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L. C. YOUNG (*)

Partial Area.

PART II: Contours on Hypersurfaces. (**)

8. – This part can be read independently of Part I, to which it is a sequel, but uses its results. We also use the theory of the papers [1] and [2], which we reformulate here in the relevant higher-dimensional case without proofs.

$[n]$ is now Euclidean n-space and $[m]$ a hyperplane of dimension $m \leqslant n$; the Cartesian coordinates in these will be fixed. Further $[n]^m$ denotes the normed space of the multivectors expressible as skew products of m vectors in $[n]$, and we term toroid the Cartesian product of a ball $R[n]$ of radius R in $[n]$ with the unit sphere $I[n]^m$ in $[n]^m$. The hyperplane in $[n]$ determined by a non-singular linear map $x(w)$ from $[m]$, whose constant Jacobian is $J \neq 0$, will be termed parallel to J; and in any such hyperplane, the termes: measure, almost everywhere, and so on, refer to LEBESGUE m-dimensional measure.

We term m-integrable a function $f(x, J)$, defined for each $J \in I[n]^m$ and for almost all x of any hyperplane parallel to J, which possesses, in each such hyperplane Π, a LEBESGUE integral, and coincides with the derivative of the latter by m-dimensional cubes, wherever this derivative exists in Π. An m-integrable function $f(x, J)$ which is bounded on each toroid is termed *m-integrand* if its definition is extended to all $J \in [n]^m$ by the homogeneity relation $f(x, tJ) = t\, f(x, J)$ for $t \geqslant 0$.

Any m-integrand can evidently be integrated on an m-dimensional simplex in $[n]$, and therefore, by addition, on an m-dimensional weighted *σ-polytope*: this last term designated here a countable sum of m-dimensional simplices

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(*) Address: Department of Mathematics, University of Wisconsin, Madison, Wisconsin, U.S.A..

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with positive real coefficients, such that the simplices lie in a same ball $R[n]$ and that the corresponding sum of their (m-dimensional) areas multiplied by these same coefficients converges (to a number that we term the area of the weighted σ-polytope). The prefix σ- is omitted when the countable sum reduces to a finite sum; moreover the term weighted is omitted in the case of positive *integer* coefficients. The integral of an m-integrand f over a weighted σ-polytope will be termed *classical integral.* We also define formally the classical integral of f for any bounded Lipschitzian map $x(w)$ from a subset W of $[m]$ into a bounded subset of $[n]$ as the expression

$$\int_W f[x(w),\ J(x)]\ dw,$$

where $J(w)$ denotes, almost everywhere in W, the Jacobian of $x(w)$. This classical integral need not exist *a priori* for an arbitrary m-integrand $f(x, J)$, for we do not know that $f[x(w), J(w)]$ is defined almost everywhere in W. However in special cases we shall verify that it does exist.

An m-integrand f will be termed *homologous* to 0, and we write $f \sim 0$, if its classical integral, over the elementary polytope consisting of the oriented elementary boundary of an arbitrary $(m+1)$-dimensional simplex in $[n]$, vanishes. More generally, two m-integrands are homologous if their difference is homologous to 0. In this section we shall be most concerned with m-integrands which are homologous to continuous ones, or in particular, are themselves continuous. The space of continuous m-integrands will be given the topology of uniform convergence in each toroid and denoted by F^m. By a *generalized hypersurface* we shall mean a non-negative linear functional $L = L(f)$ defined for $f \in F^m$: we *omit the term generalized* if there exist a bounded Lipschitzian map $x(w)$ from a subset W of $[m]$ into a bounded subset of $[n]$, for which $L(f)$ is given by the classical integral defined above, for every $f \in F^m$ (1).

