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On the variety representing the E_2 of S_r. (**)

1. – The purpose of this paper is to determine algebraic bases of all relevant dimensions on the non-singular variety W (defined to within unexceptional birational equivalence) which represents the curve elements of second order (E_2) of S_r, to find the intersection numbers of the base varieties of complementary dimensions on W, and to point out and exemplify methods of deriving enumerative applications. These investigations will be based on the properties of W obtained by C. LONGO [1] together with a wider use of the method involving degenerate collineations which I have employed elsewhere [3] to find a base for surfaces on W in the the case $r = 2$.

2. – LONGO shows ([1] § 10) that a minimal model of W may be represented parametrically by

$$(1) \qquad \begin{aligned} X_{abcdef} &= u_a\, x_b\, x_c\, x_d\, x_e\, y_f \\ Y_{ghijk} &= v x_g\, y_h\, y_i\, y_j\, y_k \end{aligned}$$

where

(*i*) writing $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, for the vectors of the X_{abcdef} and of the Y_{ghijk}, the pair $(\boldsymbol{X}, \boldsymbol{Y})$ forms a single set of homogeneous coordinates for a point of W;

(*ii*) $\boldsymbol{x} = (x_0, x_1, \dots, x_r)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_{(r^2+r-2)/2})$, and $\boldsymbol{u} = (u_0, u_1, \dots, u_{(r^3-r-6)/6})$ are respective coordinate vectors of the origin, the tangent, and the

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plane of the E_2, say E, represented by $(\boldsymbol{X}, \boldsymbol{Y})$, and the suffices $a, b, \ldots, k$ run through all values compatible with these facts, except that any admissible combination (b, c, d, e) or (h, i, j, k) is represented by only one of its permutations;

(*iii*) writing $\boldsymbol{u} = u\boldsymbol{u}'$, where in $\boldsymbol{u}'$ not only the ratios of the elements but their individual values are taken to be fixed for a given plane $\boldsymbol{u}$, $u : v$ is a position parameter determining E among the E_2 with origin $\boldsymbol{x}$, tangent $\boldsymbol{y}$, and plane $\boldsymbol{u}$ in such a manner that E is inflexional if $u = 0$ and cuspidal if $v = 0$.

Throughout the present paper we take W to be given in this form. Then the set of E_2 having the same given origin, tangent, and plane is represented by a straight line t, which joins a point of the variety C representing the cuspidal E_2 to a point of the variety F representing the inflexional E_2. Through a point of W which is not on F there passes a unique t; the t through a point of F generate a linear space L of dimension $r-1$, representing all E_2 with given origin and tangent; through any point of W there passes a unique L. If the respective dimensions of F, C, and W are f, c, and w, then

$$f = 2r-1, \qquad c = 3r-3, \qquad w = 3r-2\,. \tag{2}$$

The parametrisation (1) may be used to show that there is a group of collineations in the space of W, arising from the collineations of S_r, which leaves W invariant and which is transitive on $W - C - F$, on C, and on F; hence a proof that W is non-singular only requires a demonstration that there exist a single point of C and a single point of F which are simple on W. That such points exist can be shown by numbering the various coordinates in such a way that $\boldsymbol{X} = (1, 0, 0, \ldots, 0)$, $\boldsymbol{Y} = \boldsymbol{O}$ defines a point A of C, and $\boldsymbol{X} = \boldsymbol{O}$, $\boldsymbol{Y} = (1, 0, 0, \ldots, 0)$ a point B of F, and then using parametrisations of W near A and near B, which may be derived from (1), to prove that A and B are simple on W.

3. – Since, for all values of λ and μ, $(\lambda\boldsymbol{X}, \mu\boldsymbol{Y})$ lies on W when $(\boldsymbol{X}, \boldsymbol{Y})$ does, the equations of W may be chosen in the form

$$g_j(\boldsymbol{X}, \boldsymbol{Y}) = 0 \qquad (j = 1, 2, \ldots)$$

where each g_j is homogeneous separately in $\boldsymbol{X}$ and in $\boldsymbol{Y}$ [1]. Let V be an irreducible variety on W with equations

$$f_i(\boldsymbol{X}, \boldsymbol{Y}) = g_j(\boldsymbol{X}, \boldsymbol{Y}) = 0 \qquad (i = 1, 2, \ldots) \tag{3}$$

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[1] I am indebted to Professor J. G. Semple for pointing out that in my earlier paper already quoted ([3] § 3) a statement of this kind should by substituted for the unqualified assertion that the equations of W are homogeneous separately in $\boldsymbol{X}$ and in $\boldsymbol{Y}$.

which does not lie on C, whose dimension is v, and whose intersection with F has the natural dimension $v+f-w$ in each component. Then the essential part of the argument involving degenerate collineations which is referred to in § 1 ([3] § 3) can be applied to the present case with only minor verbal modifications to show that there exist varieties V' and V'' satisfying, respectively, equations

$$f_i(\boldsymbol{O},\ \boldsymbol{Y}) = g_j(\boldsymbol{X},\ \boldsymbol{Y}) = 0$$

and

$$f_i(\boldsymbol{X},\ \boldsymbol{O}) = g_j(\boldsymbol{X},\ \boldsymbol{Y}) = 0$$

which are each algebraically equivalent to V on W.

