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A Creep Law for Strain-hardening Materials and its Consequences Particularly under Conditions of Primary Creep. (**)

1. - Introduction.

The bases for a study of creep deformations in structures at constant temperature are gross hypotheses, which lead to constitutive equations relating stress, strain and strain rate. Often these equations are specialized into « power laws » to simplify the approach to special problems; but such specializations do not always lead to satisfactory predictions. Here some results are sought of fairly general character.

A group of hypotheses is accepted throughout the paper:

(i) Only « small » deformations are considered: so that an appropriate measure of strain is

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{1.1}$$

(u_k, components of displacement from a natural reference state); also, the strain rate is given by

$$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}\,(v_{i,j} + v_{i,j})$$

(v_k, components of speed).

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(ii) The strain is thought of as the sum of an elastic and a permanent component

$$\varepsilon_{ij} = \varepsilon_{ij}^{(e)} + \varepsilon_{ij}^{(p)}, \tag{1.2}$$

but volume changes are taken to be purely elastic

$$\varepsilon_{hh} = \varepsilon_{hh}^{(e)} . \tag{1.3}$$

(iii) The elastic component of strain is related to the stress τ_{ij} through the classical linear formulae. In terms of the deviators

$$s_{ij} = \tau_{ij} - \frac{1}{3} \tau_{hh} \delta_{ij}, \qquad \gamma_{ij} = \varepsilon_{ij} - \frac{1}{3} \varepsilon_{hh} \delta_{ij} \tag{1.4}$$

and the mean normal values

$$\tau = \frac{1}{3} \tau_{hh}, \qquad \varepsilon = \frac{1}{3} \varepsilon_{hh}$$

those relations can be written

$$s_{ij} = 2G\gamma_{ij}^{(e)}, \qquad \tau = 3K\varepsilon, \tag{1.5}$$

(G elastic shear modulus, K elastic bulk modulus; the summation convention for repeated indices is accepted; δ_{ij} is the KRONECKER index).

(iv) It is assumed that the stress does not influence the permanent component of strain directly; but rather, that proportionality exists between s_{ij} and the rate of strain $\dot{\gamma}_{ij}^{(p)}$:

$$\dot{\gamma}_{ij}^{(1)} = F \cdot s_{ij}, \tag{1.6}$$

the coefficient F depending itself on the stress level and also on the amount of permanent strain and, perhaps, on time.

The last hypothesis is the most restrictive of the four and is accepted mainly on grounds of simplicity; the specification of the function F is also very critical and the problems connected with that specification will have a prominent part in the following Sections.

2. - Experiments of simple tension.

To illuminate the significance of the hypotheses of Sect. 1, let us discuss first some simple consequences.

Consider, for instance, an experiment of pure tension of a cylindrical rod: from the point of view taken in this paper the aim of the experiment could be said to be that of determining the solution of the differential equation

$$\frac{\mathrm{d}\varepsilon_{33}^{(p)}}{\mathrm{d}t} = \frac{2}{3}\,\tau_{33}\,F_1\,(\tau_{33},\ \varepsilon_{33}^{(p)},\ t), \tag{2.1}$$

which corresponds to the initial condition

$$\left[\varepsilon_{33}^{(p)}\right]_{t=0} = 0\ . \tag{2.2}$$

The conventions in writing (2.1), (2.2) are that the pull and the axis of the rod are parallel to the third axis of reference and that F_1 is the determination of F which follows from specifying levels of stress and permanent strain through τ_{33} and $\varepsilon_{33}^{(p)}$ respectively.

