Riv. Mat. Univ. Parma 10 (1959), 275-282

H. D. SINGH (*)

A Generalization of the Extension of a Covariant Differentiation Process.

Considering tensors $T^{\alpha\ldots}_{\beta\ldots}$ whose components are functions of n variables given by x and their m derivatives x', x'', x''', ..., $x^{(m)}$, CRAIG [1] obtained the covariant derivative

$$(1) \qquad T^{\alpha\ldots}_{\beta\ldots x^{(m-1)\gamma}} - m\, T^{\alpha\ldots}_{\beta\ldots x^{(m)\lambda}} \begin{Bmatrix}\lambda\\ \gamma\end{Bmatrix} \qquad (m \geqslant 2),$$

where

$$(2) \qquad \begin{Bmatrix}\lambda\\ \gamma\end{Bmatrix} = x'^{\alpha}\, T^{\lambda}_{\gamma\alpha} + \frac{1}{2}\, x''^{\beta}\, f_{\gamma\delta\beta}\, f^{\delta\lambda},$$

in which partial derivatives are indicated by subscripts and primes have been employed to denote differentiation with respect to the parameter. The curves involved in the discussions are supposed to be given in parametric form. Throughout, a repeated letter in one term denotes a sum of n terms.

The above process was extended by MARIE M. JOHNSON [2] and H. D. SINGH [3] to derive another tensor of one higher covariant order. The results of the above writers were further extended by H. D. SINGH [4] to obtain a similar tensor. The purpose of the present paper is to generalize the extensions of the above process to derive another tensor form $T^{\alpha\ldots}_{\beta\ldots}$ whose covariant rank is one larger. First of all, the general process will be illustrated clearly by taking the tensor $T^{\alpha}_{\gamma}(x, x', x'', x''', x^{(4)}, x^{(5)})$ into consideration.

(*) Address: Department of Mathematics, Balwant Rajput College, Agra, India.

The extended point transformation

$$x^\alpha = x^\alpha (y^1, y^2, y^3, \ldots, y^n),$$

$$x'^\alpha = \frac{\partial x^\alpha}{\partial y^i} y'^i, \qquad x''^\alpha = \frac{\partial x^\alpha}{\partial y^i} y''^i + \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^i y'^j ,$$

gives the following transformation in T^α_γ,

$$\text{(3)} \qquad \overline{T}^i_j (y, y', y'', y''', y^{(4)}, y^{(5)}) = T^\alpha_\gamma (x, x', x'', x''', x^{(4)}, x^{(5)}) \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\gamma}{\partial y^j},$$

in which y indicates n variables $y^1, y^2, y^3, \ldots, y^n$ and a similar notation is used for the derivatives $y', y'', y''', y^{(4)}$ and $y^{(5)}$.

Differentiating (3) with respect to y'^h, we get

$$\overline{T}^i_{jy'^h} = \Big(T^\alpha_{\gamma x'^\beta} \frac{\partial x^\beta}{\partial y^h} + T^\alpha_{\gamma x''^\beta} \frac{\partial x''^\beta}{\partial y'^h} + T^\alpha_{\gamma x'''} \frac{\partial x'''^\beta}{\partial y'^h} + $$

$$+ T^\alpha_{\gamma x^{(4)\beta}} \frac{\partial x^{(4)\beta}}{\partial y'^h} + T^\alpha_{\gamma x^{(5)\beta}} \frac{\partial x^{(5)\beta}}{\partial y'^h} \Big) \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^\gamma}{\partial y^j},$$

which by virtue of the following general formulas

$$\text{(4)} \quad \begin{cases} \dfrac{\partial x^{(m-1)\beta}}{\partial y^{(m-2)h}} = (m-1) \dfrac{\partial x'^\beta}{\partial y^h}, & \dfrac{\partial x^{(m)\beta}}{\partial y^{(m-2)h}} = \dfrac{m(m-1)}{2} \dfrac{\partial x''^\beta}{\partial y^h}, \\[2ex] \dfrac{\partial x^{(m+1)\beta}}{\partial y^{(m-2)h}} = \dfrac{(m+1)\, m\, (m-1)}{3!} \dfrac{\partial x'''^\beta}{\partial y^h}, & \dfrac{\partial x^{(m+2)\beta}}{\partial y^{(m-2)h}} = \dfrac{(m+2)(m+1)\, m\, (m-1)}{4!} \dfrac{\partial x^{(4)\beta}}{\partial y^h}. \end{cases}$$

