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A. C. SHAMIHOKE (*)

Curves in a four-space with non-positive-definite metric-form (**)

1. – Introduction.

Forsyth in his book on Differential Geometry and Macduffee in a paper published in 1957 had earlier studied curves in a four space equipped with a metric of signature 4 and — 2 respectively . In the present paper we have considered the case of remaining two signatures viz. Two and zero. We have obtained Frenet's Formulae and also solved the converse problem in a particular case when K, T, S are non-zero real numbers.

2. – Frenet's Formulae when the signature is + 2.

Let the metric of the space be given by

$$ds^2 = \sum_{i,j} g_{ij}\, dx_i\, dx_j \tag{2.1}$$

where g_{ij}'s are real numbers. We assume g_{ij}'s to be symmetric so that the matrix $[g_{ij}]$ is also symmetric. We firstly consider the case of signature two. The matrix being of signature 2, (2.1) must be reducible to

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \tag{2.2}$$

(*) Indirizzo dell'A.: Dept. of Mathematics, University of Delhi.
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The matrix of the space is given by

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.3}$$

The set of all linear transformations leaving (2.2) invariant form the total Lorentzian group, i.e. the translations

$$\bar{x} = x + \alpha_1, \quad \bar{y} = y + \alpha_2, \quad \bar{z} = z + \alpha_3, \quad \bar{t} = t + \alpha_4 \tag{2.4}$$

along with the restricted Lorentzian transformations

$$\begin{aligned} \bar{x} &= a_{11}\, x + a_{12}\, y + a_{13}\, z + a_{14}\, t \\ \bar{y} &= a_{21}\, x + a_{22}\, y + a_{23}\, z + a_{24}\, t \\ \bar{z} &= a_{31}\, x + a_{32}\, y + a_{33}\, z + a_{34}\, t \\ \bar{t} &= a_{41}\, x + a_{42}\, y + a_{43}\, z + a_{44}\, t \end{aligned} \tag{2.5}$$

where,

$$A^T JA = J, \qquad \text{A being equal to } [a_{rs}]. \tag{2.6}$$

A curve is defined as an ordered set of four functions

$$\{ x(u), \quad y(u), \quad z(u), \quad t(u) \}$$

subject to the restriction

$$x'^2 + y'^2 + z'^2 - t'^2 = 1 \tag{2.7}$$

the primes denoting differentiation with respect to the arc length s.

The distance betwenn two points on the same curve is defined by

$$s = \int_{u_1}^{u} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \dot{t}^2}\, \mathrm{d}u \tag{2.8}$$

Let v and w be two vectors, by (2.5),

(2.9) $\bar{v} = Av$, $\bar{w} = Aw$ $\bar{w}^T = w^T A^T$, so that $\bar{w}^T J\bar{v} = w^T A^T JAv = w^T Jv$

which implies that $w^T Jv$ remains invariant, or if,

$$v = (\lambda, \mu, \nu, k)^T \quad \text{and} \quad w = (\lambda_1, \mu_1, \nu_1, k_1)^T$$

then $\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 - kk_1$ remains invariant. Therefore

$$h_{ij} = x^{(i)} x^{(j)} + y^{(i)} y^{(j)} + z^{(i)} z^{(j)} - t^{(i)} t^{(j)} \tag{2.10}$$

is an invariant for all values of i and $j \geqslant 1$. Let

$$\varDelta = \begin{bmatrix} x' & x'' & x''' & x^{iv} \\ y' & y'' & y''' & y^{iv} \\ z' & z'' & z''' & z^{iv} \\ t' & t'' & t''' & t^{iv} \end{bmatrix} \tag{2.11}$$

Then,

$$\varDelta^T J \varDelta = H\,[= |h_{rs}|] \tag{2.12}$$

Now $\overline{H} = H$, and also

$$\begin{aligned} &h_{11} = 1, \quad h_{12} = 0, \quad h_{13} = -h_{22}, \quad h_{14} = -\frac{3}{2}h'_{22}, \\ &h_{23} = \frac{1}{2}h'_{22}, \quad h_{24} = \frac{1}{2}h''_{22} - h_{33}, \quad h_{34} = \frac{1}{2}h'_{33}, \end{aligned} \tag{2.13}$$

so that

$$H = \begin{bmatrix} 1 & 0 & -h_{22} & -\frac{3}{2}h'_{22} \\ 0 & h_{22} & \frac{1}{2}h'_{22} & \frac{1}{2}h''_{22} - h_{33} \\ -h_{22} & \frac{1}{2}h'_{22} & h_{33} & \frac{1}{2}h'_{33} \\ -\frac{3}{2}h'_{22} & \frac{1}{2}h''_{22} - h_{33} & \frac{1}{2}h'_{33} & h_{44} \end{bmatrix} \tag{2.14}$$

J being of rank 4 and signature 2, the canonical form of H must be given by

$$(2.15)\qquad F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K^2 & 0 & 0 \\ 0 & 0 & K^2\,T^2 & 0 \\ 0 & 0 & 0 & -K^2\,T^2\,S^2 \end{bmatrix}$$