In eqn (2.1) τ_{33} must be considered as a fixed parameter; experiments carried out for a group of values of τ_{33} will lead to the determination of a class of solutions of (2.1) all satisfying also the initial condition (2.2)

$$\varepsilon_{33}^{(p)} = f\,(\tau_{33},\ t), \qquad f\,(\tau_{33},\ 0) = 0\ . \tag{2.3}$$

It is interesting to note that, in general, knowledge of (2.3) does not suffice to determine the right-hand side of (2.1); for that purpose one should know for each value of τ_{33} a class of solutions corresponding to initial conditions varying within an appropriate range. Sufficiency obtains only in special, though important cases: i. e., when F_1 does not depend on time or on strain. In the first case an elementary property assures that, given a solution of (2.1), say (2.3), a class of them can be obtained by taking

$$\varepsilon_{33}^{(p)} = f\,(\tau_{33},\ t + T)\ , \tag{2.4}$$

where T is an arbitrary parameter; at the same time F_1 is given by

$$F_1 = \frac{3}{2\tau_{33}}\,\frac{\partial f}{\partial t}\,, \tag{2.5}$$

provided that here $t + T$ is eliminated and ε_{33} introduced using (2.4). In the second case the procedure is even simpler: the class of solutions is

$$\varepsilon_{33}^{(p)} = f(\tau_{33},\ t) + E$$

(E arbitrary parameter) and F_1 is given directly by (2.5).

We conclude: Hypothesis (iv) is general enough to suit any group of (consistent) results of experiments of simple tension under constant stress; in fact, adequacy is assured even in the special cases of pure strain-hardening or of pure time-hardening.

Experiments of tension of a rod can be carried out with varying stress: stepwise varying stress, for instance. It is easy to realize that experiments of this type, if appropriately planned, can provide the missing information for a full determination of F_1. Alternatively the validity of the hypotheses of pure strain-hardening or pure time-hardening could be put to test; for instance, in the case of tension under stress $\tau_{33}^{(1)}$ during the interval of time $(0, t_1)$ and stress $\tau_{33}^{(2)}$ after t_1, the first hypothesis predicts, for $t \geqslant t_1$, the strain

$$\varepsilon^{(p)}(t) = f(\tau_{33}^{(2)},\ t - t_1 + t^*)$$

$[\text{if } f(\tau_{33}^{(2)},\ t^*) = f(\tau_{33}^{(1)},\ t_1)]$,

and the second hypothesis

$$\varepsilon^{(p)}(t) = f(\tau_{33}^{(2)},\ t) - f(\tau_{33}^{(2)},\ t_1) + f(\tau_{33}^{(1)},\ t_1)\,.$$

In these formulae f is the function defined by (2.3).

3. - Complex stress.

To study phenomena associated with complex stress distributions a new hypothesis must be added to those put forward in Sect. 1; precisely a hypothesis regarding the arguments that enter in the function F as measures of levels of stress and permanent strain. General principles suggest that the invariants of the stress deviator and of the tensor of permanent strain may be suitable measures; then the simplest assumption is that only the second invariants are involved. This assumption is restrictive but expedient; its main advantage (as we shall see presently) is that it allows an analysis of states of complex stress on the basis of the knowledge of the function F_1 alone. Furthermore, simple checks can be suggested to determine the limits of its validity. Let us state the hypothesis explicity:

(v) The function F of formula (1.6) depends on time and on the second invariants of s_{ij} and $\varepsilon_{ij}^{(p)}$ alone

$$F \equiv F(\Pi_s, \Pi_{\varepsilon(p)}, t), \tag{3.1}$$

$$\Pi_s = -\frac{1}{2} s_{hk} s_{hk}, \qquad \Pi_{\varepsilon(p)} = -\frac{1}{2} \varepsilon_{hk}^{(p)} \varepsilon_{hk}^{(p)} . \tag{3.2}$$

Note that, because of hypotheses (ii) and (iii), $\Pi_{\varepsilon(p)}$ can also be written

$$\Pi_{\varepsilon(p)} = \Pi_\gamma + \frac{\Pi_s}{4G^2} + \frac{\gamma_{hk} s_{hk}}{2G}, \qquad \text{with } \Pi_\gamma = -\frac{1}{2} \gamma_{hk} \gamma_{hk} . \tag{3.3}$$