It is convenient to interpolate two remarks:

(8.1) *If a generalized hypersurface is a countable sum of hypersurfaces, then it is itself a hypersurface.*

(8.2) *If two Lipschitzian maps $x(w)$, $w \in W \subset [m]$, define a same hypersurface L, then, for the two corresponding multiple systems $x(W)$, the counting functions agree outside (m-dimensional) area 0.*

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(1) If the classical integral is modified by multiplying the integrand by a non-negative measurable function $\varrho(w)$ we define similarly the notion of *hypersurface with density*. This notion includes in particular that of σ-polytope.

To verify (8.1) we observe first that, because of the topology selected for F^m, the hypersurfaces can be defined by maps from subsets of $[m]$ into a same ball $R[n]$, since their sum is a linear functional. We may clearly suppose each subset bounded, and by translating them sufficiently far apart, we can combine the relevant maps into a single bounded Lipschitzian map. To verify (8.2) we merely express L as an integral with respect to (m-dimensional) area in $[n]$.

We observe further that it will be found that *our notion of hypersurface differs much less than might be supposed from the classical notions of parametric and non-parametric curves and surfaces.*

The classical integral over a simplex or (weighted) σ-polytope similarly defines a non-negative linear functional $L(f)$ for $f \in F^m$, and *the latter will be identified with the simplex, or σ-polytope, itself.* This interpretation of the notions of simplex and polytope is more in keeping with their algebraic nature and gives a precise significance to operations which always seem somewhat forced in a purely geometrical setting. The classical integral over a (weighted) σ-polytope, is however defined also for certain m-integrands which need not be continuous, and we then write it

$$\int_L f \quad ,$$

whereas the notation $L(f)$ is reserved for $f \in F^m$. The same is true when L is a hypersurface: however we must verify, of course, that when two maps define a same hypersurface L, the two classical integrals agree for *any* m-integrand, and this can be seen by comparing them with the RIESZ representation of L as a STIELTJES integral on a toroid; this representation is determined by $L(f)$ for $f \in F^m$ and agrees with the classical integral for any m-integrand for which it exists.

The RIESZ representation will also used when L is an arbitrary generalized hypersurface, and by restricting in it the range of integration to a set Q which we shall suppose Borelian, we define the *intersection* L_Q of L with the set Q. In particular if Q is a half-space in which a vertical coordinate remains $< z$ or $\leqslant z$, L_Q is termed L truncated below, or above, the level z, and if these two truncations agree, which they clearly must for all but countably many levels z, L_Q is termed L *truncated at the level* z. We verify that

(8.3) (Lim L_ν) truncated at (lim z_ν) *coincides with the limit of* (L_ν truncated above, below, and if relevant at, z_ν).

Convergence of generalized hypersurfaces L_ν ($\nu = 1, 2, \ldots$) is here defined, in accordance with the so-called w^*-topology, to mean convergence of the corresponding numbers $L_\nu(f)$ for each $f \in F^m$. On occasion we may also use a norm defined as the *area* of the generalized hypersurface L, i.e. the value of $L(f)$ for the function $f = 1$ on the relevant toroid.

We shall also use, for an arbitrary generalized hypersurface, another extension of the symbol $L(f)$ to certain m-integrands, viz.

$$(L, f) = \lim_{h \to 0} (L, \bar{f}),$$

where $\bar{f}(x, J)$ is the mean value of $f(x + h\xi, J)$ in the unit n-dimensional cube of ξ, centered at the origin.

Infinitely differentiable m-integrands linear in J will be termed *m-forms* and we term *exact* any m-form homologous to 0. We term, respectively, *track* and *boundary* of a generalized hypersurface L, the restriction of the functional $L(f)$ to m-forms and to exact m-forms, respectively. Moreover we term L *singular* if its track vanishes, *closed* if its boundary does so. A generalized hypersurface L will be said to have *Lipschitzian track* if its track coincides with that of some hypersurface: the symbol

$$[\text{track } L, f]$$

will then denote, for an arbitrary m-integrand f, the classical integral of f over this hypersurface. A similar convention is made if the track of L coincides with that of a (weighted) σ-polytope. Further a generalized hypersurface will be said to possess a *Lipschitzian resolution* if it has the form $\int L' \, d\alpha$, where $d\alpha$ is a measure in the class of generalized surfaces which have Lipschitzian tracks.

