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The Orr hypergeometric Identity and Associated Series. (**)

1. - Introductory.

In 1899 ORR [1] gave the identity

$$(1)\qquad F\left[\begin{matrix}\alpha,\ \beta;\ x\\ \alpha+\beta-\frac{1}{2}\end{matrix}\right] F\left[\begin{matrix}\alpha,\ \beta-1;\ x\\ \alpha+\beta-\frac{1}{2}\end{matrix}\right] = {}_3F_2\left[\begin{matrix}2\alpha,\ 2\beta-1,\ \alpha+\beta-1,\ x\\ 2\alpha+2\beta-2,\ \alpha+\beta-\frac{1}{2}\end{matrix}\right],$$

It was quoted by BAILEY [2].

In this paper, series-suggested by the study of (1) of products of generalized hypergeometric functions will be given. They are of the same type as an extensiv list of ather series given by BURCHNALL and CHANDY [3], [4], [5]. Their method in proving the results is based upon the study of the differential equation satisfied by the product of two generalized hypergeometric functions. They introduced a certain type of operators and deduced their results by an application of these operators. My method is straitforward and is based upon the derangement of series.

The series will be stated in 2 and proved in 3. The constants and the variable are such that the functions involved exist. A clas of null series is given in 4.

2. - Series.

The formulae to be proved are:

$$\sum_{r=0}^{\infty} \frac{\{(\alpha,\ r)\}^2 (\beta;\ r)\ (\beta-1;\ r)\ (b;r)\ (\gamma-b;\ r)}{r!\{(\alpha+\beta-\frac{1}{2};\ r)\}^2(\gamma+r-1;\ r)\ (\gamma;\ 2r)}\, x^{2r}$$

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(2) $$ {}_3F_2\left[\begin{matrix} b+r,\ \alpha+r,\ \beta+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] {}_3F_2\left[\begin{matrix} b+r,\ \alpha+r,\ \beta-1+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] $$

$$ = {}_4F_3\left[\begin{matrix} 2\alpha,\ 2\beta-1,\ \alpha+\beta-1,\ b;\ x \\ 2\alpha+2\beta-2,\ \alpha+\beta-\tfrac{1}{2},\ \gamma \end{matrix}\right], \cdots $$

(3) $$ \sum_{r=0}^{\infty} \frac{\{(\alpha;\ r)\}^2 (\beta;\ r)\ (\beta-1;\ r)\ (\gamma-1;\ r) \{(\tfrac{1}{2}+\tfrac{1}{2}\gamma;\ r)\}^2}{r!\{(\alpha+\beta-\tfrac{1}{2};\ r)\ (\gamma;\ 2r)\}^2} x^{2r} $$

$$ {}_3F_2\left[\begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}\gamma+r,\ \alpha+r,\ \beta+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] {}_3F_2\left[\begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}\gamma+r,\ \alpha+r,\ \beta-1+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] $$

$$ = {}_4F_3\left[\begin{matrix} 2\alpha,\ 2\beta-1,\ \alpha+\beta-1,\ \tfrac{1}{2}+\tfrac{1}{2}\gamma;\ x \\ 2\alpha+2\beta-2,\ \alpha+\beta-\tfrac{1}{2},\ \gamma \end{matrix}\right], \cdots $$

(4) $$ \sum_{r=0}^{\infty} \frac{\{(\alpha;\ r)\ (\tfrac{1}{2}\gamma;\ r)\}^2 (\beta;\ r)\ (\beta-1;\ r)}{r!\{(\alpha+\beta-\tfrac{1}{2};\ r)\}^2 (\gamma+r-1;\ r)(\gamma;\ 2r)} x^{2r} $$

$$ {}_3F_2\left[\begin{matrix} \tfrac{1}{2}\gamma+r,\ \alpha+r,\ \beta+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] {}_3F_2\left[\begin{matrix} \tfrac{1}{2}\gamma+r,\ \alpha+r,\ \beta-1+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] $$

$$ = {}_4F_3\left[\begin{matrix} 2\alpha,\ 2\beta-1,\ \alpha+\beta-1,\ \tfrac{1}{2}\gamma;\ x \\ \alpha+\beta-\tfrac{1}{2}.\ 2\alpha+2\beta-2,\ \gamma \end{matrix}\right], \cdots $$