Suppose now that $(\boldsymbol{X}', \boldsymbol{Y}')$ lies on V' and that $\boldsymbol{Y}' \neq \boldsymbol{O}$; then, remenbering the separate homogeneity of the g_j in $\boldsymbol{X}$ and in $\boldsymbol{Y}$, $(\boldsymbol{O}, \boldsymbol{Y}')$ satisfies (3). Thus those points of V' which do not lie on C lie on the spaces L through the points common to V and F; since the dimensions of this intersection and of an L are $v+f-w$ and $r-1$, respectively, it follows from (2) and the dimension v of V' that each component of V' either lies on C or is composed of spaces L. By a precisely analogous reasoning, each component of V'' either lies on F or is composed of lines t. Thus we arrive at

Lemma I. *A variety on W which has no component on C and meets F only in varieties of the natural dimension is algebraically equivalent on W firstly to a variety on C together with a variety generated by spaces L and secondly to a variety on F together with a variety generated by lines t.*

On F let $(a_0,\ a_1;\ i)$, where $a_0,\ a_1,\ i$ are integers and

$$0 \leqslant a_0 < a_1 \leqslant r, \qquad 0 \leqslant i \leqslant 1,$$

represent the inflexional E_2 whose tangents belong to a given Schubert variety $(a_0,\ a_1)$ and whose origins lie in the S_{a_i} used to define this variety. On C let $(a_0,\ a_1,\ a_2;\ i_0,\ i_1;\ j)$, where $a_0,\ a_1,\ a_2,\ i_0,\ i_1,\ j$ are integers and

$$0 \leqslant a_0 < a_1 < a_2 \leqslant r, \qquad 0 \leqslant i_0 < i_1 \leqslant 2, \qquad 0 \leqslant j \leqslant 1,$$

represent the cuspidal E_2 whose planes belong to a given Schubert variety $(a_0,\ a_1,\ a_2)$, say G, whose tangents belong to the Schubert variety $(a_{i_0},\ a_{i_1})$, say G', determined by the $S_{a_{i_0}}$ and the $S_{a_{i_1}}$ used in the definition of G, and whose origins lie in the $S_{a_{i_j}}$ used in the definition of G'. Let further $[a_0,\ a_1,\ i]$ be the variety generated by the L meeting a given $(a_0,\ a_1;\ i)$, and let $[a_0,\ a_1,\ a_2;\ i_0,$

$i_1; j]$ be the variety generated by the t meeting a given $(a_0, a_1, a_2; i_0, i_1; j)$. Then, writing $a_0 + a_1 = h$, $a_0 + a_1 + a_2 = k$, and $i_0 + i_1 = k'$, the dimensions of $(a_0, a_1; i)$, $[a_0, a_1; i]$, $(a_0, a_1, a_2; i_0, i_1; j)$, and $[a_0, a_1, a_2; i_0, i_1; j)]$ are $h + i - 1$, $h + i + r - 2$, $k + k' + j - 4$, and $k + k' + j - 3$, respectively.

A set of results concerning these varieties, due with one trivial exception to E. MARTINELLI [2] and LONGO [1], is formulated in

LEMMA II. *The varieties $(a_0, a_1; i)$ of dimension d form an algebraic base of dimension d on F; the varieties $(a_0, a_1, a_2; i_0, i_1; j)$ of dimension d form an algebraic base of dimension d on C. For these bases, the intersection number of two base varieties of complementary dimensions on F is unity if the pair of varieties is of the form $(a_0, a_1; i)$, $(r - a_1, r - a_0; 1 - i)$ but zero otherwise, and the intersection number of two base varieties on C is unity if the pair of varieties is of the form $(a_0, a_1, a_2; i_0, i_1; j)$, $(r - a_2, r - a_1, r - a_0; 2 - i_1, 2 - i_0; 1 - j)$ but zero otherwise. An $[a_0, a_1; i]$ meets C in an $(a_0, a_1, r; 0, 1; i)$ if $a_1 < r$, but in an $(a_0, r - 1, r; 0, 2; i)$ if $a_0 + 1 < a_1 = r$, and in an $(r - 2, r - 1, r; 1, 2; i)$ if $a_0 + 1 = a_1 = r$; an $[a_0, a_1, a_2; i_0, i_1; j]$ meets F in an $(a_{i_0}, a_{i_1}; j)$.*

If V, still an irreducible and v- dimensional subvariety of W, either lies on C or has an intersection of excessive dimension with F, let V_1 be the intersection of W with $w - v$ generic primals of order n, and $V + V_2$ the intersection of W with $w - v$ generic primals of order n through V. Then, for sufficiently large n, neither V_1 nor V_2 either has a component on C or has an intersection of excessive dimension with F, so that V is algebraically equivalent on W to the variety $V_1 - V_2$ without components on C or intersections of excessive dimension with F. Hence Lemmas I and II, together with the fact that the t are in unexceptionally birational correspondence with the points of C, and the L with the points of F, lead to

THEOREM I. *For each d such that $0 < d < w$, one algebraic base of dimension d on W is formed by the $(a_0, a_1; i)$ and the $[a_0, a_1, a_2; i_0, i_1; j]$ of dimension d, and another by the $[a_0, a_1; i]$ and the $(a_0, a_1, a_2; i_0, i_1; j)$ of dimension d.*

For $d = w - 1$ (the only case considered by him) LONGO arrives at the result of Th. I in a different manner ([1] § 14).