We denote by Λ_R^m the normed space of the boundaries λ of m-dimensional σ-polytopes in $R[n]$, and by Λ^m their union for positive R, when the norm $\|\lambda\|$ of λ is defined as the infimum of the areas of σ-polytopes with the boundary $\lambda \in \Lambda^m$.

All the above definitions may be used with m replaced by any integer k subject to $0 < k \leqslant n$, and some also when $k = 0$. However, the terms may be modified to preserve analogies, just as area is replaced by length for $k < m$ and by volume for $k > m$. In particular for $k < m$ the terms σ-polytope and hypersurface become σ-polyhedron and graph. The case $k = 0$ is important for the purposes of induction.

Our k-forms differ only in notation from those of the theory of exterior differential forms and we term *derivative* Df of a k-form f, the $(k+1)$-form which corresponds in that theory to the differential. Accordingly, an *exact* $(k+1)$-form reduces to the derivative of a k-form. Further, we term *rectifiable* a boundary $\lambda(Df)$ which, as functional of the exact m-form Df, reduces to the track $L(f)$ of a generalized $(m-1)$-dimensional graph L, where f is any appropriate $(m-1)$-form f; and when this is the case the boundary λ and the track of L will be said to be *isomorphic*.

9. – We shall use a variant of the main theorem (1.1) of [**1**]:

(9.1) *A generalized hypersurface is closed if and only if it is the* w^* *limit of a sequence of closed weighted polytopes.*

We couple this statement with an elementary lemma, which corresponds to (3.3) of [**1**]: a closed weighted polytope is a linear combination, with positive coefficients, of closed polytopes.

We quote further the main results of [**2**], rephrased with m in place of 2:

(9.2) (i) (L, f) *exists for every* m*-integrand* f *homologous to a continuous one, and for every generalized hypersurface* L *whose boundary belongs to* Λ^m. (ii) *If* $L = \int L' \, d\alpha$ *is a Lipschitzian resolution, then the boundary of* L *belongs* Λ^m *and we have* $(L, f) = \int [\text{track } L', f] \, d\alpha$ *for every* m*-integrand* f *homologous to* 0.

(9.3) *There is a one-to-one correspondence between* m*-integrands* $f(x, J)$ *homologous to* 0 *and linear functionals* $g(\lambda)$ *on* Λ^m, *such that* $g(\lambda) = (L, f)$ *for every generalized hypersurface* L *whose boundary is* $\lambda \in \Lambda^m$.

(9.4) *Let* f_0 *be an* m*-integrand homologous to a continuous one and let* $\lambda_0 \in \Lambda^m$. *Suppose further that* L_0 *is a generalized hypersurface with boundary* λ_0 *such that* $(L_0, f_0) \leqslant (L, f_0)$ *for all generalized hypersurfaces* L *with the same boundary* λ_0. *Then there exists a non-negative* m*-integrand* f_1 *homologous to* f_0 *such that* $(L_0, f_1) = 0$.

Finally we note a lemma, rephrased from (6.3) of [**2**], which has to be used in proving (9.2) (ii): let $x(w)$ be a Lipschitzian map, with Jacobian $J(w)$ almost everywhere, from the unit cube of $[m]$ into $[n]$, let Δ denote a variable m-dimensional parallelogram and let λ_Δ be the boundary of the hypersurface defined by restricting the map $x(w)$ to Δ; further let $g(\lambda)$ and $f(x, J)$ be associated in accordance with (9.3), where $f \sim 0$. Then $g(\lambda_\Delta)$ is absolutely continuous in Δ and has for almost any w the derivative $f[x(w), J(w)]$ with respect to regular sequences of Δ.

