Riv. Mat. Univ. Parma (2) 2 (1961), 191-196

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The Maximum Term in the Taylor series of an Integral Function. (**)

Let $f(z) = \sum_0^\infty a_n z^n$ be an integral function. Let

$$M(r) = \max_\theta | f(re^{i\theta}) |, \quad m(r) = \max_n | a_n | r^n$$

and let $N(r)$ be that value of n (or the greatest value if there is more than one) for which $| a_n | r^n = m(r)$.

An account of the properties of $M(r)$, $m(r)$ and $N(r)$ has been given, for example, by VALIRON (1); and perhaps the most striking result is that if $f(z)$ is of finite order, but not otherwise, $\log m(r) \sim \log M(r)$ as r tends to infinity. Now if we are content to restrict ourselves to the case where $f(z)$ is of finite order, it is possible to prove this and allied theorems a little more directly than Valiron, and such is the purpose of this note.

Our main result, Theorem 3, is the one quoted above. The first two theorems, both well-known, appear here chiefly to prove the central result; and the last two theorems, also standard, give a certain completeness to the discussion, as we shall see.

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(**) Ricevuto il 3 luglio 1961.

(1) G. VALIRON, *Lectures on the General Theory of Integral Functions* (Toulouse, 1923; Chelsea Pub. Co. 1949). This work appears in Chapter 2.

We shall need the following result.

$$m(\lambda r) = |a_{N(\lambda r)}| (\lambda r)^{N(\lambda r)}$$

(1) $$\leqslant \lambda^{N(\lambda r)} m(r)$$

since $|a_n| r^n \leqslant m(r)$ for all values of n. (This has been taken from a paper by LITTLEWOOD and OFFORD [2], and so in essence has the proof of Theorem 2, included here for completeness).

Theorem 1. If $f(z)$ is of finite order ϱ, then $N(r) = O(r^{\varrho+\varepsilon})$ as $r \to \infty$, for every $\varepsilon > 0$.

Proof. If we replace r by $2r$ and take $\lambda = 1/2$ in (1), we obtain

$$m(r) \leqslant 2^{-N(r)} m(2r).$$

Therefore, since $m(r) \geqslant 1$ if r is large enough,

(2) $$2^{N(r)} \leqslant m(2r)$$

for sufficiently large values of r. But from Cauchy's inequality, $m(2r) \leqslant M(2r)$ and so

$$N(r) \log 2 \leqslant \log M(2r) = O(r^{\varrho+\varepsilon}).$$

The result follows at once.

Theorem 2. If $f(z)$ is of finite order ϱ, then for every $\varepsilon > 0$,

$$M(r) = O\{r^{\varrho+\varepsilon} m(r)\}$$

as $r \to \infty$.

Proof. Let $k = N(2r)$. Then

$$M(r) \leqslant \sum_0^\infty |a_n| r^n$$
$$= \sum_0^k |a_n| r^n + \sum_{k+1}^\infty |a_n| r^n,$$
$$= \sum\nolimits_1 + \sum\nolimits_2 \text{ say}.$$

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[2] J. E. LITTLEWOOD and A. C. OFFORD, *On the distribution of zeros and a-values of a random integral function (II)*, Annals of Mathematics **49** (1948) 885-952.

Since no term in $\sum_1$ exceeds $m(r)$ and there are $k+1$ terms, $\sum_1 \leqslant (k+1)\, m(r)$.

Consider $\sum_2$. Making use of (1), we have

$$|a_n|\,(2r)^n \leqslant m(2r) \leqslant 2^k\, m(r),$$

and this gives us

$$|a_n|\, r^n \leqslant 2^{k-n}\, m(r).$$

We find at once from this that

$$\sum_2 \leqslant 2^k\, m(r) \sum_{k+1}^{\infty} 2^{-n} = m(r).$$

These inequalities for $\sum_1$ and $\sum_2$ give

$$M(r) \leqslant (k+2)\, m(r), \tag{3}$$

and since by Theorem 1, $k = O(r^{\varrho+\varepsilon})$ as $r \to \infty$, the result now follows.