reduces to

$$\text{(5)} \qquad \overline{T}^i_{jy'^h} = \Big(T^\alpha_{\gamma x'^\beta} \frac{\partial x^\beta}{\partial y^h} + 2\, T^\alpha_{\gamma x''^\beta} \frac{\partial x'^\beta}{\partial y^h} + 3\, T^\alpha_{\gamma x'''^\beta} \frac{\partial x''^\beta}{\partial y^h} + $$

$$+ 4\, T^\alpha_{\gamma x^{(4)\beta}} \frac{\partial x'''^\beta}{\partial y^h} + 5\, T^\alpha_{\gamma x^{(5)\beta}} \frac{\partial x^{(4)\beta}}{\partial y^h} \Big) \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\gamma}{\partial y^j},$$

in which $\partial x'^\beta / \partial y^h$ are eliminated by [**5**, p. 255]

$$\text{(6)} \qquad \overline{\begin{Bmatrix} l \\ h \end{Bmatrix}} \frac{\partial x^\beta}{\partial y^l} = \frac{\partial x'^\beta}{\partial y^h} + \begin{Bmatrix} \beta \\ \delta \end{Bmatrix} \frac{\partial x^\delta}{\partial y^h}.$$

To eliminate $\partial x''^{\beta}/\partial y^h$, we first write x''^{β} in the form

$$x''^{\beta} = \frac{\partial x^{\beta}}{\partial y^j} y''^{j} + \overline{T}^{r}_{jh} y'^{j} y'^{h} \frac{\partial x^{\beta}}{\partial y^r} - T^{\beta}_{\alpha\delta} x'^{\alpha} x'^{\delta} , \tag{7}$$

with the help of (2), (6) and [**5**, p. 248] $f_{\alpha\beta\gamma} x'^{\beta} = 0$. It is necessary to have [**6**]

$$\frac{\partial^2 x^{\beta}}{\partial y^j \partial y^h} = \overline{\Lambda}^{t}_{jh} \frac{\partial x^{\beta}}{\partial y^t} - \Lambda^{\beta}_{\alpha\delta} \frac{\partial x^{\alpha}}{\partial y^j} \frac{\partial x^{\delta}}{\partial y^h} , \tag{8}$$

where

$$\Lambda^{\beta}_{\alpha\delta} = T^{\beta}_{\alpha\delta} - \frac{1}{2} f^{\beta\gamma} \left(f_{\delta\gamma\tau} \begin{Bmatrix} \tau \\ \alpha \end{Bmatrix} + f_{\gamma\alpha\tau} \begin{Bmatrix} \tau \\ \delta \end{Bmatrix} - f_{\alpha\delta\tau} \begin{Bmatrix} \tau \\ \gamma \end{Bmatrix} \right),$$

and so, by means of formulas (6) and (8) and the tensor

$$T^{*\beta}(x, x', x'') = x''^{\beta} + T^{\beta}_{\alpha\delta} x'^{\alpha} x'^{\delta}, \tag{9}$$

the partial derivatives of (7) have the form

$$\frac{\partial x''^{\beta}}{\partial y^h} = - \begin{vmatrix} \beta \\ \gamma \end{vmatrix} \frac{\partial x^{\gamma}}{\partial y^h} + \overline{\begin{vmatrix} r \\ h \end{vmatrix}} \frac{\partial x^{\beta}}{\partial y^r} - 2 \begin{Bmatrix} \beta \\ \alpha \end{Bmatrix} \overline{\begin{Bmatrix} l \\ h \end{Bmatrix}} \frac{\partial x^{\varkappa}}{\partial y^l} + 2 \overline{\begin{Bmatrix} r \\ i \end{Bmatrix}} \, \overline{\begin{Bmatrix} i \\ h \end{Bmatrix}} \frac{\partial x^{\beta}}{\partial y^r}, \tag{10}$$

in which we have the non-tensor form

$$\begin{vmatrix} \beta \\ \gamma \end{vmatrix} = T^{*\beta}_{x^{\gamma}} - T^{*\beta}_{x'^{\alpha}} \begin{Bmatrix} \alpha \\ \gamma \end{Bmatrix} + T^{*\varkappa} \Lambda^{\beta}_{\varkappa\gamma} . \tag{11}$$