(5) $$ \sum_{r=0}^{\infty} \frac{(-1)^r\ \{(\alpha;\ r)\}^2 (\beta;\ r)\ (\beta-1;\ r)}{r!\{(\alpha+\beta-\tfrac{1}{2};\ r)\}^2 (\gamma+r-1;\ r)\ (\gamma;\ 2r)} x^{2r} $$

$$ {}_2F_2\left[\begin{matrix} \alpha+r,\ \beta+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] {}_2F_2\left[\begin{matrix} \alpha+r,\ \beta-1+r,\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \gamma+2r \end{matrix}\right] $$

$$ = {}_3F_3\left[\begin{matrix} 2\alpha\ \ 2\beta-1,\ \alpha+\beta-1;\ x \\ 2\alpha+2\beta-2,\ \alpha+\beta-\tfrac{1}{2},\ \gamma \end{matrix}\right], \cdots $$

(6) $$ \sum_{r=0}^{\infty} \frac{(-1)^r\ \{(\alpha;\ r)\}^2 (\beta;\ r)\ (\beta-1;\ r)}{r!\ \{(\alpha+\beta-\tfrac{1}{2};\ r)\}^2 (\alpha+\beta-2+r;\ r)\ (\alpha+\beta-1;\ 2r)} x^{2r} $$

$$ {}_2F_2\left[\begin{matrix} \alpha+r,\ \beta+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \alpha+\beta-1+2r \end{matrix}\right] {}_2F_2\left[\begin{matrix} \alpha+r,\ \beta-1+r;\ x \\ \alpha+\beta-\tfrac{1}{2}+r,\ \alpha+\beta-1+2r \end{matrix}\right] $$

$$ = {}_2F_2\left[\begin{matrix} 2\alpha,\ 2\beta-1;\ x \\ 2\alpha+2\beta-2,\ \alpha+\beta-\tfrac{1}{2} \end{matrix}\right], \cdots $$

Series of products of two transcendental hypergeometric functions of GAUSS: The series are

$$(7)\qquad \sum_{r=0}^{\infty} \frac{\{(\alpha;\ r)\}^2(\beta;\ r)\ (\beta-1;\ r)\ (\gamma-\alpha-\beta+\tfrac{1}{2};\ r)}{r!\ (\alpha+\beta-\tfrac{1}{2};\ r)\ (\gamma+r-1;\ r)\ (\gamma;\ 2r)}\, x^{2r}$$

$$ {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta+r;\ x\\ \gamma+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta-1+r;\ x\\ \gamma+2r\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}2\alpha,\ 2\beta-1,\ \alpha+\beta-1;\ x\\ 2\alpha+2\beta-2,\ \gamma\end{matrix}\right],\ \ldots$$

$$(8)\qquad \sum_{r=0}^{\infty} \frac{\{(\alpha;\ r)\}^2\ (\beta;\ r)\ (\beta-1;\ r)\ (-\tfrac{1}{2};\ r)}{r!\ (\alpha+\beta-\tfrac{1}{2};\ r)\ (\alpha+\beta-2+r;\ r)\ (\alpha+\beta-1;\ 2r)}\, x^{2r}$$

$$ {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta+r;\ x\\ \alpha+\beta-1+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta-1+r;\ x\\ \alpha+\beta-1+2r\end{matrix}\right]$$

$$= {}_2F_1\left[\begin{matrix}2\alpha,\ 2\beta-1;\ x\\ 2\alpha+2\beta-2\end{matrix}\right],\ \ldots$$

$$(9)\qquad \sum_{r=0}^{\infty} \frac{\{(\alpha;\ r)\}^2\ (\beta;\ r)\ (\beta-1;\ r)\ (\alpha-\beta+\tfrac{1}{2};\ r)}{r!\ (\alpha+\beta-\tfrac{1}{2};\ r)\ (2\alpha+r-1;\ r)\ (2\alpha;\ 2r)}\, x^{2r}$$

$$ {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta+r,\ x\\ 2\alpha+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta-1+r;\ x\\ 2\alpha+2r\end{matrix}\right]$$

$$= {}_2F_1\left[\begin{matrix}2\beta-1,\ \alpha+\beta-1;\ x\\ 2\alpha+2\beta-2\end{matrix}\right],\ \ldots$$

$$(10)\qquad \sum_{r=0}^{\infty} \frac{\{(\alpha;\ r)\}^2\ (2\alpha+2\beta-3;\ r)\ (\beta;\ r)\ (\beta-1;\ r)}{r!\{(2\alpha+2\beta-2;\ 2r)\}^2}\, x^{2r}$$