This lemma, *which easily justifies the statement at the end of Part* I, is established by rephrasing the proof of its counterpart for $m = 2$, quoted above, with the help of an elementary fact which we rephrase as follows: let $x(w)$ have LIPSCHITZ constant $\leqslant 1$ on the faces of a Δ of centre the origin, and satisfy the inequality $|x(w)| < \varepsilon |w|$ on these faces; then there exists a continuation of $x(w)$ to the whole of Δ for which the area of the map $x(w)$ on Δ is $< \varepsilon |\Delta|$.

10. – We now write $n = p + q$, and as in Part I, there will be q vertical coordinates z in $[n]$. The horizontal projection of a multivector $J \in [n]^m$ is written J_p and clearly belongs to $[p]^m$ when $p \geqslant m \geqslant q$; the horizontal $(m - q)$-dimensional multivector with the same components as the vertical projection j of J

is written j_p. We observe that $j_p \in [p]^{m-q}$: in fact j vanishes unless exactly $m-q$ independent linear combinations of the vectors whose product is J are horizontal, and we may therefore express J as a product of $m-q$ horizontal vectors y' and q further vectors of the form $y''+z''$ where y'' is horizontal and z'' vertical; j is then the product of the y' and z'', so that j_p is the product of the y' with a scalar consisting of the determinant of the z''. Conversely any $j_p \in [p]^{m-q}$ arises from a coresponding $J \in [n]^m$: it is sufficient to adjoin to the factors of j_p, q further factors which are unit vectors along the vertical axes.

Besides the m-integrands $f(x, J)$, which we speak of as m-integrands in $[n]$, we shall consider m-integrands $f_p(y, J_p)$ in $[p]$ and also $(m-q)$-integrands $g_p(y, j_p)$ in $[p]$. We write $P^{-1} f_p$ and $P^{-1} g_p$ for the m-integrands in $[n]$ which depend only on y and J_p, or only on y and j, and have the same values as f_p and g_p.

We observe that any m-integrands in $[n]$ which depend only on y and J_p, or only on y and j, may be put in the form $P^{-1}f_p$ or $P^{-1}g_p$, their constancy with respect to certain of the components of x or J being sufficient to ensure that they are defined almost everywhere on the relevant m- or $(m-q)$-dimensional hyperplanes in $[p]$. It is moreover evident that $f_p \sim 0$ is equivalent to $P^{-1} f_p \sim 0$. We shall show that if f denotes an m-integrand of the form $\Phi(z)\ P^{-1}\ g_p$, where $g_p = g_p(y, j_p)$,

(10.1) $P^{-1}\ g_p \sim 0$ *implies* $g_p \sim 0$; *and* $g_p \sim 0$ *implies* $f \sim 0$.

To see this suppose first that $P^{-1}g_p \sim 0$ and let β be the oriented boundary of any $(m-q+1)$-dimensional simplex T in $[p]$. For any cube $Q \in [q]$, the Cartesian product $T \times Q$ has an oriented boundary B consisting of $\beta \times Q$ together with a sum B' of faces in each of which at least one linear combination of the vertical coordinates z remains constant. Since the vertical projections of the m-dimensional multivectors associated with the faces of B' vanish, we have $\int_{B'} f = 0$. Further, since $f \sim 0$ and B is expressible as a finite sum of oriented boundaries of simplices, in which certain pairs of faces cancel out, we have also $\int_B f = 0$ and therefore $\int_{\beta \times Q} f = 0$. However, this last integral evidently reduces to $|Q|$ times that of g_p on β. Consequently $\int_\beta g_p = 0$ and therefore $g_p \sim 0$.