K, T, S being real function ofs. From (2.15)

$$K^2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} > 0, \qquad K^4\,T^2 = \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} > 0$$

and $-K^6\,T^4\,S^2 = |\,H\,| < 0$

Also,

$$(2.16)\qquad G^T\,HG = F$$

where,

$$G = \begin{bmatrix} 1 & 0 & K^2 & KK' - K^2\dfrac{T'}{T} \\ 0 & 1 & -\dfrac{K'}{K} & T^2 + K^2 - \dfrac{K''}{K} + \dfrac{2K'^2}{K^2} + \dfrac{K'T'}{KT} \\ 0 & 0 & 1 & -\dfrac{2K'}{K} - \dfrac{T'}{T} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us form a matrix M by dividing the second column of G by K, third by KT and the fourth by KTS, i.e.,

$$(2.17)\qquad M = \begin{bmatrix} 1 & 0 & \dfrac{K}{T} & \dfrac{K'}{TS} - \dfrac{KT'}{T^2S} \\ 0 & \dfrac{1}{K} & -\dfrac{K'}{K^2T} & \dfrac{T}{KS} + \dfrac{K}{TS} - \dfrac{K''}{K^2TS} + \dfrac{2K'^2}{K^3TS} + \dfrac{K'T'}{K^2T^2S} \\ 0 & 0 & \dfrac{1}{KT} & \dfrac{2K'}{K^2TS} - \dfrac{T'}{KT^2S} \\ 0 & 0 & 0 & \dfrac{1}{KTS} \end{bmatrix}$$

From (2.17) we have

$$M^T H M = J \tag{2.18}$$

Calculating the inverse of M, we have

$$\Phi = M^{-1} = \begin{bmatrix} 1 & 0 & -K^2 & -3KK' \\ 0 & K & K' & K'' - K^3 - KT^2 \\ 0 & 0 & KT & 2K'T + KT' \\ 0 & 0 & 0 & KTS \end{bmatrix} \tag{2.19}$$

We have

$$\Delta^T J \Delta = \Phi^T J \Phi \Longrightarrow (\Delta\Phi^{-1})^T J (\Delta\Phi^{-1}) = J$$

so that,

$$\Delta\,\Phi^{-1} \text{ must be Lorentzian}$$

Let us define,

$$\Lambda = \Delta\,\Phi^{-1} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} \tag{2.20}$$

From (2.20), we find

$$\begin{aligned}
&x' = \lambda_1, \quad x'' = K\lambda_2, \quad x''' = -K^2\lambda_1 + K'\lambda_2 + KT\lambda_3, \\
&y' = \mu_1, \quad y'' = K\mu_2, \quad y''' = -K^2\mu_1 + K'\mu_2 + KT\mu_3, \\
&z' = \nu_1, \quad z'' = K\nu_2, \quad z''' = -K^2\nu_1 + K'\nu_2 + KT\nu_3, \\
&t' = k_1, \quad t'' = Kk_2, \quad t''' = -K^2 k_1 + K' k_2 + KTk_3, \\
&x^{iv} = -3KK'\lambda_1 + (K'' - K^3 - KT^2)\lambda_2 + (2K'T + KT')\lambda_3 + KTS\lambda_4 \\
&y^{iv} = -3KK'\mu_1 + (K'' - K^3 - KT^2)\mu_2 + (2K'T + KT')\mu_3 + KTS\mu_4 \\
&z^{iv} = -3KK'\nu_1 + (K'' - K^3 - KT^2)\nu_2 + (2K'T + KT')\nu_3 + KTS\nu_4 \\
&t^{iv} = -3KK'k_1 + (K'' - K^3 - KT^2)k_2 + (2K'T + KT')k_3 + KTSk_4
\end{aligned} \tag{2.21}$$

From (2.21), we obtain

$$
\begin{aligned}
&\lambda_1' = K\lambda_2, \qquad \lambda_2' = -K\lambda_1 + T\lambda_3, \qquad \lambda_3' = -T\lambda_2 + S\lambda_4\\
&\mu_1' = K\mu_2, \qquad \mu_2' = -K\mu_1 + T\mu_3, \qquad \mu_3' = -T\mu_2 + S\mu_4\\
&\nu_1' = K\nu_2, \qquad \nu_2' = -K\nu_1 + T\nu_3, \qquad \nu_3' = -T\nu_2 + S\nu_4\\
&k_1' = Kk_2, \qquad k_2' = -Kk_1 + Tk_3, \qquad k_3' = -Tk_2 + Sk_4
\end{aligned}
\tag{2.22}
$$

