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On the Order and Type of Integral Functions. (**)

1. – Let $f(z) = \sum_0^\infty a_n z^n$ be an integral function of order ϱ and lower order λ, $M(r)$, $\mu(r)$ denote respectively the maximum modulus of $f(z)$ and the maximum term of the series for $|z| = r$ and $\nu(r)$ denote the rank of this term. Further, let $W(x)$ be an indefinitely increasing positive function continuous in adjacent intervals. It is known that ([1], [2])

$$\log M(r) = \log M(r_0) + \int_{r_0}^{r} \frac{W(x)}{x}\, dx, \tag{1.1}$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log r} = \frac{\varrho}{\lambda} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log W(r)}{\log r}. \tag{1.2}$$

Also, $f(z)$ is said to be of type T and lower type t, if

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\varrho} = \frac{T}{t}. \tag{1.3}$$

If the limit in (1.3) exists (i.e., $T = t$), $f(z)$ is said to be of perfectly regular growth. Further, SHAH [3] has shown that, for $0 \leqslant \varrho \leqslant \infty$,

$$\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} \leqslant \frac{1}{\varrho} \leqslant \frac{1}{\lambda} \leqslant \limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} \tag{1.4}$$

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and for every integral function of infinite order

$$\liminf_{r\to\infty} \frac{\log M(r)}{\nu(r)} = 0. \tag{1.5}$$

In this paper we derive relations between the types of two or more integral functions of the same finite order. Using (1.1) and (1.2), we derive the analogues of (1.1) and (1.5) for $M(r)$ and $W(r)$ and establish some relations between the order and type of an integral function.

We require two lemmas.

2. – Lemma 1. *If* $\varphi(x) \sim \psi(x)$ *then for any finite constant* k *and* $x > 0$, $\{\varphi(x)\}^{k/x} \sim \{\psi(x)\}^{k/x}$.

For, let $P = \{\varphi(x)/\psi(x)\}^{k/x}$, then

$$\log P = (k/x)\left[\log\{\varphi(x)/\psi(x)\}\right] \to 0 \quad \text{as} \quad x \to \infty,$$

since $\varphi(x)/\psi(x) \to 1$ and so $P \to 1$ as $x \to \infty$.

Lemma 2. *If $\varphi(x)$ and $\psi(x)$ be non-decreasing positive functions, continuous almost everywhere for $x \geqslant 0$ and $\varphi(x) \sim \psi(x)$, then*

$$\int_{x_0}^{x} \varphi(x)\, dx \sim \int_{x_0}^{x} \psi(x)\, dx, \qquad x_0 \geqslant 0.$$

For, since $\varphi(x) \sim \psi(x)$, we have for any $\varepsilon > 0$,

$$(1-\varepsilon)\ \psi(x) < \varphi(x) < (1+\varepsilon)\ \psi(x),$$

for $x > x_0$, and so

$$(1-\varepsilon)\int_{x_0}^{x} \psi(x)\, dx < \int_{x_0}^{x} \varphi(x)\, dx < (1+\varepsilon)\int_{x_0}^{x} \psi(x)\, dx$$

and the result follows.

Theorem 1. *If* $f_1(z) = \sum_0^\infty a_n z^n$, $f_2(z) = \sum_0^\infty b_n z^n$ *be integral functions of the same order* ϱ $(0 < \varrho < \infty)$, *and types* T_1 $(0 < T_1 < \infty)$ *and* T_2 $(0 < T_2 < \infty)$ *respectively and* $f(z) = \sum_0^\infty c_n z^n$, *where* $|c_n| \sim |\sqrt{a_n b_n}|$, *then* $f(z)$ *is an integral function of order* ϱ *and type* T *such that* $T \leqslant \sqrt{T_1 T_2}$.