The derivatives $\partial x'''^{\beta}/\partial y^h$ are simplified by first writing

$$x'''^{\beta} = \left(y'''^{r} + \overline{T}^{*j} \overline{\begin{Bmatrix} r \\ j \end{Bmatrix}} + \overline{T}^{*r}_{y^i} y'^{i} + \overline{T}^{*r}_{y'^i} y''^{i} \right) \frac{\partial x^{\beta}}{\partial y^r} - \tag{12}$$

$$- \left(T^{*\alpha} \begin{Bmatrix} \beta \\ \alpha \end{Bmatrix} + T^{*\beta}_{x^{\gamma}} x'^{\gamma} + T^{*\beta}_{x'^{\gamma}} x''^{\gamma} \right)$$

by differentiating (7) with respect to the parameter and using the tensor (9).

By means of formulas (6), (8) and (10) and using the tensor

$$(13)\qquad Q^{\beta}(x,\ x',\ x'',\ x''') = x'''^{\beta} + T^{*\alpha}\begin{Bmatrix}\beta\\ \gamma\end{Bmatrix} + T^{*\beta}_{x^{\delta}}\ x'^{\delta} + T^{*\beta}_{x'^{\delta}}\ x''^{\delta}\,,$$

the partial derivatives of (12) have the form

$$(14)\qquad \frac{\partial x'''^{\beta}}{\partial y^{h}} = -\left\|\begin{matrix}\beta\\ \gamma\end{matrix}\right\|\frac{\partial x^{\gamma}}{\partial y^{h}} + \overline{\left\|\begin{matrix}r\\ h\end{matrix}\right\|}\frac{\partial x^{\beta}}{\partial y^{h}} +$$

$$+ 3\left[\left(\overline{\left|\begin{matrix}l\\ h\end{matrix}\right|}\,\overline{\begin{Bmatrix}r\\ l\end{Bmatrix}} + \overline{\begin{Bmatrix}l\\ h\end{Bmatrix}}\,\overline{\left|\begin{matrix}r\\ l\end{matrix}\right|} + 2\,\overline{\begin{Bmatrix}r\\ i\end{Bmatrix}}\,\overline{\begin{Bmatrix}i\\ l\end{Bmatrix}}\,\overline{\begin{Bmatrix}l\\ h\end{Bmatrix}}\right)\frac{\partial x^{\beta}}{\partial y^{r}} -\right.$$

$$\left. -\left(\overline{\left|\begin{matrix}l\\ h\end{matrix}\right|}\begin{Bmatrix}\beta\\ \delta\end{Bmatrix} + \overline{\begin{Bmatrix}l\\ h\end{Bmatrix}}\left|\begin{matrix}\beta\\ \delta\end{matrix}\right| + 2\,\overline{\begin{Bmatrix}l\\ i\end{Bmatrix}}\,\overline{\begin{Bmatrix}i\\ h\end{Bmatrix}}\begin{Bmatrix}\beta\\ \delta\end{Bmatrix}\right)\frac{\partial x^{\delta}}{\partial y^{l}}\right],$$

in which we have the non-tensor form

$$(15)\qquad \left\|\begin{matrix}\beta\\ \gamma\end{matrix}\right\| = Q^{\beta}_{x^{\gamma}} - Q^{\beta}_{x'^{\alpha}}\begin{Bmatrix}\alpha\\ \gamma\end{Bmatrix} - Q^{\beta}_{x''^{\alpha}}\begin{Bmatrix}\alpha\\ \gamma\end{Bmatrix} + Q^{\alpha}A^{\beta}_{\alpha\gamma}\,.$$

To eliminate $\partial x^{(4)\beta}/\partial y^{h}$, we first have $x^{(4)\beta}$ in the form

$$(16)\qquad x^{(4)\beta} = \left(y^{(4)r} + \overline{Q}^{j}\,\overline{\begin{Bmatrix}r\\ j\end{Bmatrix}} + \overline{Q}^{r}_{y^{i}}\,y'^{i} + \overline{Q}^{r}_{y'^{i}}\,y''^{i} + \overline{Q}^{r}_{y''^{i}}\,y'''^{i}\right)\frac{\partial x^{\beta}}{\partial y^{r}} -$$

$$-\left(Q^{\alpha}\begin{Bmatrix}\beta\\ \alpha\end{Bmatrix} + Q^{\beta}_{x^{\gamma}}\,x'^{\gamma} + Q^{\beta}_{x'^{\gamma}}\,x''^{\gamma} + Q^{\beta}_{x''^{\gamma}}\,x'''^{\gamma}\right)$$

by differentiating (12) and using the tensor (13).