4. – Denote by ${}_dA_\alpha$ $(\alpha = 1, 2, \ldots)$ the varieties $(a_0, a_1; i)$ of dimension d arranged in some order, and by ${}_dA'_\lambda$ $(\lambda = 1, 2, \ldots)$ the varieties $[a_0, a_1, a_2,; i_0, i_1; j]$ of dimension d arranged in some order. For given values of the integers involved, $[a_0, a_1; i]$ and $(r - a_1, r - a_0; 1 - i)$ have complementary dimensions on W, and so have $(a_0, a_1, a_2; i_0, i_1; j)$ and $[r - a_2, r - a_1, r - a_0; 2 - i_1, 2 - i_0; 1 - j]$; hence we may, and do, define ${}_dB_\beta$ to be the $[a_0, a_1; i]$ of dimension d such that

$(r-a_1,\ r-a_0;\ 1-i)$ is ${}_{w-d}A_\beta$, and ${}_dB'_\mu$ the $(a_0,\ a_1,\ a_2;\ i_0,\ i_1;\ j)$ of dimension d such that $[r-a_2,\ r-a_1,\ r-a_0;\ 2-i_1,\ 2-i_0;\ 1-j]$ is ${}_{w-d}A'_\mu$. We say that the ${}_dA_\alpha$ and the ${}_dA'_\lambda$ together form the A-base of dimension d on W, and that the ${}_dB_\beta$ and the ${}_dB'_\mu$ together form the B-base of dimension d on W. The only members common to the two bases are the ${}_dB_\beta$, which also occur among the ${}_dA'_\lambda$; when $d \geqslant 2r$ there are no ${}_dA_\alpha$, and when $d \leqslant r-2$ there are no ${}_dB_\beta$.

A ${}_dB_\beta$ meets F in the ${}_{d-r+1}A_\alpha$ defined by the same symbols (in round brackets instead of square), and a ${}_dA'_\lambda$ meets C in the ${}_{d-1}B'_\mu$ defined similarly by the same symbols. Hence by Lemma II, writing $|U \cdot V|$ for the intersection number on W of varieties U and V having complementary dimensions on W,

$$|{}_dA_\alpha \cdot {}_{d-w}B_\beta| = \delta^\alpha_\beta, \qquad |{}_dA'_\lambda \cdot {}_{d-w}B'_\mu| = \delta^\lambda_\mu,$$

where δ^α_β and δ^λ_μ are Kronecker deltas. ${}_dA_\alpha$ and ${}_{w-d}B'_\mu$ cannot meet since the former lies on F and the latter on C. If d_1 and d_2 are the respective dimensions of the intersections of F with ${}_dA'_\lambda$ and with ${}_{w-d}B_\beta$, then

$$d_1 \leqslant d-1, \quad d_2 = w-d-r+1,$$

so that

$$d_1 + d_2 \leqslant w - r < f;$$

hence it may be shown (cf. [4] § 5 Cor. II) that these two intersections with F do not meet if the ${}_dA'_\lambda$ and the ${}_{w-d}B_\beta$ concerned are each generic of its kind, and consequently that in this case ${}_dA'_\lambda$ and ${}_{w-d}B_\beta$ themselves do not meet. Thus we arrive at

THEOREM II. *In the intersection matrix on W of the varieties in the A-base of dimension d with the varieties in the B-base of dimension w-d, the element in the i^{th} row and the j^{th} column is the Kronecker delta δ^i_j if the bases are arranged in corresponding orders.*

The A-base and the B-base of a given dimension each form a minimal base of that dimension on W.

The last part of Th. II is of course a consequence of the first part.

Let b_d and b'_d be, respectively, the numbers of the ${}_dB_\beta$ and of the ${}_dB'_\mu$; then by Th. II a minimal base of dimension d on W has $b_d + b'_d$ members. Also, since the ${}_dB_\beta$ and no other varieties are common to the A-base and the B-base of dimension d, the two bases together have exactly $b_d + 2b'_d$ members, so that there exist precisely b'_d independent algebraic equivalences on W connecting these members. Now LONGO has proved ([1] § 10) the algebraic equivalence on W

$$C \sim 3P - 3Q + R \tag{4}$$

where P, Q, and R are the varieties representing the E_2 having respectively their origins on a given prime, their tangents meeting a given secundum, and their planes meeting a given tertium. Intersecting (4) with a ${}_{d+1}A'_{\lambda}$, say $[a_0, a_1, a_2; i_0, i_1; j]$, and expressing the resulting right-hand member in terms of the A-base of dimension d, we derive an equivalence expressing $(a_0, a_1, a_2; i_0, i_1; j)$ in terms of this base; by intersecting (4) in this way with all ${}_{d+1}A'_{\lambda}$ in succession we obtain exactly b'_d such relations, which are moreover independent since each involves a different ${}_dB'_{\mu}$. Hence we arrive at

