+1)\,\Gamma(\mu+\nu+2+2m-2k)}\ {}_4F_3\left\{ \begin{matrix} (\mu+\nu+(3/2)+2m\pm 2m\pm(1/2))/2 \\ \nu+1,\ (\mu+\nu+(5/2)+2m-2k\pm(1/2))/2 \end{matrix};-\frac{1}{s}\right\},$$

and by (2.2),

$$h(s) = \frac{\Gamma_*((\mu+\nu+1)/2+m\pm m)}{2^{\nu} s^{(\mu+\nu-1)/2}\,\Gamma(\nu+1)\ \Gamma((\mu+\nu+2)/2+m-k)}\cdot$$

$$\cdot\, {}_2F_2\left\{ \begin{matrix} (\mu+\nu+1+2m\pm 2m)/2 \\ \nu+1,\ (\mu+\nu+2+2m-2k)/2 \end{matrix};-\frac{1}{4s}\right\},$$

so that by the theorem we get

$$W\left[t^{(\mu+\nu)/2}\ {}_2F_2\left\{ \begin{matrix} (\mu+\nu+1+2m\pm 2m)/2 \\ \nu+1,\ (\mu+\nu+2+2m-2k)/2 \end{matrix};-t\right\};\ k,\ m\right] =$$

$$= s^{-(\mu+\nu)/2}\,\frac{\Gamma_*((\mu+\nu)/2+1+m\pm m)}{\Gamma((\mu+\nu+3)/2+m-k)}\ {}_4F_3\left\{ \begin{matrix} (\mu+\nu+(3/2)+2m\pm 2m\pm(1/2))/2 \\ \nu+1,\ (\mu+\nu+(5/2)+2m-2k\pm(1/2))/2 \end{matrix};-\frac{1}{s}\right\},$$

$$\Re(\mu+\nu+2+2m\pm 2m)>0, \qquad \Re(s)>0, \qquad |s|>1\,.$$

This verifies our theorem [3, p. 383].

3. – Theorem 9. *If*

$$W\,[f(t);\; k,\; m] = \Phi\,(s:k,\; m),$$

$$W\,[f(\sqrt{t})/\sqrt{t}\,;\; k,\; m] = h\,(s)$$

and

$$W\,[\Theta(t);\; k,\; m] = \Psi\,(s:k,\; m),$$

then, provided the integrals involved converge absolutely,

$$\int_0^\infty \Psi(t:k,\, m)\, h(1/(4t))\, t^{-1/2}\, \mathrm{d}t = \sqrt{\pi}\, 2^{-2(k+m)} \int_0^\infty \Phi(\sqrt{t}:2k - 1/2,\, 2m)\, \Theta(t)\, t^{-1}\, \mathrm{d}t.$$

The theorem follows if we proceed as in Theorem 6 of my previous paper [2].

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