By means of formulas (6), (8), (10) and (14) and the tensor

$$R^{\beta}(x,\ x',\ x'',\ x''',\ x^{(4)}) = x^{(4)\beta} + Q^{\alpha}\begin{Bmatrix}\beta\\ \alpha\end{Bmatrix} + Q^{\beta}_{x^{\gamma}}\,x'^{\gamma} + Q^{\beta}_{x'^{\gamma}}\,x''^{\gamma} + Q^{\beta}_{x''^{\gamma}}\,x'''^{\gamma}\,,$$

the partial derivatives of (16) can be put in the form

$$
\begin{aligned}
(17)\qquad \frac{\partial x^{(4)\beta}}{\partial y^h} = &- \left|\!\left|\!\left| \begin{matrix}\beta\\ \gamma\end{matrix} \right|\!\right|\!\right| \frac{\partial x^\gamma}{\partial y^h} + \overline{\left|\!\left|\!\left| \begin{matrix} r\\ h\end{matrix} \right|\!\right|\!\right|} \frac{\partial x^\beta}{\partial y^r} + \\
&+ \Bigg(4 \overline{\begin{Vmatrix} l\\ h\end{Vmatrix}} \overline{\begin{Bmatrix} r\\ l\end{Bmatrix}} + 6 \overline{\begin{vmatrix} l\\ h\end{vmatrix}} \overline{\begin{vmatrix} r\\ l\end{vmatrix}} + 12 \overline{\begin{Bmatrix} r\\ i\end{Bmatrix}} \overline{\begin{Bmatrix} i\\ l\end{Bmatrix}} \overline{\begin{vmatrix} l\\ h\end{vmatrix}} + 4 \overline{\begin{Bmatrix} l\\ h\end{Bmatrix}} \overline{\begin{Vmatrix} r\\ l\end{Vmatrix}} + \\
&+ 12 \overline{\begin{Bmatrix} l\\ h\end{Bmatrix}} \overline{\begin{vmatrix} i\\ l\end{vmatrix}} \overline{\begin{Bmatrix} r\\ i\end{Bmatrix}} + 12 \overline{\begin{Bmatrix} l\\ h\end{Bmatrix}} \overline{\begin{Bmatrix} i\\ l\end{Bmatrix}} \overline{\begin{vmatrix} r\\ i\end{vmatrix}} + 24 \overline{\begin{Bmatrix} l\\ h\end{Bmatrix}} \overline{\begin{Bmatrix} r\\ i\end{Bmatrix}} \overline{\begin{Bmatrix} i\\ t\end{Bmatrix}} \overline{\begin{Bmatrix} t\\ l\end{Bmatrix}}\Bigg) \frac{\partial x^\beta}{\partial y^r} - \\
&- \Bigg(4 \overline{\begin{Vmatrix} l\\ h\end{Vmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} + 6 \overline{\begin{vmatrix} l\\ h\end{vmatrix}} \begin{vmatrix} \beta\\ \delta\end{vmatrix} + 12 \overline{\begin{vmatrix} r\\ h\end{vmatrix}} \overline{\begin{Bmatrix} l\\ r\end{Bmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} + 4 \overline{\begin{Bmatrix} l\\ h\end{Bmatrix}} \begin{Vmatrix} \beta\\ \delta\end{Vmatrix} + \\
&+ 12 \overline{\begin{Bmatrix} t\\ h\end{Bmatrix}} \overline{\begin{vmatrix} l\\ t\end{vmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} + 12 \overline{\begin{Bmatrix} t\\ h\end{Bmatrix}} \overline{\begin{Bmatrix} l\\ t\end{Bmatrix}} \begin{vmatrix} \beta\\ \delta\end{vmatrix} + 24 \overline{\begin{Bmatrix} t\\ h\end{Bmatrix}} \overline{\begin{Bmatrix} l\\ i\end{Bmatrix}} \overline{\begin{Bmatrix} i\\ t\end{Bmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix}\Bigg) \frac{\partial x^\delta}{\partial y^l}
\end{aligned}
$$