LEMMA III. *A minimal base for the algebraic equivalences on W relating the members of the A-base and the members of the B-base of dimension d consists of the equivalences obtained by intersecting* (4) *with all varieties ${}_{d+1}A'_{\lambda}$ and expressing the resulting right-hand members in terms of the A-base of dimension d.*

5. - In practice the procedure outlined in Lemma III, although elementary, is laborious owing to the many cases which must be considered, and we omit the details. In each instance the first step is to obtain the intersection of P, Q, and R with the ${}_{d+1}A'_{\lambda}$ concerned as actual sums of members of the A-base of dimension d by choosing the spaces defining P, Q, and R in suitably special position relative to the spaces defining the ${}_{d+1}A'_{\lambda}$; secondly a verification of the fact that the members of the A-base of dimension d which occur in such an intersection do so with unit multiplicity is obtained by taking the spaces defining P, Q, and R in general position, and cutting the intersections $P\cdot{}_{d+1}A'_{\lambda}$, $Q\cdot{}_{d+1}A'_{\lambda}$, and $R\cdot{}_{d+1}A'_{\lambda}$ with the appropriate members of the B-base of dimension $w-d$. The result for the case concerned then follows from (4).

In this way it may be shown that Lemma III leads to the following algebraic equivalences on W, which we shall call the base formulae on W.

$$\text{(I)}\quad (a_0, a_1, a_2; 0, 1; 0) \sim [a_0-1, a_1, a_2; 0, 1; 0] - 2[a_0, a_1-1, a_2; 0, 1; 0] + [a_0, a_1, a_2-1; 0, 1; 0]$$

where

$[a_0-1, a_1, a_2; 0, 1; 0]$ is omitted when $a_0=0$;

$[a_0, a_1-1, a_2; 0, 1; 0]$ is replaced by $[a_0-1, a_0, a_2; 0, 1; 1]$ when $a_1 = a_0+1 > 1$, but omitted when $a_1 = a_0+1 = 1$;

$[a_0, a_1, a_2-1; 0, 1; 0]$ is replaced by $[a_0, a_1-1, a_1; 0, 2; 0]$ when $a_2 = a_1 + 1 > a_0 + 2$, but by $[a_0-1, a_0, a_0+1; 1, 2; 0]$ when $a_2 = a_1 + 1 = a_0 + 2 > 2$ (²).

(II) $$(a_0, a_1, a_2; 0, 2; 0) \sim [a_0-1, a_1, a_2; 0, 2; 0] + [a_0, a_1-1, a_2; 0, 2; 0] - 2[a_0, a_1, a_2-1; 0, 2; 0] - 3[a_0, a_1, a_2; 0, 1; 0]$$

where

$[a_0-1, a_1, a_2; 0, 2; 0]$ is omitted when $a_0 = 0$;

$[a_0, a_1-1, a_2; 0, 2; 0]$ is replaced by $[a_0-1, a_0, a_2; 1, 2; 0]$ when $a_1 = a_0 + 1 > 1$, but by $(0, a_2; 0)$ when $a_1 = a_0 + 1 = 1$;

$[a_0, a_1, a_2-1; 0, 2; 0]$ is omitted when $a_2 = a_1 + 1$.

(III) $$(a_0, a_1, a_2; 1, 2; 0) \sim [a_0-1, a_1, a_2; 1, 2; 0] + [a_0, a_1-1, a_2; 1, 2; 0] - 2[a_0, a_1, a_2-1; 1, 2; 0] - 3[a_0, a_1, a_2; 0, 1; 1]$$

where

$[a_0-1, a_1, a_2; 1, 2; 0]$ is replaced by $(a_1, a_2; 0)$ when $a_0 = 0$;

$[a_0, a_1-1, a_2; 1, 2; 0]$ is omitted when $a_1 = a_0 + 1$;

$[a_0, a_1, a_2-1; 1, 2; 0]$ is replaced by $[a_0, a_1-1, a_1; 1, 2; 1]$ when $a_2 = a_1 + 1 > a_0 + 2$, but omitted when $a_2 = a_1 + 1 = a_0 + 2$.

