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# Partial Area.

## PART III: Symmetrization and the Isoperimetric and Least Area Problems. (**)

**13.** - Let $n = p + 1$, $m = p$; in the notation of the previous parts $q = 1$ and the $p$-dimensional horizontal, and one-dimensional vertical, projections of a point $x \in [n]$ will again by $y$, $z$. The terms length and graph will be used for dimensions $< p$, usually for the dimension $p - 1$, while area and hypersurface are reserved for the dimension $p = m$, and volume and solid for $p + 1 = n$. When the dimension $k$ is kept arbitrary, we shall substitute on occasion the neutral terms $k$-measure and $k$-variety. We note the following corollary of (9.1) of Part II:

(13.1) *A closed generalized solid is always singular.*

We term *adjoint* $L^-$ of a generalized variety $L$, the generalized variety derived by reversing the orientation, i.e. defined by $L^-(f) = L(f^-)$, where $f^-(x, J) = f(x, -J)$.

Given a closed generalized $(k-1)$-variety $\Gamma$, we define a generalized $k$-variety $L$, termed *cone of $\Gamma$ with vertex $x_0$*, or simply cone of $\Gamma$, as follows: we associate to any $k$-integrand $f$ the $(k-1)$-integrand

$$g(x, J) = \int_0^1 f\big(x_0 + t\,(x - x_0), \quad (x - x_0) \times J\big)\, t^{k-1}\, dt,$$

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where $x \in [n]$, $J \in [n]^{k-1}$, and we define $L$ by writing $L(f) = \Gamma(g)$. A simple calculation with corresponding exterior differential forms shows that we have $f = Dg$ whenever $f$ is exact, and hence that *the cone of $\Gamma$ is bounded by $\Gamma$.*

Again, if $\Gamma$ is any generalized $(k-1)$-variety situated in $[p]$, we define a generalized $k$-variety $L$ in $[n]$, termed *cylinder on $\Gamma$ between the levels $z_1$ and $z_2$*, or simply cylinder on $\Gamma$, as follows: we associate to any $k$-integrand $f$ in $[n]$, the $(k-1)$-integrand $g$ in $[p]$ obtained by writing, for $y \in [p]$ and $J_p \in [p]^{k-1}$,

$$g(y,\ J_p) = \int_{z_1}^{z_2} f[(y,\ z),\ \zeta \times J_p]\, dz$$

where $\zeta$ is the vertical unit vector; and we define $L$ by writing $L(f) = \Gamma(g)$. We shall only be concerned with the case in which $\Gamma$ is bounded by a generalized $(k-2)$-variety $\Gamma^*$ and a simple calculation then shows that $L$ is bounded by the sum of the corresponding cylinder on $\Gamma^*$ and two horizontal generalized $(k-1)$-varieties consisting of the translate of $\Gamma$ in the hyperplane $z = z_1$ and that of its adjoint in the hyperplane $z = z_2$.

We term *minimal* generalized $k$-variety in a given class, one of smallest $k$-measure in that class. The minimum is attained if the class is closed and not empty, provided that the $k$-measures are not increased by restricting the varieties to be situated in some fixed ball. In particular a minimal generalized $k$-variety with given boundary, or bounded by a given closed $(k-1)$-variety, exists.

**14.** – We come to a lemma which will be the main tool of this part. Let $Q_0$ be in $[n]$ a horizontal strip of height $h_0$ and let $Q$ be any horizontal substrip $z_1 \leqslant z \leqslant z_2$ whose height $z_2 - z_1$ we denote by $h$. Let $L$ and $\sum$ denote respectively a closed generalized hypersurface situated in $Q_0$ and a minimal generalized solid bounded by $L$, let $L_Q$, $\sum_Q$ be their intersections with $Q$ and let $V$, $V_Q$, $A$, $A_Q$ denote the volumes of $\sum$, $\sum_Q$ and the areas of $L$, $L_Q$. We denote further by $s(z)$, or simply by $s$, the length of the slice $s_z L$, and by $\sigma(z)$, or simply by $\sigma$, the area of a minimal generalized hypersurface with the same boundary as $L$ truncated below the level $z$.