in which we have the non-tensor form

$$
(18)\qquad \left|\!\left|\!\left| \begin{matrix}\beta\\ \gamma\end{matrix} \right|\!\right|\!\right| = R^\beta_{x\gamma} - R^\beta_{x'\alpha} \begin{Bmatrix} \alpha\\ \gamma\end{Bmatrix} - R^\beta_{x''\alpha} \begin{vmatrix} \alpha\\ \gamma\end{vmatrix} - R^\beta_{x'''\alpha} \begin{Vmatrix} \alpha\\ \gamma\end{Vmatrix} + R^x \Lambda^\beta_{\alpha\gamma}\,.
$$

Substituting the values given by (6), (10), (14) and (17) in (5) we have

$$
\begin{aligned}
\overline{T}^i_{jy'h} = &\Bigg|T^x_{\gamma x'\beta} \frac{\partial x^\beta}{\partial y^h} - 2\, T^x_{\gamma x''\beta} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} \frac{\partial x^\delta}{\partial y^h} - 3 T^x_{\gamma x'''\beta} \begin{vmatrix} \beta\\ \delta\end{vmatrix} \frac{\partial x^\delta}{\partial y^h} - \\
&\qquad - 4 T^x_{\gamma x^{(4)}\beta} \begin{Vmatrix} \beta\\ \delta\end{Vmatrix} \frac{\partial x^\delta}{\partial y^h} - 5 T^x_{\gamma x^{(5)}\beta} \left|\!\left|\!\left| \begin{matrix}\beta\\ \delta\end{matrix} \right|\!\right|\!\right| \frac{\partial x^\delta}{\partial y^h}\Bigg] \frac{\partial y^i}{\partial x^x} \frac{\partial x^\gamma}{\partial y^j} + \\
&+ 2 \overline{\begin{Bmatrix} l\\ h\end{Bmatrix}} \Bigg[T^x_{\gamma x''\beta} \frac{\partial x^\beta}{\partial y^l} + 3 T^x_{\gamma x'''\beta} \left(\overline{\begin{Bmatrix} r\\ l\end{Bmatrix}} \frac{\partial x^\beta}{\partial y^r} - \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} \frac{\partial x^\delta}{\partial y^l}\right) + \\
&\qquad + 6\, T^x_{\gamma x^{(4)}\beta} \left(- \begin{vmatrix} \beta\\ \delta\end{vmatrix} \frac{\partial x^\delta}{\partial y^l} + \overline{\begin{vmatrix} r\\ l\end{vmatrix}} \frac{\partial x^\beta}{\partial y^r} - 2 \overline{\begin{Bmatrix} t\\ l\end{Bmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} \frac{\partial x^\delta}{\partial y^t} + 2 \overline{\begin{Bmatrix} r\\ t\end{Bmatrix}} \overline{\begin{Bmatrix} t\\ l\end{Bmatrix}} \frac{\partial x^\beta}{\partial y^r}\right) + \\
&+ 10\, T^x_{\gamma x^{(5)}\beta} \Bigg\{- \begin{Vmatrix} \beta\\ \delta\end{Vmatrix} \frac{\partial x^\delta}{\partial y^l} + \overline{\begin{Vmatrix} r\\ l\end{Vmatrix}} \frac{\partial x^\beta}{\partial y^r} + 3 \left(\overline{\begin{vmatrix} t\\ l\end{vmatrix}} \overline{\begin{Bmatrix} r\\ t\end{Bmatrix}} + \overline{\begin{Bmatrix} t\\ l\end{Bmatrix}} \overline{\begin{vmatrix} r\\ t\end{vmatrix}} + 2 \overline{\begin{Bmatrix} r\\ n\end{Bmatrix}} \overline{\begin{Bmatrix} n\\ t\end{Bmatrix}} \overline{\begin{Bmatrix} t\\ l\end{Bmatrix}}\right) \frac{\partial x^\beta}{\partial y^r} - \\
&\qquad - 3 \left(\overline{\begin{vmatrix} t\\ l\end{vmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} + \overline{\begin{Bmatrix} t\\ l\end{Bmatrix}} \begin{vmatrix} \beta\\ \delta\end{vmatrix} + 2 \overline{\begin{Bmatrix} t\\ n\end{Bmatrix}} \overline{\begin{Bmatrix} n\\ l\end{Bmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix}\right) \frac{\partial x^\delta}{\partial y^t}\Bigg\}\Bigg] \frac{\partial y^i}{\partial x^x} \frac{\partial x^\gamma}{\partial y^j} + \\
&+ 3 \overline{\begin{vmatrix} l\\ h\end{vmatrix}} \Bigg[T^x_{\gamma x'''\beta} \frac{\partial x^\beta}{\partial y^l} + 4\, T^x_{\gamma x^{(4)}\beta} \left(\overline{\begin{Bmatrix} r\\ l\end{Bmatrix}} \frac{\partial x^\beta}{\partial y^r} - \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} \frac{\partial x^\delta}{\partial y^l}\right) + \\
&+ 10\, T^x_{\gamma x^{(5)}\beta} \left(- \begin{vmatrix} \beta\\ \delta\end{vmatrix} \frac{\partial x^\delta}{\partial y^l} + \overline{\begin{vmatrix} r\\ l\end{vmatrix}} \frac{\partial x^\beta}{\partial y^r} - 2 \overline{\begin{Bmatrix} r\\ l\end{Bmatrix}} \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} \frac{\partial x^\delta}{\partial y^r} + 2 \overline{\begin{Bmatrix} r\\ t\end{Bmatrix}} \overline{\begin{Bmatrix} t\\ l\end{Bmatrix}} \frac{\partial x^\beta}{\partial y^r}\right)\Bigg] \frac{\partial y^i}{\partial x^x} \frac{\partial x^\gamma}{\partial y^j} + \\
&+ 4 \overline{\begin{Vmatrix} l\\ h\end{Vmatrix}} \Bigg[T^x_{\gamma x^{(4)}\beta} \frac{\partial x^\beta}{\partial y^l} + 5 T^x_{\gamma x^{(5)}\beta} \left(\overline{\begin{Bmatrix} r\\ l\end{Bmatrix}} \frac{\partial x^\beta}{\partial y^r} - \begin{Bmatrix} \beta\\ \delta\end{Bmatrix} \frac{\partial x^\delta}{\partial y^l}\right)\Bigg] \frac{\partial y^i}{\partial x^x} \frac{\partial x^\gamma}{\partial y^j} + \\
&\qquad + 5 \overline{\left|\!\left|\!\left| \begin{matrix} l\\ h\end{matrix} \right|\!\right|\!\right|} T^x_{\gamma x^{(5)}\beta} \frac{\partial x^\beta}{\partial y^l} \frac{\partial y^i}{\partial x^x} \frac{\partial x^\gamma}{\partial y^j}\,,
\end{aligned}
$$