(IV) $$(a_0, a_1, a_2; 0, 1; 1) \sim -2[a_0-1, a_1, a_2; 0, 1; 1] + [a_0, a_1-1, a_2; 0, 1; 1] + [a_0, a_1, a_2-1; 0, 1; 1] + 3[a_0, a_1, a_2; 0, 1; 0]$$

(²) The case $a_2 = a_1+1 = a_0+2 = 2$ does not arise since $(0, 1, 2; 0, 1; 0)$ is a single point.

where

$[a_0-1,\ a_1,\ a_2;\ 0,\ 1;\ 1]$ is omitted when $a_0=0$;

$[a_0,\ a_1-1,\ a_2;\ 0,\ 1;\ 1]$ is omitted when $a_1=a_0+1$;

$[a_0,\ a_1,\ a_2-1;\ 0,\ 1;\ 1]$ is replaced by $[a_0,\ a_1-1,\ a_1;\ 0,\ 2;\ 1]$ when $a_2=a_1+1>a_0+2$, but by $[a_0-1,\ a_0,\ a_0+1;\ 1,\ 2;\ 1]$ when $a_2=a_1+1=a_0+2>2$, and by $(0,\ 1;\ 1)$ when $a_2=a_1+1=a_0+2=2$.

(V) $$(a_0,\ a_1,\ a_2;0,2;1)\sim-2\,[a_0-1,a_1,a_2;0,2;1]+[a_0,a_1-1,a_2;0,2;1]$$
$$+[a_0,\ a_1,\ a_2-1;\ 0,\ 2;\ 1]+3\,[a_0,\ a_1,\ a_2;\ 0,\ 2;\ 0]$$

where

$[a_0-1,\ a_1,\ a_2;\ 0,\ 2;\ 1]$ is omitted when $a_0=0$;

$[a_0,\ a_1-1,\ a_2;\ 0,\ 2;\ 1]$ is replaced by $[a_0-1,\ a_0,\ a_2;\ 1,\ 2;\ 1]$ when $a_1=a_0+1>1$, but by $(0,\ a_2;\ 1)$ when $a_1=a_0+1=1$;

$[a_0,\ a_1,\ a_2-1;\ 0,\ 2;\ 1]$ is omitted when $a_2=a_1+1$.

(VI) $$(a_0,\ a_1,\ a_2;\ 1,2;1)\sim[a_0-1,\ a_1,\ a_2;\ 1,2;\ 1]-2\,[a_0,\ a_1-1,\ a_2;\ 1,2;\ 1]$$
$$+[a_0,\ a_1,\ a_2-1;1,2;1]+3\,[a_0,\ a_1,\ a_2;1,\ 2;\ 0]-3\,[a_0,\ a_1,\ a_2;\ 0,\ 2;\ 1]$$

where

$[a_0-1,\ a_1,\ a_2;\ 1,\ 2;\ 1]$ is replaced by $(a_1,\ a_2;\ 1)$ when $a_0=0$;

$[a_0,\ a_1-1,\ a_2;\ 1,\ 2;\ 1]$ is omitted when $a_1=a_0+1$;

$[a_0,\ a_1,\ a_2-1;\ 1,\ 2;\ 1]$ is omitted when $a_2=a_1+1$.

Evidently the intersection number of two given members of two A-bases or two B-bases of complementary dimensions on W can be deduced from Th. II and the appropriate base formulae.

6. – We say that an E_2 is contained in a variety U of S_r if it is the E_2 determined by a curve branch on U at one of its points. Then the image U^* of U on W is defined to be the variety generated by the points J representing the E_2 contained in U and the limiting points of such J, while the image D^* on W of a system D of varieties in S_r is the variety generated by the images of the members of D and the limiting points of these images, with the convention that the component of U^* or D^* arising from a multiple component of U or D, as the case may be, is counted with the appropriate multiplicity (3). If u is the dimension of U and of a member of D, and if d, u^*, d^* are the respective dimensions of D, U^*, D^*, then

$$u^* = 3u - 2, \qquad d^* = d + 3u - 2,$$

provided in the second case that $d \leqslant 3\,(r - u)$ and that on each irreducible component of D^* a generic point lies on a finite number of images of members of D.

The base formulae on W can be used to demonstrate that on both U^* and D^* the limiting points referred to in the definitions do actually occur in quite simple cases, as is shown by the following two examples.

(i) Taking $r = 3$, let U be a quadric cone. From base formula (V)

$$(5) \qquad (0, 1, 3;\, 0, 2;\, 1) \sim (0, 3;\, 1) + [0, 1, 2;\, 0, 2;\, 1] + 3\,[0, 1, 3;\, 0, 2;\, 0],$$

and the respective intersection numbers of the members of the right-hand side with U^* are 0, 2, and 0. Thus there exist two (possibly coincident) or infinitely many cuspidal E_2 represented on U^* whose planes pass through a given line g and whose tangents pass through a given point H of g, although no such cuspidal E_2 are contained in U for general g and H. It may in fact be deduced that the threefold representing all cuspidal E_2 having the vertex of U for their origin lies, and is double, on U^*.