Proof. $f_1(z)$ and $f_2(z)$ are of order ϱ and types T_1, T_2 *if and only if*, (BOAS [6], p. 11)

$$\limsup_{n\to\infty}\{ n \mid a_n \mid^{\varrho/n} \} = \varrho e T_1, \qquad \limsup_{n\to\infty}\{ n \mid b_n \mid^{\varrho/n} \} = \varrho e T_2 .$$

Hence, for any $\varepsilon > 0$, we have for sufficiently large n

$$\frac{n}{e\varrho} \mid a_n \mid^{\varrho/n} < T_1 + \varepsilon, \qquad \frac{n}{e\varrho} \mid b_n \mid^{\varrho/n} < T_2 + \varepsilon .$$

Therefore

$$\left(\frac{n}{e\varrho}\right)^2 \mid a_n \mid^{\varrho/n} \mid b_n \mid^{\varrho/n} < T_1 T_2 + o(1),$$

since T_1, T_2 are finite. Therefore

$$\text{(2.1)} \qquad \frac{n}{e\varrho} \mid \sqrt{a_n b_n} \mid^{\varrho/n} \leqslant \frac{n}{e\varrho} \mid a_n \mid^{\varrho/(2n)} \mid b_n \mid^{\varrho/(2n)} < \sqrt{T_1 T_2} + o\,(1).$$

Thus, if $\mid c_n \mid \sim \mid \sqrt{a_n b_n} \mid$, we have from lemma 1,

$$\mid c_n \mid^{\varrho/n} \sim \mid \sqrt{a_n b_n} \mid^{\varrho/n},$$

since ϱ is finite. Therefore, from (2.1) we get

$$\frac{n}{e\varrho} |c_n|^{\varrho/n} \sim \frac{n}{e\varrho} |\sqrt{a_n b_n}|^{\varrho/n} < \sqrt{T_1 T_2} + o(1) .$$

Therefore

$$\limsup_{n\to\infty}\{ n \mid c_n \mid^{\varrho/n} \} = e\varrho T \leqslant e\varrho \sqrt{T_1 T_2},$$

or $T \leqslant \sqrt{T_1 T_2}$.

Corollary 1. *If* $\mid a_n/a_{n+1} \mid$, $\mid b_n/b_{n+1} \mid$ *are non-decreasing functions,* $f_1(z)$, $f_2(z)$ *are of perfectly regular growth and of the same finite order* ϱ, *then so is* $f(z)$ *and* $T = \sqrt{T_1 T_2}$.

For, SHAH [5] has shown that if $|a_n/a_{n+1}|$ is non-decreasing, then the lower type of $f_1(z)$ is given by

$$t_1 = \liminf_{n\to\infty} \left\{ \frac{n}{e\varrho} \mid a_n \mid^{\varrho/n} \right\}.$$

Hence, if $f_1(z)$ is of perfectly regular growth, i.e., $t_1 = T_1$, then

$$\liminf_{n\to\infty} \{ n \mid a_n \mid^{\varrho/n} \} = e\varrho t_1 = e\varrho T_1 = \limsup_{n\to\infty} \{ n \mid a_n \mid^{\varrho/n} \},$$

so that

$$\lim_{n\to\infty} \{ n \mid a_n \mid^{\varrho/n} \} = e\varrho T_1 .$$

Similarly, we have

$$\lim_{n\to\infty} \{ n \mid b_n \mid^{\varrho/n} \} = e\varrho T_2 .$$

Hence

$$\lim_{n\to\infty} \{ n \mid \sqrt{a_n\, b_n} \mid^{\varrho/n} \} = e\varrho \sqrt{T_1\, T_2}$$

and the result follows if $|c_n| \sim |\sqrt{a_n\, b_n}|$.

Corollary 2. *If*

$$f_k(z) = \sum_0^\infty a_n^{(k)} z^n \qquad (k = 1,\ 2,\ \ldots,\ m)$$

be m integral functions each of order ϱ $(0 < \varrho < \infty)$ and non-zero finite types $T_1, T_2, \ldots, T_m$ respectively, then the function $f(z) = \sum_0^\infty c_n z^n$, where $|c_n| \sim$ $\sim |a_n^{(1)} \ldots a_n^{(m)}|^{1/m}$, is an integral function of order ϱ and type T such that

$$T \leqslant (T_1\, T_2 \ldots T_m)^{1/m}. \tag{2.2}$$

Corollary 3. *If $f_1(z), f_2(z), \ldots, f_m(z)$ are each of perfectly regular growth and the same finite order ϱ, $|a_n^{(k)}/a_{n+1}^{(k)}|$ $(k = 1, 2, \ldots, m)$ are all non-decreasing functions of n, then $f(z)$ is also of perfectly regular growth and finite order ϱ and its type $T = (T_1\, T_2 \ldots T_m)^{1/m}$.*

Corollaries 2 and 3 follow as immediate generalizations of Theorems 1 and Corollary 1 respectively.