where the terms have been grouped to facilitate the next reduction. Therefore, we have

$$(19)\qquad \overline{T}^{i}_{jy'h} = \left[T^{\alpha}_{\gamma x'\beta} - 2T^{\alpha}_{\gamma x''\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - 3T^{\alpha}_{\gamma x'''\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix} - 4\, T^{\alpha}_{\gamma x^{(4)}\delta}\begin{Vmatrix}\delta\\ \beta\end{Vmatrix} - 5\, T^{\alpha}_{\gamma x^{(5)}\delta}\left|\!\left|\!\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\!\right|\!\right| \right] \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial y^{j}} \frac{\partial x^{\beta}}{\partial y^{h}} + 2\, \overline{T}^{i}_{jy''l}\overline{\begin{Bmatrix} l\\ h\end{Bmatrix}} + 3\, \overline{T}_{jy'''l}\overline{\begin{vmatrix} l\\ h\end{vmatrix}} + 4\, \overline{T}^{i}_{jy^{(4)}l}\overline{\begin{Vmatrix} l\\ h\end{Vmatrix}} + 5\, \overline{T}^{i}_{jy^{(5)}l}\overline{\left|\!\left|\!\left|\begin{matrix} l\\ h\end{matrix}\right|\!\right|\!\right|}\,.$$

Hence the new tensor of one higher covariant rank is

$$(20)\qquad T^{\alpha}_{\gamma x'\beta} - 2\, T^{\alpha}_{\gamma x''\delta}\begin{Bmatrix}\beta\\ \delta\end{Bmatrix} - 3\, T^{\alpha}_{\gamma x'''\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix} - 4\, T^{\alpha}_{\gamma x^{(4)}\delta}\begin{Vmatrix}\delta\\ \beta\end{Vmatrix} - 5\, T^{\alpha}_{\gamma x^{(5)}\delta}\left|\!\left|\!\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\!\right|\!\right|,$$

where $\begin{Bmatrix}\delta\\ \beta\end{Bmatrix}$, $\begin{vmatrix}\delta\\ \beta\end{vmatrix}$, $\begin{Vmatrix}\delta\\ \beta\end{Vmatrix}$, $\left|\!\left|\!\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\!\right|\!\right|$ are definied by (2), (11), (15), (18).