(ii) Taking $r = 2$, let D be a general pencil of conics whose line pairs k_i $(i = 1, 2, 3)$ have respective vertices K_i. Put $p = (0, 1, 2;\, 0, 2;\, 0)$, $\overline{p} = (0, 2;\, 0)$, $\overline{q} = (0, 1;\, 1)$, and (as in § 3) $t = [0, 1, 2;\, 0, 1;\, 0]$. Then by intersecting P, Q,

(3) This definition, adequate for the purposes of the examples to be considered here, could for theoretical investigations be replaced by a more formal one, requiring substantial discussion, which would be based on a definition of a generic E_2 contained in an irreducible variety, and which would then resemble the one I have given elsewhere (e. g., [4] § 8 Def. VIII) in a similar case.

and C with the image k^* on W of a generic member k of D, and from base formula (II),

$$k^* \sim 2\bar{p} + 2\bar{q}, \qquad p \sim \bar{p} - 3t\,.$$

The image on W of k_i consists of two curves $\bar{q}$; thus D^* must contain a specialisation of k^* associated with $k \to k_i$ which consists, in addition to these two q, of a curve algebraically equivalent on W to $2p + 6t$, and which may now be shown to have for its components the p associated with K_i as well as the two t which are associated with K_i and the components of k_i (the p being counted twice, and each of the the t three times, in this specialisation). The points of these curves plainly do not lie on the images of any members of D.

7. – We shall conclude by indicating two kinds of enumerative applications of our results, giving examples of each. In doing so we shall always denote a variety or system in S_r and its image on W by the same letter, with an asterisk added for the image. It will be understood that, if Φ is a character of a system D (which may be a single variety) defined as the number of times which a certain property is satisfied by D, and if this number becomes infinite in a particular case (as for instance the number of inflexions of a curve when the latter is a straight line), then Φ is to be interpreted as the virtual intersection number of D^* with the appropriate variety on W.

We shall restrict our attention to general cases in the sense that we shall, without further mention, make two assumptions: Firstly, it will be understood that every system of varieties in S_r considered is such that on each irreducible component of its image on W a generic point lies on precisely one image of a member of the system (the dimension of the system being chosen sufficiently small for this to be possible); secondly, any two systems of varieties in S_r considered together, of which one may be a single variety, and whose images on W have complementary dimensions on W in view of the first assumption, will be supposed to intersect in a finite number of points, each representing an E_2 contained in a unique member of each system, and any two distinct E_2 of this kind being contained in distinct members of each system.

Of these assumptions, the first clearly imposes a condition on the nature of the systems concerned; but from any pair of systems which satisfy this condition, and whose images on W have appropriate dimensions, a pair of systems satisfying the condition imposed by the second assumption can be derived through the transformation of either of the systems by a suitable collineation of S_r, except when one of the systems has an image on W with an intersection of excessive dimension with F. I have elsewhere ([4] § 8) given a full proof of a largely analogous result in a similar though slightly simpler case; a complete demonstration would now be disproportionately laborious in connection with the easy numerical examples to be outlined here, but the reader will find that the earlier argument can be adapted and extended to deal with the present case.

It will be seen that each of our two types of application comprises infinitely many enumerative results; moreover in applications of the first type, to be discussed in § 8, many of these results not only are theoretically obtainable but can in practice be read off without much further labour (4). Thus it may be hoped that the procedures considered below, and especially that of § 8, will provide both a fruitful source of results required in other investigations and a check on results obtained by different methods.

Confronted with an unlimited choice of possible examples, I have been guided by the view that results concerning simple objects easily accessible to the intuition are generally the most interesting, and I have kept in mind that as a check on our methods is it desirable to include examples leading to results which are either known already or easily verifiable by other means.

8. – Our first type of application consists in choosing a system D (which may be a single variety) in S_r and, denoting by d^* the dimension of D^*, to intersect with D^* the base formulae connecting varieties of dimension $w - d^*$, thereby obtaining relations between characters of D.

(i) Taking U to be an irreducible curve, and intersecting with U^* the formula (4) of § 4, itself a special case of base formula (VI), we find

$$c - i = 3\,(n - m),$$

where n, m, and c are the order, rank, and number of cusps of U, respectively, and i (also called the number of apparent inflexions of U) is the number of points J of U such that either U has an inflexion at J or else, M being a fixed S_{r-3} in general position in S_r, the osculating plane to U at J meets M. This result is of course known, being reducible (by projection onto a plane) to one of Plücker's famous equations.

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(4) It will be found that the only remaining difficulty is to determine, for a given system D (possibly a single variety) in S_r, the multiplicities of intersections of D^* with varieties ${}_dB'_\mu$, but also that even this difficulty is frequently removed by the evident absence of such intersections.

(ii) Let U be a surface in S_3. Intersecting with U^* the formula (5) of § 6 we find that, for non-singular U,

$$j = m - \mu$$

where m and μ are, respectively, the class and rank of U, and j is the number of inflexional tangents to U which pass through a fixed general point of S_3. The reason why this result applies only to non-singular surfaces is clear from example (i) of § 6. Again, the intersection of

$$(0, 1, 3; 1, 2; 0) \sim (1, 3; 0) - 2\,[0, 1, 2; 1, 2; 0] - 3\,[0, 1, 3; 0, 1; 1]$$

with U^* leads to the familiar result that U, not now a plane but otherwise arbitrary, has two inflexional tangents at a general point of itself.