$$ {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta+r;\ x\\ 2\alpha+2\beta-2+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta-1+r,\ x\\ 2\alpha+2\beta-2+2r\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}2\alpha,\ 2\beta-1,\ \alpha+\beta-1;\ x\\ 2\alpha+2\beta-2,\ 2\alpha+2\beta-2\end{matrix}\right],\ \ldots$$

$$(11)\quad \sum_{r=0}^{\infty} \frac{\{(\alpha;\,r)\}^2\,(2;\,r)\,(\beta-1;\,r)}{r!\;(2\alpha+2\beta-2+r;\;r)\,(2\alpha+2\beta-1;\;2r)}\,x^{2r}$$

$$ {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta+r;\ x\\ 2\alpha+2\beta-1+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ \beta-1+r;\ x\\ 2\alpha+2\beta-1+2r\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}2\alpha,\ 2\beta-1,\ \alpha+\beta-1;\ x\\ 2\alpha+2\beta-2,\ 2\alpha+2\beta-1\end{matrix}\right], \quad \ldots$$

The following formulae are required in the proofs:

$$(12)\quad {}_4F_3\left[\begin{matrix}-n,\ \alpha,\ \beta,\ \frac{3}{2}-\alpha-\beta-n;\ 1\\ \alpha+\beta-\frac{1}{2},\ 1-\alpha-n,\ 2-\beta-n\end{matrix}\right] =$$

$$= \frac{(2\alpha;n)\,(2\beta-1;n)\,(\alpha+\beta-1;\ n)}{(\alpha;\ n)\,(\beta-1,\ n)\,(2\alpha+2\beta-2;\ n)}, \quad \ldots$$

which can easily deduced from (1) by equatiny the coefficients of x^n on both sides;

(DOUGALL [6]),

$$(13)\quad {}_5F_4\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c,\ d,\ e;\ 1\\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e\end{matrix}\right] =$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\ \Gamma(1+a-c-d-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-c-d)\,\Gamma(1+a-c-e)}, \quad \ldots$$

(WHIPPLE, [7], p. 255),

$$(14)\quad {}_4F_3\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c,\ d;\ 1\\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d\end{matrix}\right] =$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-c-d)}{\Gamma(1+a)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-c)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-d)\,\Gamma(1+a-c-d)}, \quad \ldots$$

(WIPPLE, [7], p. 251)

$$(15)\quad {}_4F_3\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c,\ d;\ -1\\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d\end{matrix}\right] = \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma(1+a)\,\Gamma(1+a-c-d)}, \quad \ldots$$

(DUPONN [8]),

$$(16)\qquad F\left[\begin{matrix} a,\ b,\ c;\ 1 \\ 1+a-b \quad 1+a-c\end{matrix}\right] =$$

$$= \frac{\Gamma(1+\frac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+\frac{1}{2}a-c)\,\Gamma(1+a-b-c)}, \quad \ldots$$

3. - Proofs of (2).

The L. H. S. of (2) can be put in the form

$$\sum_{r=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\{(\alpha;\ r)\}^2 (\beta;\ r)(\beta-1;\ r)(\gamma-b;\ r)(b;\ r)}{r!\,\mu!\,\nu!\{(\alpha+\beta-\frac{1}{2};\ r)\}^2(\gamma+r-1;r)(\gamma;\ 2r)}$$

$$\frac{(b+r;\ \mu)(b+r;\ \nu)(\alpha+r;\ \mu)(\alpha+r;\ \nu)(\beta+r;\ \mu)(\beta-1+r;\ \nu)}{(\alpha+\beta-\frac{1}{2}+r;\ \mu)(\alpha+\beta-\frac{1}{2}+r;\ \nu)(\gamma+2r;\ \mu)(\gamma+2r;\ \nu)}\, x^{\mu+\nu+2r}$$

Here put $\mu = p - r$, $v = q - r$ and change the order of rummation putting the first summation last and get