Conversely, suppose that $g_p \sim 0$. Let B denote the oriented elementary boundary of any $(m+1)$-dimensional simplex in $[n]$, let B' be the sum of the faces of B if any, on which at least one linear combination of the vertical coordinates remains constant, and let $\beta(z)$ be the section of $B-B'$ by the hyperplane $z = $ constant, where z is any point of $[q]$. Whe find that there is an elementary measure μ in $[q]$ such that

$$\int_{B-B'} f = \int \Phi(z) \{ \int_{\beta(z)} g_p \}\, d\mu\,.$$

Since $g_p \sim 0$ the righthand side vanishes, and therefore also the lefthand side. Moreover it is clear that $\int_{B'} f = 0$, since the multivectors in $I[n]^m$ associated with the faces of B' have vanishing vertical projections. Hence by addition $\int_B f = 0$ and consequently $f \sim 0$.

We shall term horizontal and vertical parts of a generalized hypersurface L in $[n]$, the restrictions of the linear functional $L(f)$ to continuous integrands of the form $P^{-1} f_p$ and $P^{-1} g_p$, i.e. to m-integrands in F^m which depend only on y, J_p or on y, j_p. These parts of L define in their turn a generalized hypersurface PL and a generalized graph sL, both situated in $[p]$, which we term respectively the *horizontal projection* and the *horizontal dissection* of L. They are defined by the linear functionals

$$P\,L(f_p) = L(P^{-1} f_p)\,, \qquad s\,L(g_p) = L(P^{-1} g_p)\,,$$

where f_p and g_p denote continuous m- and $(m-q)$-integrands in $[p]$. In the case $q = 1$, we term further *truncated projection* of L above, or below, or when relevant *at*, the level $z \in [q]$, the horizontal projection of the truncation of L above, or below, or when relevant at, this level, and we write these

$$P_{z+} L, \qquad P_{z-} L, \qquad P_z L\,.$$

We recall that, on account of the topology of F^m, the linear functional $L(f)$ is unaltered by altering $f(x, J)$ outside some toroid of sufficiently large radius and therefore has a RIESZ representation as the integral of $f(x, J)$ with respect to a bounded measure on this toroid, and so on the Cartesian product of a ball in $[q]$ with a certain toroidal set E of values of (y, J). We can express the measure concerned as the integral of a measure in E, which depends on a parameter $z \in [q]$, with respect to a measure in $[q]$ of the form $\mathrm{d}z + \mathrm{d}\mu$, where $\mathrm{d}\mu$ has its derivative in $\mathrm{d}z$ vanishing almost everywhere in $[q]$. Denoting by ${}_zL(f)$ the integral of $f(x, J)$ with respect to the inner measure on E, and by N the set of $z \in [q]$ at which $\mathrm{d}\mu/\mathrm{d}z$ does not exist or is $\neq 0$, it follows from FUBINI's theorem that

$$L(f) = \int {}_zL(f)\,\mathrm{d}z + \int_N {}_zL(f)\,\mathrm{d}\mu, \qquad \text{where} \qquad |N| = 0\,.$$

From this formula it is clear that ${}_zL(f)$ is uniquely defined for almost every z and, moreover, for μ-almost every z of N where μ is fixed, although of course μ may be chosen in more than one way. It follows that

$$sL = \int s\,{}_zL\,\mathrm{d}z + \int_N s\,{}_zL\,\mathrm{d}\mu, \qquad \text{where} \qquad |N| = 0\,. \tag{10.2}$$

We shall term $s_z L$ the *slice at the level* z of L dissected. This slice consists of an $(m-q)$-dimensional generalized graph uniquely determined for almost every z; the levels in N will be termed *sparse*.

(10.3) (i) *If the boundary of* L *belongs to* Λ^m, *the slice* $s_z L$ *has its boundary in* Λ^{m-q} *for almost every level* z *and is closed for* μ-*almost every sparse level.* (ii) *If* L *is closed, so is, at almost every level* z, *the slice* $s_z L$, *and in the case* $q = 1$ *the track of this slice is isomorphic with the boundary of the truncated projection of* L *at this same level.*

(10.4) *If* L *is a hypersurface, the slice* $s_z L$ *is a graph for almost every level and vanishes for the sparse levels; further if* M *is the multiple system* $x(W)$, *associated with a Lipschitzian map* $x(w)$, $w \in W \subset [m]$ *which defines* L, *then the length of the slice at almost any level* z *coincides with that of the section* M_z.