Considering the m times extended point transformation, we extend the process to tensor whose components contain derivatives of any order. We can easily verify by virtue of the general relations in (4) that the covariant rank of the tensor

$$(21)\qquad T^{\alpha\,\cdots}_{\gamma\,\cdots\, x^{(m-4)}\beta} - (m-3)\, T^{\alpha\,\cdots}_{\gamma\,\cdots\, x^{(m-3)}\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - \frac{(m-2)\,(m-3)}{2}\, T^{\alpha\,\cdots}_{\gamma\,\cdots\, x^{(m-2)}\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix} - \frac{(m-1)\,(m-2)\,(m-3)}{3!}\, T^{\alpha\,\cdots}_{\gamma\,\cdots\, x^{(m-1)}\delta}\begin{Vmatrix}\delta\\ \beta\end{Vmatrix} - \frac{m\,(m-1)\,(m-2)\,(m-3)}{4!}\, T^{\alpha\,\cdots}_{\gamma\,\cdots\, x^{(m)}\delta}\left|\!\left|\!\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\!\right|\!\right| \qquad (m \geqslant 5)$$

is one greater than that of the original tensor $T^{\alpha\,\cdots}_{\gamma\,\cdots}$ whose components are functions of $(x, x', x'', \ldots, x^{(m)})$.

Some of the obvious properties of the above process are:

(a) If the components of the tensor $T^{\varkappa}_{\gamma}(x, x', x'', x''', x^{(4)}, x^{(5)})$ do not contain the derivatives $x^{(5)}$, then (20) reduces to the result obtained by H. D. SINGH [4].

(b) If the components of the tensor do not contain $x^{(5)}$ and the tensor equations for $\overline{T}^{i}_{j}(y, y', y'', y^{m}, y^{(4)})$ are differentiated with respect to y''^{h}, then (20) reduces to the result given by H. D. SINGH [3] by putting $m = 6$.

(c) If the components of the tensor do not contain $x^{(4)}$ and $x^{(5)}$, then (20) gives the result obtained by MARIE M. JOHNSON [2].

(d) If there are no x''', $x^{(4)}$ and $x^{(5)}$ derivatives, then the result is CRAIG's covariant derivative (1).

(e) If there are no x'', x''', $x^{(4)}$ and $x^{(5)}$, then the result in partial differentiation with respect to x'.

(f) The usual rules for the derivatives of a sum of tensors of the same rank and kind and for the product of any tensors are conserved.

(g) If $m = 4$, a scalar $T(x, x', x'', x''', x^{(4)})$ will give a covariant tensor which is similar to that in (20), when the tensor equations for $\overline{T}(y, y', y'', y''', y^{(4)})$ are differentiated with respect to y^{h} instead of y'^{h}. The tensor so obtained is

$$T_{x\beta} - T_{x'\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - T_{x''\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix} - T_{x'''\delta}\begin{Vmatrix}\delta\\ \beta\end{Vmatrix} - T_{x^{(4)}\delta}\left|\left|\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\right|\right| .$$

(h) If $m = 4$, a tensor $T^{\varkappa}(x, x', x'', x''', x^{(4)})$ will give

$$T^{\varkappa}_{x\beta} - T^{\varkappa}_{x'\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - T^{\varkappa}_{x''\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix} - T^{\varkappa}_{x'''\delta}\begin{Vmatrix}\delta\\ \beta\end{Vmatrix} - T^{\varkappa}_{x^{(4)}\delta}\left|\left|\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\right|\right| + T^{\delta} A^{\varkappa}_{\delta\beta} ,$$

when the tensor equations for $\overline{T}^{i}(y, y', y'', y''', y^{(4)})$ are differentiated with respect to y^{h}.