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(\alpha;\ p)(\alpha;\ q)\ (\beta;\ p)(\beta-1;\ q)(b;\ p)(b;\ q)\,x^{p+q}}{p!\,q!\,(\alpha+\beta-\frac{1}{2};\ p)(\alpha+\beta-\frac{1}{2};\ q)(\gamma;\ p)(\gamma;\ q)}$$

$${}_5F_4\left[\begin{matrix}\gamma-1,\ \frac{1}{2}+\frac{1}{2}\gamma,\ \gamma-b,\ -p,\ -q;\ 1 \\ \frac{1}{2}\gamma-\frac{1}{2},\ b,\ \gamma+p,\ \gamma+q\end{matrix}\right].$$

Now sum the ${}_5F_4$ by means of (13) and the last expression becomes

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(\alpha;\ p)(\alpha;\ q)(\beta;\ p)(\beta-1;\ q)(b;\ p+q)\,x^{p+q}}{p!\,q!\,(\alpha+\beta-\frac{1}{2};\ p)(\alpha+\beta-\frac{1}{2};\ q)\ (\gamma;\ p+q)}$$

$$= \sum_{n=0}^{\infty} \frac{(b;\ n)(\alpha;\ n)(\beta-1;\ n)}{n!\ (\gamma;\ o)\ (\alpha+\beta-\frac{1}{2};\ n)} \times {}_4F_3\left[\begin{matrix} -n,\ \alpha,\ \beta,\ \frac{3}{2}-\alpha-\beta-n;\ 1 \\ \alpha+\beta-\frac{1}{2},\ 1-\alpha-n,\ 2-\beta-n\end{matrix}\right] x^n;$$

and the result follows from (12). In (2) take $b = \alpha + \beta - \frac{1}{2}$ and so obtain (7). Also in (7) take $\gamma = \alpha + \beta - 1$, 2α resp. and get (8) and (9). Proceeding as before and using (14) instead of (13), then (4) can be obtained. Also (4) can be obtained by taking $b = \frac{1}{2}\gamma$ in (2). (11) is a particular case of (4) obtained by taking $\gamma = 2\alpha + 2\beta - 1$.

Proof of (3): The L.H.S. of (3) is equal to

$$\sum_{r=0}^{\infty} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{\{(\alpha;\, r)\,(\tfrac{1}{2}+\tfrac{1}{2}\gamma;\, r)\}^2\,(\beta;\, r)\,(\beta-1;\, r)\,(\gamma-1;\, r)}{r!\,\mu!\,\nu!\{(\alpha+\beta-\tfrac{1}{2};\, r)\}^2\,\{(\gamma\,;\, 2r)\}^2}\, x^{2r+\mu+\nu}$$

$$\frac{(\tfrac{1}{2}+\tfrac{1}{2}\gamma+r;\, \mu)\,(\tfrac{1}{2}+\tfrac{1}{2}\gamma+r;\, \nu)\,(\alpha+r;\, \mu)\,(\alpha+r;\, \mu)\,(\beta+r;\, \mu)\,(\beta-1+r;\, \nu)}{(\alpha+\beta-\tfrac{1}{2}+r;\, \mu)\,(\alpha+\beta-\tfrac{1}{2}+r;\, \nu)}.$$

Here put $\mu = p - r$, $\nu = q - r$ and change the order of summation, putting the first summation last and get

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha;\, p)\,(a;\, q)\,(\beta;\, p)\,(\beta-1;\, q)\,(\tfrac{1}{2}+\tfrac{1}{2}\gamma;\, p)\,(\tfrac{1}{2}+\tfrac{1}{2}\gamma;\, q)\, x^{p+q}}{p!\,q!\,(\alpha+\beta-\tfrac{1}{2};\, p)\,(\alpha+\beta-\tfrac{1}{2};\, q)\,(\gamma;\, p)\,(\gamma;\, q)}$$

$$ {}_3F_2\left[\begin{matrix}\gamma-1,\ -p,\ -q;\ 1\\ \gamma+p,\ \gamma+q\end{matrix}\right].$$

Now sum the ${}_3F_2$ by (16) and the last expression becomes

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha;\, p)\,(\alpha;\, q)\,(\beta;\, p)\,(\beta-1;\, q)\,(\tfrac{1}{2}+\tfrac{1}{2}\gamma;\, p+q)\, x^{p+q}}{p!\,q!\,(\alpha+\beta-\tfrac{1}{2};\, p)\,(\alpha+\beta-\tfrac{1}{2};\, q)\,(\gamma;\, p+q)}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha;\, n)\,(\beta-1;\, n)\,(\tfrac{1}{2}+\tfrac{1}{2}\gamma;\, n)}{n!\,(\alpha+\beta-\tfrac{1}{2};\, n)\,(\gamma;\, n)}\Bigg|\ {}_4F_3\left[\begin{matrix}-n,\ \alpha,\ \beta,\ \tfrac{3}{2}-\alpha-\beta-n;\ 1\\ \alpha+\beta-\tfrac{1}{2},\ 1-\alpha-n,\ 2-\beta-n\end{matrix}\right] x^n,$$

and the result follaw from (12). In (3) take $\gamma = 2\alpha + 2\beta - 2$ and so obtain (10).