Proof of (10.3). Since (i) is clearly valid for a weighted σ-polytope, and the slices at sparse levels then vanish, we suppose, by adding such a weighted σ-polytope, that L is closed; and by an easy induction we may set $q = 1$. The slice at μ-almost any sparse level z_0 is the limit as $h \to 0$ of a graph $L^{(h)}(g_p)$ in $[p]$, where $L^{(h)}(g_p)$ is the value of $L(f)$ for an f of the form

$$f(x, J) = \Phi_h(z)\, g(y, j), \qquad \text{where} \qquad g = P^{-1} g_p,$$

where $x = (y, z)$ and where $\Phi_h(z)$ is an infinitely differentiable function suitably constructed. When g_p is an exact $(m-q)$-form we find that f is an exact m-form by (10.1) and therefore that $L(f) = 0$ since L is closed. This implies $L^{(h)}(g_p) = 0$ for any exact g_p, and so $L^{(h)}$ is closed. Making $h \to 0$ we find that our slice at z_0 is closed also. Similarly we see that the slice at almost any level is also closed, or this can be deduced from the second assertion of (ii) which we shall now establish.

To this effect, we denote by L_ν a weighted closed polytope with the limit L, in accordance with (9.1), and by a slight rotation which does not affect this limit we may suppose that no constituent simplex of L_ν is horizontal. Since L_ν is a linear combination of closed polytopes, we may regard as elementary the fact that L_ν has no sparse levels, and can be truncated at any level, and that the boundary of its truncated projection at this level is isomorphic with the track of the corresponding slice; this implies that

$$(s_z L_\nu)(g_p) = (P_z L_\nu)(Dg_p) \tag{10.5}$$

for any $(m-1)$-form g_p in $[p]$. Further for almost any level z_0, L can be truncated at z_0, and therefore also at z_0+h for almost all h, and there is a sequence of h for which

$$s_{z_0} L = \lim_h \left[\frac{1}{h}\left\{ (sL)_{\text{truncated at } z_0+h} - (s\,L)_{\text{truncated at } z_0}\right\}\right].$$

Hence for any $(m-1)$-form g_p in $[p]$, if we denote by $G(z)$, $G_\nu(z)$ the quantities $(s\,{}_zL)(g_p)$, $(s_z\,L_\nu)(g_p)$, we deduce from (8.3) that, for our sequence of h,

$$G(z_0) = \lim_h \lim_\nu \left[\frac{1}{h}\int_{z_0}^{z_0+h} G_\nu(z)\,\mathrm{d}z\right].$$

By applying the mean-value theorem, we find that there exists a sequence of ν and a corresponding sequence of levels z_ν, such that $z_\nu \to z_0$ and $G_\nu(z_\nu) \to G(z_0)$. But by (10.5) $G_\nu(z_\nu)$ is the value of $(P\,{}_zL_\nu)(Dg_p)$ for $z=z_\nu$ and this tends to $(P\,{}_zL)(Dg_p)$ for $z=z_0$ by (8.3). We thus have at $z=z_0$,

$$(P\,{}_zL)(Dg_p) = G(z) = (s\,{}_zL)(g_p)$$

for an arbitrary $(m-1)$-form g_p in $[p]$, and this establishes the asserted isomorphism.