3. – Theorem 2. *If $f(z)$ be an integral function of order ϱ and lower order λ, then*

$$\liminf_{r\to\infty} \frac{\log M(r)}{W(r)} \leqslant \frac{1}{\varrho} \leqslant \frac{1}{\lambda} \leqslant \limsup_{r\to\infty} \frac{\log M(r)}{W(r)} . \tag{3.1}$$

Proof. From (1.2) we have, for any $\varepsilon > 0$,

$$r^{\lambda-\varepsilon} < W(r) < r^{\varrho+\varepsilon} \quad \text{for} \quad r \geqslant r_0 = r_0(f) .$$

Hence

$$\int_{r_0}^{r} x^{\lambda-\varepsilon-1}\, dx < \int_{r_0}^{r} \frac{W(x)}{x}\, dx < \int_{r_0}^{r} x^{\varrho+\varepsilon-1}\, dx .$$

Using (1.1) we therefore get

$$(r^{\lambda-\varepsilon} - r_0^{\lambda-\varepsilon})/(\lambda-\varepsilon) < \log\{ M(r)/M(r_0) \} < (r^{\varrho+\varepsilon} - r_0^{\varrho+\varepsilon})/(\varrho+\varepsilon) .$$

On dividing by $W(r)$ and proceeding to limits the result follows since $M(r_0)$ and r_0 are finite and

$$r^{\lambda-\varepsilon}/W(r) < 1 < r^{\varrho+\varepsilon}/W(r) .$$

Corollary. *For every integral function of infinite order*

$$\liminf_{r\to\infty} \frac{\log M(r)}{W(r)} = 0 . \tag{3.2}$$

This follows immediately from the first inequality in (3.1) since $\varrho = \infty$.

Remark. An alternative proof of Theorem 2 can be given on the same lines as given by Singh ([7], p. 10) for the functions $\mu(r)$ and $\nu(r)$.

4. – Let

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{W(x)}{x^{\varrho}} = \begin{matrix}\alpha\\ \beta\end{matrix} . \tag{4.1}$$

In what follows, we give some relations between the order ϱ $(0 < \varrho < \infty)$ of the integral function $f(z)$ and the numbers α, β, T and t.

Theorem 3. *If $\alpha = \beta$, then $f(z)$ is of perfectly regular growth and its type $T = \alpha/\varrho$.*

Proof. Since $\alpha = \beta$, we have from (4.1) $W(x) \sim \alpha\, x^{\varrho}$. Hence, in view of lemma 2, we have

$$\log \frac{M(r)}{M(r_0)} = \int_{r_0}^{r} \frac{W(x)}{x}\, \mathrm{d}x \sim \int_{r_0}^{r} \alpha\, x^{\varrho - 1}\, \mathrm{d}x = \frac{\alpha}{\varrho}\,(r^{\varrho} - r_0^{\varrho}).$$

Therefore

$$\lim_{r \to \infty} \frac{\log M(r)}{r^{\varrho}} = \frac{\alpha}{\varrho}\,,$$

so that $t = T = \alpha/\varrho$.

Theorem 4. *We find:*

(4.2) $$\alpha \leqslant e\varrho T,$$

(4.3) $$\beta \leqslant \varrho T,$$

(4.4) $$\alpha + \beta \leqslant e\varrho T,$$

equality cannot simultaneously hold in (4.3) *and* (4.4).

Starting with the relations (1.1), the above relations can be established in the same way as established by SHAH [4] and SINGH [7] for the functions $\mu(r)$ and $\nu(r)$.

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