Proof of (5): Prèceeding as before, it is easily seen that the L. H. S. of (5) is equal to

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha;\, p)\,(\alpha;\, q)\,(\beta;\, p)\,(\beta-1;\, q)\, x^{p+q}}{p!\,q!\,(\alpha+\beta-\tfrac{1}{2};\, p)\,(\alpha+\beta-\tfrac{1}{2};\, q)\,(\gamma;\, p)\,(\gamma;\, q)}$$

$$ {}_4F_3\left[\begin{matrix}\gamma-1,\ \tfrac{1}{2}\gamma+\tfrac{1}{2},\ -p,\ -q;\ -1\\ \tfrac{1}{2}\gamma-\tfrac{1}{2},\ \gamma+p,\ \gamma+q\end{matrix}\right].$$

Hepre apply to sum the ${}_4F_3$ and get

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha;\, p)\,(\alpha;\, q)\,(\beta;\, p)\,(\beta-1;\, q)\, x^{p+q}}{p!\,q!\,(\alpha+\beta-\tfrac{1}{2};\, p)\,(\alpha+\beta-\tfrac{1}{2};\, q)\,(\gamma;\, p+q)}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha;\, n)\,(\beta-1;\, n)}{n!\,(\alpha+\beta-\tfrac{1}{2};\, n)\,(\gamma;\, n)\ (\gamma;\, n)}\, x\ {}_4F_3\left[\begin{matrix}-n,\ \alpha,\ \beta,\ \tfrac{3}{2}-\alpha-\beta-n;\ 1\\ \alpha+\beta-\tfrac{1}{2},\ 1-\alpha-n,\ 2-\beta-n\end{matrix}\right] x^n;$$

and the result follows from (12). Take $\gamma = \alpha + \beta - 1$ in (5) and so obtain (6).

Many other particular cases can be obtained from (2), (3), (4), (5) by taking γ equal to, 2α, $2\beta - 1$, $\alpha + \beta - 1$ respectively.

4. - A class of null series.

We are now in a position to obtain a class of series, the sum of each is equal to zero. Thus taking $\beta = 1/2$ in the formulae (2) to (11) respec, we obtain

$$(17)\qquad 0 = -1 + \sum_{r=0}^{\infty} \frac{(\frac{1}{2};\, r)\,(-\frac{1}{2};\, r)\,(b;\, r)\,(\gamma - b;\, r)}{r!\,(\gamma + r - 1;\, r)\,(\gamma;\, 2r)}\, x^{2r}$$

$$\cdot {}_2F_1\left[\begin{matrix} b + r,\ \frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right] {}_2F_1\left[\begin{matrix} b + r,\ -\frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right], \ \ldots$$

$$(18)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{(\frac{1}{2};\, r)\,(-\frac{1}{2};\, r)\,(\gamma - 1;\, r)\,\{(\frac{1}{2} + \frac{1}{2}\gamma;\, r)\}^2}{r!\{(\gamma;\, 2r)\}^2}\, x^{2r}\right.$$

$$\left. \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2} + \frac{1}{2}\gamma + r,\ \frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right] {}_2F_1\left[\begin{matrix} \frac{1}{2} + \frac{1}{2}\gamma + r,\ -\frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right]\right\}, \ \ldots$$

$$(19)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{\{(\frac{1}{2}\gamma;\, r)\}^2\,(\frac{1}{2};\, r)\,(-\frac{1}{2};\, r)}{r!\,(\gamma + r - 1;\, r)\,(\gamma;\, 2r)}\, x^{2r}\right.$$

$$\left. \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}\gamma + r,\ \frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right] {}_2F_1\left[\begin{matrix} \frac{1}{2}\gamma + r,\ -\frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right]\right\}, \ \ldots$$