Proof of (10.4). Let L_N be the intersection of L with the set N of the sparse levels, where $|N|=0$. This intersection is a hypersurface defined by restricting the map $x(w)$ to a certain set W_N. For any m-integrand g of the form $P^{-1}g_p$, where g_p is a continuous $(m-q)$-integrand in $[p]$, the absolute value of the classical integral defining $L_N(g)$ cannot exceed the product of the norm of g and the partial area of $x(w)$, $w \in W_N$. This partial area vanishes by (4.2) and (4.3) since $|N|=0$, and consequently $s\,L_N=0$. The slices of sL in N, which are also those of $s\,L_N$, therefore vanish for (μ-almost all) sparse levels.

In the remaining assertions, we again set $q=1$ without loss of generality and we may suppose, in view of what has just been proved, that the vertical component $z(w)$ of our map is differentiable and that its gradient $Z(w)$ does not vanish. Further, by (8.1) it is enough to express L as a countable sum of hypersurfaces for which our assertions are valid, so that we can use the operations of removing from W a set of measure 0 and of splitting W into countably many subsets, to make certain simplifying assumptions. We may suppose first that the partial derivative of $z(w)$ with respect to some fixed coordinate of w, which we denote by v, does not vanish, and in fact is of constant sign which we may take to be positive. We write $w=(u,v)$, taking v to be the m-th coordinate and u to consist of the remaining $m-1$. Next we may suppose, since $z_v > 0$, that z_v exceeds some fixed positive constant throughout W; by homogeneity we may suppose $z_v > 2$ and we denote by K an upper bound of $|Z(w)|$ in W. We may also suppose, by EGOROV's theorem, that the sequence of functions

$$f_\nu(w) = \sup_{\substack{|w'-w|<1/\nu \\ w' \neq w,\ w' \in W}} \frac{|z(w') - z(w) - Z(w)(w'-w)|}{|w'-w|}$$

converges uniformly to 0 and we denote by ν_0 a value of ν for which $|f_\nu(w)| \leqslant 1$ throughout W. Finally, by a further subdivision, we may suppose W to have diameter $> 1/\nu_0$.

For any two points w, w' of W, we then have, writing $z = z(w)$, $z' = z'(w)$,

$$|z_v(v'-v)| \leqslant |z' - z - z_u(u'-u)| + f_{\nu_0}(w)\,|w'-w|$$

and remembering that $z_v > 2$, $|z_u| \leqslant K$, $f_{\nu_0}(w) \leqslant 1$, if follows that

$$|v'-v| \leqslant |z'-z| + (K+1)\,|u'-u| \leqslant (K+2)\,|(u', z') - (u, z)|.$$

This shows that there exsists a Lipschitzian map onto W of a corresponding set of (u, z). This map may be supposed one-to-one (by yet another subdivision) and evidently has a positive Jacobian. The combined mapping therefore defines the same hypersurface L and by (8.2) we may substitute it for the map $x(w)$. This is equivalent to supposing the vertical component $z(w)$ of $x(w)$ to coincide with v, and with this final simplifying assumption our assertions are easily verified. This completes the proof.

Note: The ideas on which the foregoing section is based have much in common with a device used by HADAMARD in the theory of partial differential equations, see [3].

11. – As in Part I, § 7, we suppose given a map $z(y)$ from $[p]$ to $[q]$, but this time a continuously differentiable one in the first instance. We use our earlier notations: $x(y)$ is the map $[y, z(y)]$ from $[p]$ to $[p+q]$; $\mathfrak{X}$ and $\mathfrak{Z}$ are the Jacobian matrices of $x(y)$ and $z(y)$; α, β, γ denote respectively any $[m]^*$, any horizontal $[m]^*$, any horizontal $[q]^*$; $X_{\alpha\beta}$ is the determinant of the columns of α and rows of β in $\mathfrak{X}$, Z_γ is the determinant of the rows of γ in $\mathfrak{Z}$, and Z is the multivector $\in [p]^q$ with the components Z_γ.