(14.1) *For almost any $z$, the slice $s_z \sum$ is a minimal generalized hypersurface of area $\sigma(z)$ bounded by $s_z L$ and we have*

$$\text{(i)}\quad V \leqslant K h_0 A, \qquad \text{(ii)}\quad V_Q = \int_{z_1}^{z_2} \sigma\, dz, \qquad \text{(iii)}\quad A \geqslant \int_{z_1}^{z_2} \sqrt{s^2\, dz^2 + d\sigma^2},$$

*where this last integral is in the sense of* Burkill *and where $K$ denotes a certain constant factor depending only on $n$.*

Proof. We first deal with (iii). We denote for $z = z_1$, by $L_1$ the (or a), minimal generalized hypersurface whose area was defined to be $\sigma(z_1)$, and we observe that the corresponding class for $z = z_2$ for which the minimum of the area was defined to be $\sigma(z_2)$ includes the generalized hypersurface $L_1 + P\,L_Q$, where $P\,L_Q$ is the horizontal projection of $L_Q$. Hence $\sigma(z_2) \leqslant \sigma(z_1) + A(P\,L_Q)$, and by combining this with the inequality derived by symmetry

$$A(P\,L_Q) \geqslant | \, \sigma(z_2) - \sigma(z_1) \, | \, .$$

Further, by (10.2) of Part II,

$$a(L_Q) \geqslant \int_{z_1}^{z_2} s \ \mathrm{d}z \, .$$

Moreover, since $| \, J \, | \geqslant | \, j \, | \cdot \cos\alpha + | J_p \, | \cdot \sin\alpha$ for all $\alpha$, we evidently have $A(L_Q) \geqslant a(L_Q) \cdot \cos\alpha + A(P\,L_Q) \cdot \sin\alpha$, and by choice of $\alpha$,

$$A_Q = A(L_Q) \geqslant \left\{ \, a(L_Q)^2 + A(P\,L_Q)^2 \right\}^{1/2}.$$

Substituting from the previous inequalities this leads to

$$A_Q \geqslant \left\{ \, ( \int_{z_1}^{z_2} s \ \mathrm{d}z)^2 + [\sigma(z_2) - \sigma(z_1)]^2 \right\}^{1/2}$$

and since this inequality holds for every $Q$, we deduce (iii).

We next reduce the remaining assertions to (i) which we therefore assume while doing so. We consider in the first instance the substrip $Q$ for almost any $z_1$, in which case $s\,{}_zL$ may be taken to be, for $z = z_1$, a closed generalized graph $\Gamma_1$ such that $L_1$ is bounded by $\Gamma_1$. We denote by $\sum_1$ the cylinder on $L_1$ between the levels $z_1$ and $z_2$, and by $\sum_2$ a minimal generalized solid bounded by the closed generalized hypersurface consisting of the sum of the following three terms:

$L_Q$; the adjoint of the cylinder on $\Gamma_1$; the translate in $z = z_2$ of the adjoint of $P\,L_Q$.

For fixed $z_1$, the area of each of these three terms tends to 0 with $h$, and therefore by (i) the volume of $\sum_2$ is $< \varepsilon h$ where $\varepsilon \to 0$ as $h \to 0$. Also by the $p$-dimensional analogue of (13.1), $\sum_Q$ and $\sum_1 + \sum_2$ have the same boundary, so that, from the minimal property, $V_Q$ cannot exceed the sum of the volumes of $\sum_1$ and $\sum_2$. Since $V_Q$ is by (10.3) the integral from $z_1$ to $z_2$ of the area of $s\,{}_z\sum$, it follows that for almost every $z_1$, by dividing by $h$ and making $h \to 0$, $\sigma(z_1)$

coincides with the area of $s\,_z\sum$ for $z = z_1$. Hence $\sigma(z)$ is the area of $s\,_z\sum$ for almost every $z$ and this establishes (ii) for an arbitrary $Q$. It is clear also that $s\,_z\sum$ is, for almost every $z$, a minimal generalized hypersurface bounded by $s\,_zL$, since it is bounded by the latter and has the minimal area $\sigma(z)$.

It remains to establish (i), which we rename $(\mathrm{i})_n$ where the suffix denotes the dimension, and we shall do this by induction, in conjunction with the following statement, to which we shall refer as $(\mathrm{iv})_n$:

Let $\Delta$ denote the part of $[n]$ obtained by restricting certain horizontal coordinates $y_k$ to inequalities $a_k \leqslant y_k \leqslant a_k + h_0$ and let $L$ be a closed generalized hypersurface of area $A$ situated in $\Delta$. Then, for some subdivision of the $z$-axis into intervals $\delta_\nu$ of length $h_0$, there exists for each $\nu$ a closed generalized surface $L_\nu$, situated in the part $\Delta_\nu$ of $\Delta$ for which $z \in \delta_\nu$, with the following properties:

(a) $L_\nu = 0$ except for a finite number of values of $\nu$;

(b) The sum of the areas of the $L_\nu$ does not exceed a constant multiple of $A$,

(c) The sum of the $L_\nu$ may be written in the form $L + L^*$, where $L^*$ is singular.