$$(20)\qquad 0 = -1 + \sum_{p=0}^{\infty} \left\{ \frac{(-1)^r\,(\frac{1}{2};\, r)\,(-\frac{1}{2};\, r)}{r!\,(\gamma + r - 1;\, r)\,(\gamma;\, 2r)}\, x^{2r}\right.$$

$$\left. \cdot {}_1F_1\left[\begin{matrix} \frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right] {}_1F_1\left[\begin{matrix} -\frac{1}{2} + r;\ x \\ \gamma + 2r \end{matrix}\right]\right\}, \ \ldots$$

$$(21)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{(-1)^r\,(\frac{1}{2};\, r)\,(-\frac{1}{2};\, r)}{r!\,(\alpha - \frac{3}{2} + r;\, r)\,(\alpha - \frac{1}{2};\, 2r)}\, x^{2r}\right.$$

$$\left. \cdot {}_1F_1\left[\begin{matrix} \frac{1}{2} + r;\ x \\ \alpha - \frac{1}{2} + 2r \end{matrix}\right] {}_1F_1\left[\begin{matrix} -\frac{1}{2} + r;\ x \\ \alpha - \frac{1}{2} + 2r \end{matrix}\right]\right\}, \ \ldots$$

$$(22)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{(\alpha;\ r)\,(\tfrac{1}{2};\ r)\,(-\tfrac{1}{2};\ r)\,(\gamma-\alpha;\ r)}{r!\,(\gamma+r-1;\ r)\,(\gamma;\ 2r)}\, x^{2r} \cdot {}_2F_1\left[\begin{matrix}\alpha+r,\ \tfrac{1}{2}+r;\ x\\ \gamma+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ -\tfrac{1}{2}+r;\ x\\ \gamma+2r\end{matrix}\right]\right\}, \ldots$$

$$(23)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{(\alpha;\ r)\,(\tfrac{1}{2};\ r)\,\{-\tfrac{1}{2};\ r)\}^2}{r!\,(\alpha-\tfrac{3}{2}+r;\ r)\,(\alpha-\tfrac{1}{2};\ 2r)}\, x^{2r} \cdot {}_2F_1\left[\begin{matrix}\alpha+r,\ \tfrac{1}{2}+r;\ x\\ \alpha-\tfrac{1}{2}+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ -\tfrac{1}{2}+r;\ x\\ \alpha-\tfrac{1}{2}+2r\end{matrix}\right]\right\}, \ldots$$

$$(24)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{(\tfrac{1}{2};\ r)\,(-\tfrac{1}{2};\ r)\,\{(\alpha;\ r)\}^2}{r!\,(2\alpha+r-1;\ r)\,(2\alpha;\ 2r)}\, x^{2r} \cdot {}_2F_1\left[\begin{matrix}\alpha+r,\ \tfrac{1}{2}+r;\ x\\ 2\alpha+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ -\tfrac{1}{2}+r;\ x\\ 2\alpha+2r\end{matrix}\right]\right\}, \ldots$$

$$(25)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{\{(\alpha;\ r)\}^2\,(2\alpha-2;\ r)\,(\tfrac{1}{2};\ r)\,(-\tfrac{1}{2};\ r)}{r!\,\{(2\alpha-1;\ 2r)\}^2}\, x^{2r} \cdot {}_2F_1\left[\begin{matrix}\alpha+r,\ \tfrac{1}{2}+r;\ x\\ 2\alpha-1+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ -\tfrac{1}{2}+r\\ 2\alpha-1+2r\end{matrix}\right]\right\}, \ldots$$

$$(26)\qquad 0 = -1 + \sum_{r=0}^{\infty} \left\{ \frac{\{(\alpha;\ r)\}^2\,(\tfrac{1}{2};\ r)\,(-\tfrac{1}{2};\ r)}{r!\,(2\alpha-1+r;\ r)\,(2\alpha;\ 2r)}\, x^{2r} \cdot {}_2F_1\left[\begin{matrix}\alpha+r,\ \tfrac{1}{2}+r;\ x\\ 2\alpha+2r\end{matrix}\right] {}_2F_1\left[\begin{matrix}\alpha+r,\ -\tfrac{1}{2}+r;\ x\\ 2\alpha+2r\end{matrix}\right]\right\}, \ldots$$

Other class of null series can be obtained by taking $\beta = 1-\alpha$ in (2), (3), (4), (5), (7), (9), (10), (11) respectively.

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