From formula (2.1) it follows immediately that

$$F(\Pi_s, \Pi_{\varepsilon(p)}, t) \equiv F_1\left(\sqrt{-3\Pi_s}, \sqrt{-\frac{4}{3}\Pi_{\varepsilon(p)}}, t\right); \tag{3.4}$$

such simple identity would not hold, of course, if F were to depend also on the third invariants of s_{ij} and $\varepsilon_{ij}^{(p)}$. In fact in that case knowledge of F_1 would not suffice for a complete determination of F.

Hypothesis (v) has another striking consequence: deformation under constant complex stress can be described in terms of the function f of formula (2.3) alone. In fact, when the tensor s_{ij} is constant its components appear in the differential system

$$\frac{d\gamma_{ij}^{(p)}}{dt} = F(\Pi_s, \Pi_{\varepsilon(p)}, t) \cdot s_{ij} \tag{3.5}$$

as mere parameters. By putting

$$\gamma_{ij}^{(p)}(t) = s_{ij}\, \Gamma(t), \tag{3.6}$$

all equations (3.5) reduce to a single one in Γ

$$\frac{d\Gamma}{dt} = F(\Pi_s, \Gamma^2 \Pi_s, t),$$

which can also be written by (3.4),

$$\frac{\mathrm{d}\left(\sqrt{-\frac{4}{3}\,\Pi_s\,\Gamma}\right)}{\mathrm{d}t}=\frac{2}{3}\sqrt{-3\Pi_s}\,F_1\left(\sqrt{-3\Pi_s},\ \sqrt{-\frac{4}{3}\,\Pi_s\,\Gamma},\ t\right).$$

But this equation coincides with eqn (2.1); only the names of the quantities which appear in it have been changed. It follows that the solution of (3.5) which satisfies the initial conditions $\gamma_{ij}^{(p)}(0)=0$ is given by

$$\gamma_{ij}^{(p)}=\frac{s_{ij}}{\left(-\frac{4}{3}\,\Pi_s\right)^{1/2}}\,f\,(\sqrt{-3\Pi_s},\ t)\,. \tag{3.7}$$

This result (together with a parallel one which predicts a correlation between phenomena of uniaxial and complex stress relaxation) could be used to assess the validity of hypothesis (v).

4. - Dimensional considerations. Primary creep.

Fomulae (1.6) and (3.5) are not in a dimensionally invariant form; we intend to modify them appropriately here, introducing three material constants. In theory one could perhaps envisage bodies devoid of such constants (apart from the classic moduli), whose behaviour could be described on the basis of hypotheses (i) to (v): then the function F would necessarily have the form

$$t^{-1}\,F^*\left(\frac{\Pi_s}{G^2}\,,\ \Pi_{\varepsilon\,(p)}\right). \tag{4.1}$$

However, experimental evidence seems to justify rather the view that material constants exist. For instance, because the microscopic mechanism of creep is completely unrelated to that which causes the elastic response, it seems reasonable to assume that one of the constants, Y say, has the dimensions of stress. If the peculiar time-dependence (4.1) has to be avoided (1), one time constant, τ_1, is also necessary to assure dimensional invariance; actually we assume here that two such constants, τ_1 and τ_2, exist. The reason for this complication lies in the fact that the dependence of F on time is there to represent thermal instability of the microscopic structure, a phenomenon concurrent with, but largely independent of, creep under stress.

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(1) Though there are many examples of « logarithmic » creep; one such example was illustrated by P. Feltham at the meeting.