From the fact that Λ is Lorentzian, we obtain

$$
\begin{aligned}
&\lambda_1^2 + \mu_1^2 + \nu_1^2 - k_1^2 = 1, \quad \lambda_2^2 + \mu_2^2 + \nu_2^2 - k_2^2 = 1,\\
&\lambda_3^2 + \mu_3^2 + \nu_3^2 - k_3^2 = 1, \quad \lambda_4^2 + \mu_4^2 + \nu_4^2 - k_4^2 = -1,\\
&\lambda_i \lambda_j + \mu_i \mu_j + \nu_i \nu_j - k_i k_j = 0 \text{ for } i \neq j .
\end{aligned}
\tag{2.23}
$$

From (2.22) and (2.23), we obtain

$$
\lambda_4' = S\lambda_3, \quad \mu_4' = S\mu_3, \quad \nu_4' = S\nu_3, \quad k_4' = Sk_3 .
\tag{2.24}
$$

The Formulae (2.21) and (2.23) may also be written as

$$
\Lambda' = \Lambda Q
\tag{2.25}
$$

where,

$$
Q = \begin{bmatrix} 0 & -K & 0 & 0 \\ K & 0 & -T & 0 \\ 0 & T & 0 & S \\ 0 & 0 & S & 0 \end{bmatrix}
\tag{2.26}
$$

3. – The converse problem.

Let $S(s)$ be given of the form (2.26), to determine the curve upto a Lorentzian transformation is our problem.

Let $\Lambda(s)$ be a matrix satisfying the differential equation

(3.1) $$\Lambda'(s) = \Lambda(s) \quad Q(s) .$$

Let $\Lambda(s)$ be Lorentzian for some value s_2 in the interval $s_0 \leqslant s \leqslant s_1$, so that $\Lambda(s)$ will be Lorentzian throughout the interval $s_0 \leqslant s \leqslant s_1$.

We shall solve the equation only for the case $K' = T' = S' = 0$. In this case Q is constant. We have

(3.2) $$\Lambda' = \Lambda Q, \; \Lambda'' = \Lambda Q^2, \; \Lambda''' = \Lambda Q^3, \; \Lambda^{(iv)} = \Lambda Q^4 .$$

The characteristic equation of Q is

$$x^4 + (K^2 + T^2 - S^2)x^2 - K^2 S^2 = 0 .$$

Since Q satisfies it,

(3.3) $$Q^4 + (K^2 + T^2 - S^2)Q^2 - K^2 S^2 I = 0 .$$

Operating Λ, we have

(3.4) $$\Lambda^{(iv)} + (K^2 + T^2 - S^2)\, \Lambda'' - K^2 S^2 \Lambda = 0 .$$

Its roots are easily seen to be of the form

$$\pm \mu, \quad \pm i\nu$$

so that we have

(3.5) $$\Lambda = A \cosh \mu s + B \sinh \mu s + C \cos \nu s + D \sin \nu s .$$

Differentiating, we obtain:

(3.6) $$\Lambda\, Q = \Lambda' = \mu B \cosh \mu s + \mu A \sinh \mu s + \nu D \cos \nu s - \nu C \sin \nu s .$$

These functions being linearly independent, we have

$$AQ = \mu B, \quad BQ = \mu A, \quad CQ = \nu D, \quad DQ = -\nu C. \tag{3.7}$$

Taking $\Lambda(0) = I$ as the initial condition, we have from (3.7)

$$A = \frac{Q^2 + \nu^2 I}{\mu^2 + \nu^2}, \quad B = \frac{Q^3 + \nu^2 Q}{\mu(\mu^2 + \nu^2)}, \quad C = \frac{-Q^2 + \mu^2 I}{\mu^2 + \nu^2}, \quad D = \frac{-Q^3 + \mu^2 Q}{\nu(\mu^2 + \nu^2)} \tag{3.8}$$

Also, $\Delta = \Lambda\Phi \Longrightarrow \Delta\Phi^{-1} = \Lambda$ so that the first column of Λ must be $(x', y', z', t')^T$. Therefore

$$\begin{aligned}
\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left(\frac{\nu^2}{\mu^2 + \nu^2} \cosh \mu s + \frac{\mu^2}{\mu^2 + \nu^2} \cos \nu s \right) \\
&+ \begin{bmatrix} 0 \\ K \\ 0 \\ 0 \end{bmatrix} \left(\frac{\nu^2}{\mu^2 + \nu^2} \frac{\sinh \mu s}{\mu} + \frac{\mu^2}{\mu^2 + \nu^2} \frac{\sin \nu s}{\nu} \right) \\
&+ \begin{bmatrix} K^2 \\ 0 \\ KT \\ 0 \end{bmatrix} \left(\frac{1}{\mu^2 + \nu^2} \cosh \mu s - \frac{1}{\mu^2 + \nu^2} \cos \nu s \right) \\
&+ \begin{bmatrix} 0 \\ -K^3 - KT^2 \\ 0 \\ KTS \end{bmatrix} \left(\frac{1}{\mu^2 + \nu^2} \frac{\sinh \mu s}{\mu} - \frac{1}{\mu^2 + \nu^2} \frac{\sin \nu s}{\nu} \right)
\end{aligned} \tag{3.9}$$