**15.** - We proceed with our proof. From $(\mathrm{iv})_n$, by an easy induction combined with suitable rotations of axes, we deduce that with the hypotheses of (14.1) there exists a finite sum of closed generalized hypersurfaces $L'$, where this sum differs from $L$ only by a singular generalized hypersurface, such that:

$$\begin{cases} \text{Each } L' \text{ is situated in a cube of side } h_0 . \\ \text{The sum of the areas of the } L' \text{ does not exceed some constant multiple} \\ \quad \text{of } A. \end{cases}$$

But this in turn implies $(\mathrm{i})_n$ since we construct a generalized solid of volume $\leqslant K\, h_0\, A$ bounded by $L$, simply by adding together cones on the $L'$ with appropriate vertices.

Thus $(\mathrm{iv})_n$ implies $(\mathrm{i})_n$.

We verify next that $(\mathrm{i})_{n-1}$ implies $(\mathrm{iv})_n$ The statement of $(\mathrm{i})_{n-1}$ will be in terms of lengths and graphs instead of areas and hypersurfaces, since the dimension is lowered.

To this effect, with the hypotheses of $(\mathrm{iv})_n$, let $s(t)$, $0 \leqslant t \leqslant h_0$, be the sum of the lengths of the slices $s\,_zL$ at the levels $z = t + \nu h_0$ for $\nu = 0, \pm 1, \ldots$.

We note that this sum has only a finite number of non-vanishing terms, since $L$ has to be situated in some ball of $[n]$. By (10.2)

$$A \geqslant \int_0^{h_0} s(t)\,dt$$

and therefore there exists a value of $t$ for which $s(t) \leqslant A/h_0$. Such values of $t$ occupy positive measure, and we choose one for which the numbers $z_\nu = t + \nu h$ $(\nu = 0, \pm 1, \ldots)$ are never among the sparse levels of the formula (10.1) of Part II.

We denote by $\delta_\nu$ the interval $z_{\nu-1} \leqslant z \leqslant z_\nu$, so that $\Delta_\nu$ is the corresponding part of $\Delta$, and by $\Gamma_\nu$, $\Gamma_\nu^-$ the slice $s\,_zL$ and its adjoint, when $z = z_\nu$. By $(\mathrm{i})_{n-1}$ there exists a generalized hyperusrface $L_\nu$, bounded by $\Gamma_\nu$, whose area does not exceed a constant multiple of the product of $h_0$ and the length of $\Gamma_\nu$. We denote by $L_\nu^-$ the adjoint of $L_\nu$, bounded by $\Gamma_\nu^-$, and we write $L'$ (dropping the suffix) for the sum of

$$L_{\nu-1};\ L_\nu^-;\quad \text{the intersection of } L \text{ with } \Delta_\nu .$$

Evidently $L'$ is a closed generalized surface situated in $\Delta_\nu$ and the sum of the areas of the $L'$ (for varying $\nu$) is $\leqslant K\,h_0\,s(t) \leqslant KA$. Further the sum of the $L'$ consists, by definition, of $L$ together with the sum of singular generalized surfaces $L_\nu + L_\nu^-$. This establishes $(\mathrm{iv})_n$ as a consequence of $(\mathrm{i})_{n-1}$.

To complete the induction it now only remains to verify $(\mathrm{i})_n$ in the case $n = 1$. This is easily done with the appropriate definitions [1] of $k$-variety, etc., when $k = 0$, so that the proof of (14.1) is now complete.

**16.** – We now consider briefly the *isoperimetric property of the n-dimensional ball.* We denote by $\Theta$ the volume of the unit ball in $[n]$ and by $\theta$ that of the unit ball in $[m]$. We write $H$ for the class of generalized hypersurfaces which possess Lipschitzian tracks.

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(1) A $k$-integrand, for $k = 0$, is a function $f(x, j)$ where $j$ takes only the two values $\pm 1$, so that there is no longer any homogeneity in $j$. From well-known theorem of Banach it is easy to obtain the general form of a $k$-variety when $k=0$. In particular we find that a *closed* $k$-variety for $k = 0$ is a finite sum $\sum a_\nu \ \{ f(x_\nu, 1) + f(x'_\nu, -1) \}$, where the $a_\nu$ are $\geqslant 0$ and the $x_\nu$, $x'_\nu$ are points.