(iii) Let U be a surface lying properly in S_r $(r \geqslant 4)$. For non-singular U the intersection of

$$(r-3, r-2, r; 0, 2; 1) \sim -2\,[r-4, r-2, r; 0, 2; 1] + [r-4, r-3, r; 1, 2; 1]$$
$$+ [r-3, r-2, r-1; 0, 2; 1] + 3\,[r-3, r-2, r; 0, 2; 0]$$

with U^* leads to

$$e = m + 2\lambda - \mu,$$

where m and μ are, respectively, the class and rank of U, λ is the number of tangent planes to U which meet a general S_{r-4} of S_r (5), and e is the number of osculating planes to curve branches on U which meet a general S_{r-3} of S_r in a line. Again, without restriction on the singularities of U,

$$(r-3, r-2, r; 1, 2; 0) \sim [r-4, r-2, r; 1, 2; 0]$$
$$-2\,[r-3, r-2, r-1; 1, 2; 0] - 3\,[r-3, r-2, r; 0, 1; 1]$$

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(5) This interpretation of $\lambda = |\, U^* \cdot [r-4, r-2, r; 0, 2; 1]\,|$ is equivalent to the statement that those osculating planes to curve branches on U which pass through a given tangent line to U form a linear pencil. Hence this known result could be derived from

$$(r-4, r-1, r; 0, 2; 1) \sim -2\,[r-5, r-1, r; 0, 2; 1] + [r-4, r-2, r; 0, 2; 1]$$
$$+ 3\,[r-4, r-1, r; 0, 2; 0]$$

(with the first term on the right-hand side omitted when $r = 4$), since the respective intersection numbers of U^* with the members of the right-hand side are 0, λ, and 0.

leads by intersection with U^* to the known result that for $r > 4$ the fourfold cone formed by the osculating planes to curve branches on U at a fixed general point is a quadric cone.

(iv) Let D be a system of ∞^2 primals in S_r ($r \geqslant 3$) such that α members of D contain a curve branch having a given tangent line and a given osculating plane at a given point, β touch a given line g and have at their points of contact with g a given tangent plane through g, i have at a given point an inflexional tangent in a given plane, and j have a given inflexional tangent. Defining t, p, $\overline{p}$, $\overline{q}$ as in example (ii) of § 6, and putting $q = (0, 1, 2; 0, 1; 1)$, it follows by intersecting

$$(6) \qquad p \sim \overline{p} - 3t, \qquad q \sim \overline{q} + 3t$$

with D^* that

$$(7) \qquad i = 3\alpha$$

if the members of D, taken together, have at most ∞^1 multiple points in a given plane, and

$$(8) \qquad j = \beta - 3\alpha$$

if the members of D, taken together, have at most a finite number of multiple points in a given plane. Intersecting the members of D with a plane, it follows further that (7) and (8) remain valid when D is replaced by a system $\sum$ of ∞^2 plane curves of which α touch a given line at a given point, β have double points on a given line, i have inflexions at a given point, and j have a given inflexional tangent, provided $\sum$ is such that these numbers are all finite ([6]). These results concerning $\sum$ may be verified by means of direct algebra and correspondences.

(v) Let D be a system of ∞^1 primals in S_r ($r \geqslant 3$) in which α members pass through a given point, β touch a given line, μ touch a given plane, i have an inflexional tangent belonging to a given linear pencil of lines, and j have an

([6]) This approach is preferable to the direct application of (6) to the case $r = 2$, since it is not clear a priori whether and with what multiplicity the surface G representing the cuspidal E_2 in the plane of $\sum$ which have their origins on the Jacobian of $\sum$ is contained in $\sum^*$ (cf. example (ii) of § 6); the argument employed in the text shows that G is in general simple on $\sum^*$.

inflexional tangent at a point of a given line g and lying in a given plane through g. Then by intersecting with D^* the formulae

$$(0, 1, 2; 1, 2; 0) \sim (1, 2; 0) - 3[0, 1, 2; 0, 1; 1],$$

$$(0, 1, 2; 0, 2; 1) \sim (0, 2; 1) + 3[0, 1, 2; 0, 2; 0]$$

it follows that

$$j = 3\beta \tag{9}$$

if the members of D, taken together, have at most a finite number of singularities in a given plane, and

$$i = \mu - 3\alpha \tag{10}$$

if the members of D have no singularities in a given plane. Proceeding as in (iv), (9) and (10) remain valid if D is replaced by a system of ∞^1 plane curves in which α pass through a given point, β touch a given line, μ have double points, i have inflexional tangents passing through a given point, and j have inflexions on a given line, provided all these numbers are finite.

9. – Our second type of application consists in choosing systems D_1 and D_2 (of which one may be a single variety) in S_r such that D_1^* and D_2^* have complementary dimensions on W, expressing D_1^* in terms of the appropriate A-base or B-base, and intersecting the expression obtained with D_2^* so as to obtain a formula, in terms of the characters of D_1 and D_2, for the number $T(D_1, D_2)$ of pairs (V_1, V_2) of varieties, respectively in D_1 and D_2, such that a curve branch on V_1 has osculating contact with a curve branch on V_2. Applications of this kind are more laborious, and also of rather more limited interest, than those of § 8; we therefore give details only for two of the simplest cases.