In conclusion we specify the dimensionally invariant form of eqn (1.6) as follows

$$\frac{\mathrm{d}\gamma_{ij}^{(p)}}{\mathrm{d}t} = G\left(\frac{\Pi_s}{Y^2}, \Pi_{\varepsilon(p)}, \frac{t}{\tau_2}\right) \cdot \frac{s_{ij}}{\tau_1 Y}. \tag{4.2}$$

It appears then that strain-hardening effects will predominate over time-hardening effects, at least during the first stages of creep, if τ_2 is much larger than τ_1. We intend hereafter to concentrate our attention on that case and we begin by suggesting a specification of the function G, valid when the permanent strain is small. It is usual in similar circumstances to introduce a development in power series; but such procedure would imply regularity of G in the neighbourhood of $\Pi_{\varepsilon(p)} = 0$. It seems more advantageous to assume in our case that G is singular for $\Pi_{\varepsilon(p)} = 0$: more precisely (on grounds of simplicity) that G has a pole of order one at $\Pi_{\varepsilon(p)} = 0$

$$G\left(\frac{\Pi_s}{Y}, \Pi_{\varepsilon(p)}, \frac{t}{\tau_2}\right) \sim -\frac{M\left(-\dfrac{\Pi_s}{Y^2}\right)}{\Pi_{\varepsilon(p)}}, \quad \text{for} \quad \frac{t}{\tau_2} \to 0, \ \Pi_{\varepsilon(p)} \to 0. \tag{4.3}$$

This conjecture may appear rather arbitrary; hence an indication of its implications in some simple cases may be in order. Because the consequences of (4.3) will be discussed in some detail in the next Section we consider here only the case of tension of a rod. In that case, with the conventions of Sect. 2, one has

$$\frac{\mathrm{d}\varepsilon_{33}^{(p)}}{\mathrm{d}t} = \frac{8}{9} \frac{M\left(\dfrac{\tau_{33}^2}{3Y^2}\right)}{[\varepsilon_{33}^{(p)}]^2} \frac{\tau_{33}}{\tau_1 Y}$$

and hence

$$\varepsilon_{33}^{(p)} = \left\{\frac{8\tau_{33}}{3Y} M\left(\frac{\tau_{33}^2}{3Y^2}\right)\right\}^{1/3} \left(\frac{t}{\tau_1}\right)^{1/3},$$

a formula which expresses the Andrade-Orowan law of primary creep. Because of this coincidence we will refer henceforth to assumption (4.3) as pertaining to conditions of primary creep (2).

(2) See, for instance, the papers: A. H. Cottrell, The Time Laws of Creep; A. J. Kennedy, On the Generality of the Cubic Creep Function, both in J. Mech. Physics of Solids, **1**, (1953), 53-63, 172-181; there, experimental evidence in support of the Andrade-Orowan law is quoted and possible mechanisms for primary creep are suggested.

5. - Primary creep under complex stress.

In the restricted formulation suggested in Sect. 4 the complete constitutive equations for an elastic body subject to primary creep are

$$\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t} = \frac{1}{2G}\frac{\mathrm{d}s_{ji}}{\mathrm{d}t} - \frac{M\left(-\frac{\Pi_s}{Y^2}\right)}{\Pi_\gamma + \frac{\Pi_s}{4G^2} + \frac{\gamma_{hk}\,s_{hk}}{2G}}\,\frac{s_{ij}}{\tau_1 Y}, \tag{5.1}$$

$$\varepsilon = \frac{\tau}{3K}.$$

To tackle a problem of creep deformation, these equations must be considered together with the indefinite equations of equilibrium of continua and the corresponding boundary conditions. Initial conditions are also required; usually the elastic deformation can be considered as instantaneous: hence one may assume

$$(\gamma_{ij})_{t=0} = \frac{1}{2G}(s_{ij})_{t=0}, \qquad (\varepsilon)_{t=0} = \frac{1}{3K}(\tau)_{t=0}. \tag{5.2}$$

Of course, in special problems no mention is made of some of these equations, either because they are obviously satisfied or because they are tacitly assumed to be. Such is, for instance, the case when constant and uniform complex stress is discussed, as in Sect. 3. About that case we may remark here incidentally that during primary creep the function $\Gamma(t)$ which enters in formula (3.6) may be specified as follows

$$\Gamma(t) = \left[\frac{-3}{Y\,\Pi_s}\,M\left(-\frac{\Pi_s}{Y^2}\right)\right]^{1/3}\left(\frac{t}{\tau_1}\right)^{1/3}, \tag{5.3}$$

leading to a generalized ANDRADE-OROWAN cube-root law.