Integrating these equations sub²ect to the initial condition that $(x, y, z, t) = (0, 0, 0, 0)$ for $s = 0$, and rearranging, we obtain

$$
\begin{aligned}
&x - \frac{\nu^2 - K^2}{KT} z = \frac{1}{\nu} \sin \nu s \qquad && x + \frac{K^2 + \mu^2}{KT} z = \frac{1}{\mu} \sinh \mu s \\
&y - \frac{\mu^2 - S^2}{ST} t = \frac{K}{\nu^2}(1 - \cos \nu s) \qquad && y + \frac{\nu^2 + S^2}{ST} t = \frac{K}{\mu^2}(\cosh \mu s - 1)
\end{aligned}
\tag{3.10}
$$

The transformation,

$$
\begin{aligned}
&\bar{x} = \frac{1}{F_1} x - \frac{1}{F_1} \frac{\nu^2 - K^2}{KT} z \qquad && \bar{y} = \frac{1}{F_2} y - \frac{1}{F_2} \frac{\mu^2 - S^2}{ST} t \\
&\bar{z} = \frac{1}{F_3} x + \frac{1}{F_3} \frac{K^2 + \mu^2}{KT} z \qquad && \bar{t} = \frac{1}{F_4} y + \frac{1}{F_4} \frac{\nu^2 + S^2}{ST} t
\end{aligned}
\tag{3.11}
$$

will be Lorentzian, provided,

$$
\begin{aligned}
&F_1 = \pm \sqrt{\frac{\mu^2 + \nu^2}{\nu^2 - K^2}} \qquad && F_2 = \pm \sqrt{\frac{\mu^2 + s^2}{S^2 + \nu^2}} \\
&F_3 = \pm \sqrt{\frac{\mu^2 + \nu^2}{K^2 + \mu^2}} \qquad && F_4 = \pm \sqrt{\frac{\mu^2 + \nu^2}{S^2 - \mu^2}}
\end{aligned}
\tag{3.12}
$$

It transforms (3.10) into the equations

$$
\begin{aligned}
&\bar{x} = \frac{1}{\nu F_1} \sin \nu s, \qquad && \bar{y} = \frac{K}{\nu^2 F_2}(1 - \cos \nu s), \\
&\bar{z} = \frac{1}{\mu F_3} \sinh \mu s, \qquad && \bar{t} = \frac{K}{\mu^2 F_4}(\cosh \mu s - 1)\,.
\end{aligned}
\tag{3.13}
$$

After the translation,

$$
\xi = \bar{x}, \qquad \eta = \bar{y} - \frac{K}{\nu^2 F_2}, \qquad \zeta = \bar{z}, \qquad u = t + \frac{K}{\mu^2 F_4}
\tag{3.14}
$$

we have

$$\xi = a \sin \nu s, \quad \eta = a \cos \nu s, \quad \zeta = b \sinh \mu s, \quad u = b \cosh \mu s, \tag{3.15}$$

$$a^2 \nu^2 - b^2 \mu^2 = 1$$

where,

$$a^2 = \frac{1}{\nu^2 F_1^2} = \frac{K^2}{\nu^4 F_2^2}, \qquad \text{and} \qquad b^2 = \frac{1}{\mu^2 F_3^2} = \frac{K^2}{\mu^4 F_4^2}$$

4. – Frenet's formulae for the case of signature zero.

We now c.me to the consideration of signature zero. In this case the metric form must be reducible to

$$ds^2 = dx^2 + dy^2 - dz^2 - dt^2 \tag{4.1}$$

so that the matrix of the space is given by

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.2}$$

The set of all linear transformation leaving (4.1) invariant form the total Lorentzian group here, i.e., the translations

$$\bar{x} = x + \alpha, \quad \bar{y} = y + \alpha_2, \quad \bar{z} = z + \alpha_3, \quad \bar{t} = t + \alpha_4 \tag{4.3}$$

and the transformations

$$\begin{aligned} \bar{x} &= a_{11} x + a_{12} y + a_{13} z + a_{14} t, \\ \bar{y} &= a_{21} x + a_{22} y + a_{23} z + a_{24} t, \\ \bar{z} &= a_{31} x + a_{32} y + a_{33} z + a_{34} t, \\ \bar{t} &= {}_{41} x + a_{42} y + a_{43} z + a_{44} t \end{aligned} \tag{4.4}$$

where,

$$A^T JA = J, \quad J \text{ being given by (4.2)} \quad A = [a_{rs}]. \tag{4.5}$$

A curve is defined as an ordered set of four functions

$$\{ x(u), \quad y(u), \quad z(u), \quad t(u) \}$$

subject to the restriction

$$x'^2 + y'^2 - z'^2 - t'^2 = 1 \tag{4.6}$$

the primes denoting differentiation with respect te the arc lenght s.