We write further $P^{-1}y$ for the point $x = [y, z(y)]$, and given any multivector $J_P \in [p]^m$, we write $P^{-1} J_P$ for the multivector $J \in [p + q]^m$ with the components

$$J_\alpha = \sum_\beta X_{\alpha\beta} (J_P)_\beta ;$$

further, for any m-integrand $f(x, J)$, we denote by Pf the m-integrand in $[p]$

$$P f(y, J_P) = f(P^{-1} y, P^{-1} J_P) .$$

We associate with any generalized hypersurface L_P in $[p]$ a generalized *relief* hypersurface $L = P^{-1} L_P$ in $[p + q]$, by writing, for any $f \in F^m$, $L(f) = L_P(Pf)$. Its vertical part defines, as in the preceding section, a horizontal $(m - q)$-dimensional graph sL, the horizontal dissection of L which we shall also write $C L_P$ and term the *contour dissection* of L_P. The slice of sL at the level z will be termed *contour* of L_P, or of $C L_P$ at this level, and denoted $C_z L_P$, or sometimes simply C_z. We shall term sparse contours those whose levels belong to the corresponding set $N \subset [q]$, where $|N| = 0$.

The contour dissection of L_P can also be defined, without the intermediary of the generalized relief surface L, by using the formula (7.1) of Part I. To this effect, let $j(J)$ be the vertical part j of $J \in [n]^m$, and for any multivector $J_P \in [p]^m$ let $C J_P$ be the multivector $j_P \in [p]^{m-q}$ defined by $j(P^{-1} J_P)$. If δ denotes any horizontal $[m - q]^*$, the δ-component of $C J_P$ is

$$(11.1) \qquad (j_P)_\delta = \sum_{\beta \supset \delta} Z_{\beta/\delta} (J_P)_\beta ,$$

where β is used for a horizontal $[m]^*$. We write further, for any $g_P \in F^{m-q}$ in $[p]$, $C^{-1} g_P$ for the integrand $g \in F^m$ in $[p]$ defined by $g(y, J_P) = g_P(y, C J_P)$. Then $C L_P$ is the generalized $(m - q)$-dimensional graph in $[p]$, defined by writing, for any $g_P \in F^{m-q}$ in $[p]$,

$$(11.2) \qquad (C L_P)(g_P) = L_P (C^{-1} g_P) .$$

In the case in which L_P is a *hypersurface* in $[p]$, we can suppose that $z(y)$ is merely *Lipschitzian* instead of continuously differentiable. The relief hypersurface $P^{-1} L_P$ may then be defined by the relief map associated, as in Part I, § 7, to any, bounded Lipschitzian map, $y(w)$, $w \in W \subset [m]$ into $[p]$, which defines L_p. And the formula (10.2) remains valid provided that the definition of $C\,J_p$ given (10.1) is replaced by that obtained by writing, as in (7.2) of Part I,

$$(11.3) \qquad (j_p)_\delta = \sum_{\beta \supset \delta} \Lambda_{\beta/\delta} \, (J_p)_\beta ,$$

where Λ is the reduced Jacobian.

From theorems (10.3) and (10.4) we obtain:

(11.4) *For a hypersurface L_p in $[p]$, the contours $C_z L_p$ are $(m-q)$-dimensional graphs in $[p]$ and those at sparse levels vanish. Further if $x(w)$, $w \in W \subset [m]$ is a Lipschitzian map of the relief hypersurface L, and if M is the multiple system $x(W)$ associated with this map, then the length of the contour at the level z coincides for almost all z with the length of the section M_z of M.*

(11.5) *Suppose that $q = 1$. Let L_p be a closed hypersurface in $[p]$ and let $z(y)$ be a Lipschitzian map from $[p]$ to $[q]$; or alternately, let L_p be a closed generalized hypersurface in $[p]$ and let $z(y)$ be a continuously differentiable map from $[p]$ to $[q]$. Then for almost every level $z = z_0$, the track of the contour $C_z L_p$ is isomorphic to the boundary of the intersection of L_p with the set of y for which $z(y) \leqslant z_0$.*

References for Part II.

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