In general the solution of eqns (5.1), (5.2) and of the associated equations of equilibrium present grave analytical difficulties: in fact, the mere explicit statement of the initial conditions (5.2) requires in general the solution of a non-trivial problem within the classical theory of elasticity. We want to show now that the general problem itself can be reduced to another one which is of the type of those discussed within that theory; an approximation is involved in the process, which seems, however, clearly justified in this context.

Let us take for the components of stress and strain the developments

$$
\begin{aligned}
&\gamma_{ij} = A_{ij} + B_{ij}\left(\frac{t}{\tau_1}\right)^{1/3} + \dots, \qquad && \varepsilon = A + B\left(\frac{t}{\tau_1}\right)^{1/3} + \dots\,; \\
&s_{ij} = C_{ij} + D_{ij}\left(\frac{t}{\tau_1}\right)^{1/3} + \dots, \qquad && \tau = C + D\left(\frac{t}{\tau_1}\right)^{1/3} + \dots\,;
\end{aligned}
\tag{5.4}
$$

where A_{ij}, C_{ij}, A, C are functions which describe the initial elastic distribution of strain and stress.

Introduction of these expansions (5.4) in eqns (5.1), yields for B_{ij}, D_{ij}, B and D the conditions

$$
B_{ij} - \frac{D_{ij}}{2G} = \frac{6M\left(-\dfrac{\Pi_c}{Y^2}\right)}{Y} \frac{C_{ij}}{\sum\limits_{h,k}\left(B_{hk} - \dfrac{D_{hk}}{2G}\right)^2},
$$

$$
B = \frac{D}{3K}\,.
$$

Taking the second invariants of both members of the first of these equations one obtains

$$
\sum_{h,k}\left(B_{hk} - \frac{D_{hk}}{2G}\right)^2 = \left[\frac{6}{Y}\,M\left(-\frac{\Pi_c}{Y^2}\right)\right]^{2/3}(-2\,\Pi_c)^{1/3};
$$

hence one has the system

$$
B_{ij} = \frac{D_{ij}}{2G} + \left[-\frac{3M\left(-\dfrac{\Pi_c}{Y^2}\right)}{Y\Pi_c}\right]^{1/3} C_{ij}\,,
$$

$$
B = \frac{D}{3K}\,.
$$

These relations are parallel to the linear stress-strain relations of the theory of elasticity [cfr. eqns (5.2)], except for the non-homogeneous addendum depending on C_{ij}. Actually a term of that type appears also in the classical theory when account must be taken, for instance, of concurrent phenomena of thermal expansion. Of course, the tensor $D_{ij} + \delta_{ij} D$ must also satisfy Cauchy's equations of equilibrium together with appropriate boundary conditions; hence the process for determining B_{ij} and B does not differ substantially from the solution of a purely elastic problem.

Once B_{ij}, B are known, our original problem can be considered solved, because the first time-dependent term in expansions (5.4) is the only one which seems worth determining in view of the hypothesis accepted at the beginning on the constitutive equation.

There are many cases where the suggested process can be applied usefully; elementary examples are the problems of the bending of a bar, the torsion of a shaft, the expansion of a tube under pressure.

6. - Acknowledgement.

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Sunto.

Si propone una equazione reologica di stato per la descrizione di fenomeni di scorrimento e se ne ricavano conseguenze (valide per scorrimento primario) che generalizzano leggi ben note (la legge della radice cubica di Andrade-Orowan, ad esempio).