As in the section 2, we obtain that if (λ, μ, ν, k) and $(\lambda_1, \mu_1, \nu_1, k_1)$ be two vectors, then $\lambda\lambda_1 + \mu\mu_1 - \nu\nu_1 - kk_1$ remains invariant. Therefore,

$$\begin{aligned} &\bar{h}_{ij} = h_{ij}, \ if \\ &h_{ij} = x^{(i)} x^{(j)} + y^{(i)} y^{(j)} - z^{(i)} z^{(j)} - t^{(i)} t^{(j)} \end{aligned} \tag{4.7}$$

for all values od i and $j \geqslant 1$.

Let,

$$\Delta = \begin{bmatrix} x' & x'' & x''' & x^{iv} \\ y' & y'' & y''' & y^{iv} \\ z' & z'' & z''' & z^{iv} \\ t' & t'' & t''' & t^{iv} \end{bmatrix} \tag{4.8}$$

Then,

$$\Delta^T J \Delta = H$$

Also,

$$H = \begin{bmatrix} 1 & 0 & -h_{22} & -\frac{3}{2} h'_{22} \\ 0 & h_{22} & \frac{1}{2} h'_{22} & \frac{1}{2} h'_{22} - h_{33} \\ -h_{22} & \frac{1}{2} h'_{22} & h_{33} & \frac{1}{2} h'_{33} \\ -\frac{3}{2} h'_{22} & \frac{1}{2} h''_{22} - h_{33} & \frac{1}{2} h'_{33} & h_{44} \end{bmatrix} \tag{4.9}$$

New J being of signature zero, the canonical form of H must be given by

$$(4.10)\qquad F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K^2 & 0 & 0 \\ 0 & 0 & -K^2T^2 & 0 \\ 0 & 0 & 0 & -K^2T^2S^2 \end{bmatrix}$$

where,

$$(4.11)\qquad \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = K^2 > 0$$

$$\begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} = -K^4T^2 < 0, \quad |H| = K^6T^4S^2 > 0$$

Also,

$$(4.12)\qquad G^T HG = F$$

where,

$$(4.13)\qquad G = \begin{bmatrix} 1 & 0 & K^2 & KK' - K^2\dfrac{T'}{T} \\ 0 & 1 & -\dfrac{K'}{K} & -T^2 + K^2 - \dfrac{K''}{K} + \dfrac{2K'^2}{K^2} + \dfrac{K'T'}{KT} \\ 0 & 0 & 1 & -\dfrac{2K'}{K} - \dfrac{T'}{T} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us form a matrix M by dividing the second column of G by K, third by KT and the fourth by KTS, i.e.,

$$(4.14)\qquad M = \begin{bmatrix} 1 & 0 & \frac{K}{T} & \frac{TS}{K'} - \frac{T^2 S}{KT^2} \\ 0 & \frac{1}{K} & -\frac{K'}{K^2 T} & -\frac{T}{KS} + \frac{K}{TS} - \frac{K''}{K^2 TS} + \frac{2K''^2}{K^3 TS} + \frac{K'T'}{K^2T^2S} \\ 0 & 0 & \frac{1}{KT} & -\frac{2K'}{K^2 TS} - \frac{KT^2 S}{T'} \\ 0 & 0 & 0 & \frac{1}{KTS} \end{bmatrix}$$

Then we have,

$$(4.15)\qquad M^T HM = J .$$

Denoting by Φ, the inverse of M, we obtain

$$(4.16)\qquad \Phi = M^{-1} = \begin{bmatrix} 1 & 0 & -K^2 & -3KK' \\ 0 & K & K' & K'' - K^3 + KT^2 \\ 0 & 0 & KT & 2K'T + KT' \\ 0 & 0 & 0 & KTS \end{bmatrix}$$

Now,

$$\Delta^T J\Delta = H = \Phi^T J\Phi \Longrightarrow (\Delta\Phi^{-1})^T J(\Delta\Phi) = J, \text{ i.e. } \Delta\Phi^{-1} \text{ in Lorentzian.}$$

Let us define

$$(4.17)\qquad \Lambda = \Delta\Phi^{-1} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

From (4.17), we find

$$
\begin{aligned}
&x' = \lambda_1, && x'' = K\lambda_2, && x''' = -K^2\lambda_1 + K'\lambda_2 + KT\lambda_3, \\
&y' = \mu_1, && y'' = K\mu_2, && y'' = -K^2\mu_1 + K'\mu_2 + KT\mu_3, \\
&z' = \nu_1, && z'' = K\nu_2, && z''' = -K^2\nu_1 + K'\nu_2 + KT\nu_3, \\
&t' = k_1, && t'' = Kk_2, && t''' = -K^2 k_1 + K' k_2 + KTk_3,
\end{aligned}
$$