(16.1) *Let $A$ be the area of a closed generalized hypersurface $L \in H$ and let $V$ be the volume of a minimal generalized solid bounded by $L$. Then*

$$\left\{ \frac{A}{n\Theta} \right\}^n \geqslant \left\{ \frac{V}{\Theta} \right\}^{n-1}.$$

We prove this by induction, the case $n = 1$ being easily verified. With the notation of (14.1) the inductive hypothesis ensures that, for almost every level $z$,

$$\{ s/(m\theta) \}^m \geqslant \{ \sigma/\theta \}^{m-1}.$$

Writing $\sigma = \theta u^m$, it follows that $s \geqslant m\, \theta\, u^{m-1}$ so that (ii) and (iii) of (14.1) yield

$$V = \theta \cdot \int u^m \, \mathrm{d}z, \qquad A \geqslant m\theta \cdot \int u^{m-1} \sqrt{\mathrm{d}z^2 + \mathrm{d}u^2}.$$

Here the two right-hand members are, however, the volume and area of the solid and the hypersurface derived from the curve $u = u(z)$ by revolution about the axis of $z$ in $n$-space. The corresponding minimum problem being well known, we deduce the validity of our assertion for $n$, and so the truth of our theorem. The argument used is the well-known symmetrization process of SCHWARZ.

**17.** – Finally we come to our *application to the least area problem*. Let $L$ be a generalized hypersurface in $H$ with the boundary $\Gamma_1 + \Gamma_2$ where $\Gamma_1$ and $\Gamma_2$ are separated by a strip $z_1 < z < z_2 (= z_1 + h)$, that is to say $\Gamma_1$ is situated in the half-space $z \leqslant z_1$ and $\Gamma_2$ in the half-space $z \geqslant z_2$. We denote by $A_1$ and $A_2$ the least areas of two generalized hypersurfaces, bounded respectively by $\Gamma_1$ and $\Gamma_2$; and we write $A^*(A_1, A_2, h)$, or simply $A^*$, for the least area of a hypersurface of *revolution*, whose boundary is the sum of those of two $m$-dimensional balls of areas $A_1$, $A_2$ at a distance $h$ apart. The determination of $A^*$ in terms of $A_1$, $A_2$, $h$ is an elementary variational problem, and in particular so is that of formulating the conditions under which $A^* = A_1 + A_2$. We observe that the function $A^*$, as an immediate consequence of its definition as a minimum, is subject to an inequality of the form

$$A^*(A_1, A_2, h) \leqslant A^*(B_1, B_2, h) + | A_1 - B_1 | + | A_2 - B_2 | .$$

We shall prove that

(17.1) *The area of $L$ is at least $A^*$.*

In particular if $A^* = A_1 + A_2$ we necessarily have a similar degeneracy in the least area problem for the boundary $\Gamma_1 + \Gamma_2$.

Proof of (17.1). Let $Q$ be the strip $z_1 \leqslant z \leqslant z_2$ and let $A_Q$ be the area of the intersection of $L$ with $Q$. We shall apply (14.1) to the closed generalized hypersurface obtained by adding to $L$ a minimal hypersurface bounded by $\Gamma_1^-$ and a minimal hypersurface bounded by $\Gamma_2^-$, where $\Gamma_1^-$ and $\Gamma_2^-$ are the adjoints of $\Gamma_1$ and $\Gamma_2$. We find that

$$A_Q \geqslant \int_{z_1}^{z_2} \sqrt{s^2\, dz^2 + d\sigma^2}$$

and we again substitute $\sigma = \theta u^m$, so that, by the isoperimetric inequality, $s \geqslant m\theta u^{m-1}$ and therefore

$$A_Q \geqslant m\theta \cdot \int_{z_1}^{z_2} u^{m-1} \sqrt{dz^2 + du^2}.$$

It follows that $A_Q \geqslant A^*(B_1, B_2, h)$, where $B_1$, $B_2$ are the values of $\theta u^m$ for $z = z_1$ and for $z = z_2$, i.e. the corresponding values of $\sigma(z_1)$, $\sigma(z_2)$.

Now if $A_-$ and $A_+$ are the areas of the intersection of $L$ with the half spaces $z < z_1$ and $z > z_2$ we evidently have

$$B_1 + A_- \geqslant A_1 \qquad \text{and} \qquad A_1 + A_- \geqslant B_1,$$

i.e. $A_- \geqslant |A_1 - B_1|$. Similarly $A_+ \geqslant |A_2 - B_2|$. Since the area of $L$ is $A_- + A_Q + A_+$, we find that it is

$$\geqslant A^*(B_1, B_2, h) + |A_1 - B_1| + |A_2 - B_2| \geqslant A^*(A_1, A_2, h),$$

which completes the proof.

## References for Part III.

[1] W. H. Fleming and L. C. Young, *Representations of generalized surfaces as mixtures*, Rend. Circ. Mat. Palermo (2) 5 (1956), 117-144.

[2] W. H. Fleming and L. C. Young, *Generalized surfaces with prescribed elementary boundaries*, Red. Circ. Mat. Palermo (2) 6 (1957), 320-340.