(i) Let D_1 be a system of ∞^1 curves which, taken together, have singularities at most at a finite number of points, and of which α_1 meet a given secundum and β_1 touch a given prime. For $r \geqslant 3$ let λ_1 members of D_1 have tangents meeting a given S_{r-3}, and i_1 have osculating planes meeting a given S_{r-2} in a line; for $r = 2$ put $\lambda_1 = 0$ and take i_1 to be the number of curves in D_1 which have inflexional tangents through a given point. By intersecting D_1^* with the B-base of dimension $w - 2$,

$$D_1^* \sim \mu_1[0, 1, 2; 0, 2; 0] + \beta_1(1, 2; 0) + \alpha_1(0, 2; 1) + \lambda_1(0, 3; 0)$$

where by base formula (V)

$$\mu_1 = | D_1^* \cdot (r-2,\ r-1,\ r;\ 0,\ 2;\ 1) | = 3\alpha_1 + i_1 - 2\lambda_1 .$$

Let D_2 be a system of ∞^1 primals of which α_2 pass through a given point, β_2 touch a given line, i_2 have inflexional tangents belonging to a given linear pencil of lines, and j_2 have inflexional tagents at points of a given line g lying in a given plane through g. Then, intersecting D_2^* with the expression for D_1^*,

$$T(D_1,\ D_2) = \mu_1 \alpha_2 + \beta_1 j_2 + \alpha_1 i_2 + 2\lambda_1 \alpha_2,$$

so that from § 8 example (v)

$$T(D_1,\ D_2) = 3\alpha_1 \alpha_2 + 3\beta_1 \beta_2 + \alpha_1 i_2 + \alpha_2 i_1 \tag{11}$$

if the members of D_2 taken together have at most a finite number of multiple points in a given plane. In particular, taking $r = 2$, (11) gives the number of 3-point contacts between the members of two systems of ∞^1 curves in a plane which, taken together, have only a finite number of multiple points. (7)

(ii) Let U be a primal of order n_1 whose general plane section has the class μ_1 and the number i_1 of inflexions; for $r \geqslant 3$ let a general solid section of U have the class m_1 and the number j_1 of inflexional tangents passing through a general point of the intersecting solid. Then by cutting U^* with the members of the A-base of dimension 3,

$$U^* \sim n_1 (r-2,\ r-1,\ r;\ 0,\ 1;\ 1) + \mu_1 (r-2,\ r-1,\ r;\ 0,\ 2;\ 0)$$
$$+ i_1 [r-2,\ r-1;\ 0] + 2n_1 [r-3,\ r-1;\ 1] + j_1 [r-3,\ r;\ 0],$$

the last two terms being omitted when $r = 2$. Let D be a system of ∞^2 curves among which α_2 touch a given prime N at points of a given S_{r-2} lying in N. For $r \geqslant 3$ let β_2 members of D meet a given S_{r-3}, λ_2 have tangent lines in N meeting a given S_{r-3} of N, i_2 have at points of a given S_{r-2} osculating planes

(7) A simple verification by elementary methods is available when D_1 and D_2 are both pencils of conics. For by considering the representation of the conics in the web containing both D_1 and D_2 on the points of an S_3, it may be shown that in this case $T(D_1,\ D_2)$ is the number of lines of a general linear congruence in S_3 which are inflexional tangents to the cubic surface in which the chord locus of a Veronese surface G is cut by the S_3 when the latter is general in the space of G.

which meet this S_{r-2} in lines, and j_2 have osculating planes in N. For $r = 2$ put $\beta_2 = \lambda_2 = 0$, and let i_2 members of D have an inflexion at a given point, while j_2 have a given inflexional tangent. Then, intersecting the expression for U^* with D^*, and using base formulae (IV) and (II) to evaluate $| D^* \cdot (r-2, r-1, r; 0, 1; 1) |$ and $| D^* \cdot (r-2, r-1, r; 0, 2; 0) |$, we find

$$T(U, D) = n_1(-2\lambda_2 + j_2 + 3\alpha_2) + \mu_1(\beta_2 + i_2 - 3\alpha_2) + i_1\alpha_2 + 2n_1\lambda_2 + j_1\beta_2$$

$$= n_1 j_2 + \mu_1 i_2 + (3n_1 - 3\mu_1 + i_1)\alpha_2 + (\mu_1 + j_1)\beta_2 .$$

In view of examples (i) and (ii) of § 8 this result reduces to

$$T(U, D) = n_1 j_2 + \mu_1 i_2 + m_1 \beta_2$$

when $r \geqslant 3$ and U has no singularities in a general solid, and to

$$T(U, D) = n_1 j_2 + \mu_1 i_2 + c_1 \alpha_2$$

when $r = 2$ and the curve U has c_1 cusps.

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