(4.18)

$$
\begin{aligned}
x^{iv} &= -3KK'\lambda_1 + (K'' - K^3 + KT^2)\lambda_2 + (2K\,T' + KT')\lambda_3 + KTS\lambda_4 \\
y^{iv} &= -3KK'\mu_1 + (K'' - K^3 + KT^2)\mu_2 + (2K'\,T + KT')\mu_3 + KTS\mu_4 \\
z^{iv} &= -3KK'\nu_1 + (K'' - K^3 + KT^2)\nu_2 + (2K'\,T + KT')\nu_3 + KTS\nu_4 \\
t^{iv} &= -3KK'k_1 + (K'' - K^3 + KT^2)k_2 + (2K'\,T + KT)k_2 + KTSk_4
\end{aligned}
$$

From (4.18), we obtain

$$
(4.19)\qquad
\begin{aligned}
&\lambda_1' = K\lambda_2, && \lambda_2' = -K\lambda_1 + T\lambda_3, && \lambda_3' = T\lambda_2 + S\lambda_4 \\
&\mu_1' = K\mu_2, && \mu_2' = -K\mu_1 + T\mu_3, && \mu_3' = T\mu_2 + S\mu_4 \\
&\nu_1' = K\nu_2, && \nu_2' = -K\nu_1 + T\nu_3, && \nu_3' = T\nu_2 + S\nu_4 \\
&k_1' = Kk_2, && k_2' = -Kk_1 + Tk_3, && k_3' = Tk_2 + Sk_4\,.
\end{aligned}
$$

From the fact that Λ is Lorentzian, we obtain

$$
(4,20)\qquad \lambda_i\lambda_j + \mu_i\mu_j - \nu_i\nu_j - k_i k_j = \begin{cases} 1 & \text{for } i = j = 1 \quad i = j = 2 \\ 0 & \text{for } i \neq j \\ -1 & \text{for } i = j = 3 \quad i = j = 4 \end{cases}
$$

From (4.18), (4.19), and (4.20), we obtain

$$
(4.21)\qquad \lambda_4' = -\lambda S_3, \qquad \mu_4' = -S\mu_3, \qquad \nu_4' = -S\nu_3, \qquad k_4' = -Sk_3\,.
$$

The formulae (4.19) (4.21) may also be written in the matrix form:

$$A' = AQ \tag{4.22}$$

where,

$$Q = \begin{bmatrix} 0 & -K & 0 & 0 \\ K & 0 & T & 0 \\ 0 & T & 0 & -S \\ 0 & 0 & S & 0 \end{bmatrix} \tag{4.23}$$

5. – The converses problem.

Let Q in the form (4.23) be given. Further, let $A(s)$ be a matrix satisfying the differential equation

$$A'(s) = A(s)\,Q(s)\,. \tag{5.1}$$

Let $A(s)$ be Lorentzian for some value s_2 in the interval $s_0 \leqslant s \leqslant s_1$, so that $A(s)$ will be Lorentzian throughout the interval $s_0 \leqslant s \leqslant s_1$.

Let K, T, S be non-zero real numbers, i.e., constants. Then

$$A' = AQ,\ A'' = AQ^2,\ A''' = AQ^3,\ A^{(iv)} = AQ^4. \tag{5.2}$$

The characteristic equation of Q is

$$x^4 + (K^2 + S^2 - T^2)x^2 + K^2 S^2 = 0 \tag{5.3}$$

and since Q satisfies it,

$$Q^4 + (K^2 + S^2 - T^2)\,Q^2 + K^2 S^2 I = 0\,. \tag{5.4}$$

Operating A, we obtain

$$A^{iv} + (K^2 + S^2 - T^2)\,A'' + K^2 S^2 A = 0\,. \tag{5.5}$$

Now two cases arise:

Case (i):

Let $K^2+S^2-2 \geqslant 0$, so that the roots of (5.5) are of the form $\pm i\mu$, $\pm i\nu$. Therefore,

$$\Lambda = A\cos\mu s + B\sin\mu s + C\cos\nu s + D\sin\nu s\,. \tag{5.6}$$

A, B, C, D being real constant matrices. Differentiating (5.6),

$$\Lambda Q = \Lambda' = \mu B\cos\mu s - \mu A\sin\mu s + \nu D\cos\nu s - \nu C\sin\nu s\,. \tag{5.7}$$

These functions being linearly independent, we must have

$$AQ = \mu B,\quad BQ = -\mu A,\quad CQ = \nu D,\quad DQ = -\nu C\,. \tag{5.8}$$

Taking $\Lambda(0) = I$ as the initial condition, (5.8) gives

$$A = \frac{Q^2+\nu^2 I}{\nu^2-\mu^2},\quad B = \frac{Q^2+\nu^2 Q}{\mu(\nu^2-\mu^2)},\quad C = \frac{-Q^2-\mu^2 I}{\nu^2-\mu^2},\quad D = \frac{-Q^3-\mu^2 Q}{\nu(\nu^2-\mu^2)} \tag{5.9}$$

Also,

$\Lambda\Phi^{-1} = \Lambda$ implies mat $(x', y', z', t')^T$ is un first column of Λ.