(i) However, if $m = 4$, and a tensor $T^{\varkappa}_{\gamma}(x, x', x'', x^{m}, x^{(4)})$ is used under the process (h), the new tensor of one higher covariant rank is

$$T^{\varkappa}_{\gamma x\beta} - T^{\varkappa}_{\gamma x'\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - T^{\varkappa}_{\gamma x''\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix} - T^{\varkappa}_{\gamma x'''\delta}\begin{Vmatrix}\delta\\ \beta\end{Vmatrix} - T^{\varkappa}_{\gamma x^{(4)}\delta}\left|\left|\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\right|\right| + T^{\delta}_{\gamma} A^{\varkappa}_{\delta\beta} - T^{\varkappa}_{\delta} A^{\delta}_{\gamma\beta}$$

and so, a tensor $T_{\gamma}(x, x', x'', x''', x^{(4)})$ will give the new tensor

$$T_{\gamma x\beta} - T_{\gamma x'\delta}\begin{Bmatrix}\delta\\ \beta\end{Bmatrix} - T_{\gamma x''\delta}\begin{vmatrix}\delta\\ \beta\end{vmatrix} - T_{\gamma x'''\delta}\begin{Vmatrix}\delta\\ \beta\end{Vmatrix} - T_{\gamma x^{(4)}\delta}\left|\left|\left|\begin{matrix}\delta\\ \beta\end{matrix}\right|\right|\right| - T_{\delta} A^{\delta}_{\gamma\beta} .$$

Casting our glance on the results obtained by CRAIG [1], MARIE M. JOHNSON [2], H. D. SINGH [3, 4] and in virtue of the equation (21), we can generalise the extensions of the process to tensors whose components involve derivatives of any order. Thus if the components $T^{x\ldots}_{\gamma\ldots}$ of a given tensor are functions of x, x', x'', ..., $x^{(m)}$, then the quantities

$$T^{x\ldots}_{\gamma\ldots x^{(m-n+1)}\beta} - (m-n+2)\, T^{x\ldots}_{\gamma\ldots x^{(m-n+2)}\delta} \left\{ \begin{matrix} \delta \\ \beta \end{matrix} \right\} -$$

$$- \frac{(m-n+3)(m-n+2)}{2!} T^{x\ldots}_{\gamma\ldots x^{(m-n+3)}\delta} \left| \begin{matrix} \delta \\ \beta \end{matrix} \right| -$$

$$- \frac{(m-n+4)(m-n+3)(m-n+2)}{3!} T^{x\ldots}_{\gamma\ldots x^{(m-n+4)}\delta} \left\| \begin{matrix} \delta \\ \beta \end{matrix} \right\| -$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$- \frac{(m+1)\, m\, (m-1) \ldots (m-n+2)}{n!} T^{x\ldots}_{\gamma\ldots x^{(m+1)}\delta} \left\| \ldots \left| \begin{matrix} \delta \\ \beta \end{matrix} \right| \ldots \right\|$$

$(m \geqslant n+1)$ are the components of the tensor.

The number of bars in the general term, i.e. $(n+1)^{\text{th}}$, being $n-1$ and the value of the term associated with bars is obtained by virtue of the relations given by (11), (15) and (18).

Obviously, this process yields the properties similar to those stated above.

In the end I record my respectful thanks to Prof. R. S. MISHRA, M. Sc., Ph. D., D. Sc., University of Delhi for his kind interest and valuable suggestions in the preparation of this paper. I am also grateful to Dr. R. K. SINGH, Principal B. R. College, Agra, who encouraged me for advanced studies.

References.

1. H. V. CRAIG, *On a covariant differentiation process*, Bull. Amer. Math. Soc. **37** (1931), 731-734.

2. MARIE M. JOHNSON, *An extension of a covariant differentiation process*, Bull. Amer. Math. Soc. **46** (1940), 269-271.

3. H. D. SINGH, *An extension of a covariant differentiation process*, Tensor (New series) **7** (1957), 137-140.

4. H. D. SINGH, *The second extension of a covariant differentiation process*, Riv. Mat. Univ. Parma **8** (1957), 397-404.

5. J. H. TAYLOR, *A generalization of Levi Civita's parallelism and Frenet's formulas*, Trans. Amer. Math. Soc. **27** (1925), 246-264.

6. H. D. SINGH, *An extension of a covariant differentiation process*, Tensor (New Series) **8** (1958), 90-94.