Therefore,

$$\begin{aligned}
\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\left(\frac{\nu^2}{\nu^2-\mu^2}\cos\mu s - \frac{\mu^2}{\nu^2-\mu^2}\cos\nu s\right) \\
&+ \begin{bmatrix} 0 \\ K \\ 0 \\ 0 \end{bmatrix}\left(\frac{\nu^2}{\nu^2-\mu^2}\,\frac{\sin\mu s}{\mu} - \frac{\mu^2}{\nu^2-\mu^2}\,\frac{\sin\nu s}{\nu}\right) \\
&+ \begin{bmatrix} -K^2 \\ 0 \\ KT \\ 0 \end{bmatrix}\left(\frac{1}{\nu^2-\mu^2}\cos\mu s - \frac{1}{\nu^2-\mu^2}\cos\nu s\right) \\
&+ \begin{bmatrix} 0 \\ -K^3+KT^2 \\ 0 \\ KTS \end{bmatrix}\left(\frac{1}{\nu^2-\mu^2}\,\frac{\sin\mu s}{\mu} - \frac{1}{\nu^2-\mu^2}\,\frac{\sin\nu s}{\nu}\right)
\end{aligned} \tag{5.10}$$

Integrating these equations subject to the initial condition that $(x, y, z, t) = (0, 0, 0, 0)$ for $s = 0$, and rearranging, we obtain

$$(5.11)\qquad \begin{aligned} &x - \frac{\nu^2 - K^2}{KT} z = \frac{1}{\nu} \sin \nu s & & x + \frac{K^2 - \mu^2}{KT} z = \frac{1}{\mu} \sin \mu s \\ &y - \frac{S^2 - \mu^2}{ST} t = \frac{K}{\nu^2}(1 - \cos \nu s) & & y + \frac{\nu^2 - S^2}{ST} t = \frac{K}{\mu^2}(1 - \cos \mu s) \end{aligned}$$

The transformation,

$$(5.12)\qquad \begin{aligned} &\bar{x} = \frac{1}{F_1} x - \frac{1}{F_1} \frac{\nu^2 - K^2}{KT} z, & & \bar{y} = \frac{1}{F_2} y - \frac{1}{F_2} \frac{S^2 - \mu^2}{ST}; \\ &\bar{z} = \frac{1}{F_3} x + \frac{1}{F_3} \frac{K^2 - \mu^2}{KT} z; & & \bar{t} = \frac{1}{F_4} y + \frac{1}{F_4} \frac{\nu^2 - S^2}{ST} t; \end{aligned}$$

will be Lorentzian, provided,

$$(5.13)\qquad \begin{aligned} &F_1 = \pm \sqrt{\frac{\nu^2 - \mu^2}{K^2 - \mu^2}} & & F_2 = \pm \sqrt{\frac{\nu^2 - \mu^2}{\nu^2 - S^2}} \\ &F_3 = \pm \sqrt{\frac{\nu^2 - \mu^2}{\nu^2 - K^2}} & & F_4 = \pm \sqrt{\frac{\nu^2 - \mu^2}{S^2 - \mu^2}} \end{aligned}$$

It transforms (5.11) into

$$(5.14)\qquad \begin{aligned} &\bar{x} = \frac{1}{\nu F_1} \sin \nu s & & \bar{y} = \frac{K}{\nu^2 F_2} (1 - \cos \nu s) \\ &\bar{z} = \frac{1}{\mu F_3} \sin \mu s & & \bar{t} = \frac{K}{\mu^2 F_4} (1 - \cos \mu s) . \end{aligned}$$

After translation,

$$(5.15)\qquad \xi = \bar{x} \qquad \eta = \bar{y} - \frac{K}{\nu^2 F} \qquad \zeta = \bar{z} \qquad u = \bar{t} - \frac{K}{\mu^2 F_4}$$

we have

$$\xi = a \sin \nu s \quad \eta = a \cos \nu s \quad \zeta = b \sin \mu s \quad b = h \cos \mu s, \tag{5.16}$$

$$a^2 \nu^2 - h^2 \mu^2 = 1$$

where,

$$a^2 = \frac{1}{\nu^2 F_1^2} = \frac{K^2}{\nu^4 F_2^2} \qquad \text{and} \qquad b^2 = \frac{1}{\mu^2 F_3^2} = \frac{K^2}{\mu^4 F_4^2}\,.$$

Case (ii).

Let $K^2 + S^2 - T^2 > 0$, so that the roots of the equation (5.5) are of the form

$$\pm \mu, \qquad \pm \nu$$

Therefore,

$$\Lambda = A \cosh \mu s + B \sinh \mu s + C \cosh \nu s + D \sinh \nu s \tag{5.18}$$

where A, B, C, D are real constant matrices. Differentiating (5.18),

$$\Lambda Q = \Lambda' = \mu B \cosh \mu s + \mu A \sinh \mu s + \nu D \cosh \nu s + \nu C \sinh \nu s \tag{5.19}$$

These functions being linearly independent, we must have

$$AQ = \mu B, \; BQ = \mu A, \; CQ = \nu Q = \nu D, \; DQ = \nu C\,. \tag{5.20}$$

Taking $\Lambda(0) = I$ as the initial condition we obtain

$$\begin{aligned} A &= \frac{-Q^2 + \nu^2 I}{\nu^2 - \mu^2} & \qquad B &= \frac{-Q^3 + \nu^2 Q}{\mu(\nu^2 - \mu^2)} \\ C &= \frac{Q^2 - \mu^2 I}{\nu^2 - \mu^2} & \qquad D &= \frac{Q^3 - \mu^2 Q}{\nu(\nu^2 - \mu^2)} \end{aligned} \tag{5.21}$$

Also,

(5.22)
$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left(\frac{\nu^2}{\nu^2 - \mu^2} \cosh \mu s - \frac{\mu^2}{\nu^2 - \mu^2} \cosh \nu s \right)$$
$$+ \begin{bmatrix} 0 \\ k \\ 0 \\ 0 \end{bmatrix} \left(\frac{\nu^2}{\nu^2 - \mu^2} \frac{\sinh \mu s}{\mu} - \frac{\mu^2}{\nu^2 - \mu^2} \frac{\sinh \nu s}{\nu} \right)$$
$$+ \begin{bmatrix} -K^2 \\ 0 \\ KT \\ 0 \end{bmatrix} \left(\frac{1}{\nu^2 - \mu^2} \cosh \nu s - \frac{1}{\nu^2 - \mu^2} \cosh \mu s \right)$$
$$+ \begin{bmatrix} 0 \\ -K^3 + KT^2 \\ 0 \\ KTS \end{bmatrix} \left(\frac{1}{\nu^2 - \mu^2} \frac{\sinh \nu s}{\nu} - \frac{1}{\nu^2 - \mu^2} \frac{\sinh \mu s}{\mu} \right)$$

Integrat ingthese equations subject to the initial condition that $(x, y, z, t) = (0, 0,0\ 0)$ for $s = 0$ and rearranging we obtain

(5.23)
$$x + \frac{K^2 + \mu^2}{KT} z = \frac{1}{\mu} \sinh \mu s \qquad x + \frac{\nu^2 + K^2}{KT} z = \frac{1}{\nu} \sinh \nu s$$
$$y - \frac{\nu^2 + S^2}{ST} t = \frac{K}{\mu^2} (\cosh \mu s - 1) \qquad y - \frac{S^2 + \mu^2}{ST} t = \frac{K}{\nu^2} (\cosh \nu s - 1).$$

The transformation

(5.24)
$$\bar{x} = \frac{1}{F_1} x + \frac{1}{F_1} \frac{\nu^2 + K^2}{KT} z \qquad \bar{y} = \frac{1}{F_2} y - \frac{1}{F_2} \frac{\nu^2 + S^2}{ST} t$$
$$\bar{t} = \frac{1}{F_3} x + \frac{1}{F_3} \frac{K^2 + \mu^2}{KT} z \qquad \bar{t} = \frac{1}{F_4} y - \frac{1}{F_4} \frac{S^2 + \mu^2}{ST} t$$

will be Lorentzian provided,

$$(5.25)\qquad F_1 = \pm\sqrt{\frac{\mu^2 - \nu^2}{K^2 + \mu^2}} \qquad F_2 = \pm\sqrt{\frac{\mu^2 - \nu^2}{S^2 + \mu^2}}$$

$$F_3 = \pm\sqrt{\frac{\mu^2 - \nu^2}{K^2 + \nu^2}} \qquad F_4 = \pm\sqrt{\frac{\mu^2 - \nu^2}{S^2 + \nu^2}}$$

It transforms (5.23) into

$$(5.26)\qquad \bar{x} = \frac{1}{\nu F_1}\sinh \nu s \qquad \bar{y} = \frac{K}{\mu^2 F_2}(\cosh \mu s - 1)$$

$$\bar{z} = \frac{1}{\mu F_3}\sinh \mu s \qquad \bar{t} = \frac{K}{\nu^2 F_4}(\cosh \nu s - 1).$$

After translation

$$(5.27)\qquad \xi = \bar{x}, \qquad \eta = \bar{y} + \frac{K}{\mu^2 F_2}, \qquad \zeta = \bar{z}, \qquad u = \bar{t} + \frac{K}{\nu^2 F_4}$$

we obtain:

$$(5.28)\qquad \xi = a \sinh \nu s, \quad \eta = b \cosh \mu s, \quad \zeta = b \sinh \mu s, \quad u = a \cosh \nu s,$$

$$b^2 \mu^2 - a^2 \nu^2 = 1$$

where,

$$(5.29)\qquad a^2 = \frac{1}{\nu^2 F_1^2} = \frac{K^2}{\nu^4 F_4^2}, \qquad \text{and} \qquad b^2 = \frac{K^2}{\mu^4 F_2^2} = \frac{1}{\mu^2 F_3^2}$$

So far we have assumed that $\mu \neq \nu$. In the case when $\mu = \nu$ calculations may easily be done in the same manner.

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