Lemma. (i) Suppose that $\Phi(x)$ is convex for $x \geqslant 0$ and that for a sequence of values of x tending to infinity, $\Phi(x) < K x$ where K is a constant. Then $\Phi(x) < Kx$ for all sufficiently large values of x.

(ii) If $\Phi(x)$ is convex for $x \geqslant 0$ and if $x^{-1}\Phi(x) \to \infty$ as $x \to \infty$ through some sequence of values of x, then $x^{-1}\Phi(x) \to \infty$ as $x \to \infty$ without restriction.

Proof. If $\Phi(x_1) < K x_1$ and $\Phi(x_1) < K x_2$ it is easily seen that $\Phi(x) < Kx$ between x_1 and x_2 in virtue of the convexity. From this, (i) follows at once.

In (ii), if $x^{-1}\Phi(x)$ does not tend to infinity, there must be a constant K and a sequence (x_n) tending to infinity such that $\Phi(x_n) < Kx_n$. By (i), $\Phi(x) < Kx$ for all large values of x, thus contradicting the hypothesis in (ii).

Theorem 3. If $f(z)$ is of finite order, $\log M(r) \sim \log m(r)$ as r tends to infinity.

Proof. Let ϱ by the order of $f(z)$. Since $m(r) \leqslant M(r)$ and in virtue of Theorem 2, it is easily seen ([3]) that for any $\varepsilon > 0$

$$1 - (\varrho + \varepsilon)\frac{\log r}{\log M(r)} \leqslant \frac{\log m(r)}{\log M(r)} \leqslant 1 \tag{4}$$

([3]) It is worth remembering at this point that since ε is arbitrary, the constant implied by the O in Theorem 2 may be taken to be unity.

if r is large enough, depending upon ε. We shall consider separately the two cases $\varrho > 0$ and $\varrho = 0$.

Suppose first that $\varrho > 0$. We shall show that $\log r/\log M(r) \to 0$ as $r \to \infty$, which together with (4) will prove that $\log m(r)/\log M(r) \to 1$. Let ε be chosen so that $\varrho - \varepsilon > 0$. For a sequence of values of r tending to infinity,

$$\log M(r) > r^{\varrho - \varepsilon}$$

(an elementary property of the order), and therefore $\log M(r) / \log r \to \infty$ as $r \to \infty$ through this sequence. But by HADAMARD's three circles theorem ([4]), $\log M(r)$ is a convex function of $\log r$; and so by Lemma (ii), $\log M(r)/\log r \to \infty$ as $r \to \infty$, which is what we require.

If $\varrho = 0$ we can see from (4) that, since ε may be as small as we please, it is sufficient to show that $\log r/\log M(r)$ is bounded as $r \to \infty$. If it is not bounded, there is a sequence of values of r tending to infinity for which

$$\log M(r)/\log r < 1/2.$$

(This is true with any positive number on the right-hand side; for our purpose, any number less than 1 is sufficient). But by Lemma (i), and again appealing to the three circles theorem, this inequality holds for all sufficiently large values of r. Thus as $r \to \infty$, $M(r) = O(r^{1/2})$, and consequently ([5]) $f(z)$ is identically constant. B.t in this case, Theorem 3 is trivially true and therefore the result is pro ed completely.

Now consider a familiar example, finding the order of the function

$$g(z) = \sum_0^\infty z^n (n!)^{-\alpha} \equiv \sum_0^\infty \mu_n z^n.$$

If we use Stirling's formula, we can easily see that if n and r are large,

$$\log \mu_n r^n \sim \alpha n - \alpha (n + 1/2) \log n - n \log r.$$

This is a maximum when $n \sim r^{1/\alpha}$; and so in this case,

$$\log m(r) \sim \alpha r^{1/\alpha}.$$

From this we may deduce that the order of $g(z)$ is $1/\alpha$.

([4]) See for example, E. C. TITCHMARSH, *The Theory of Functions*, 2nd Ed. Oxford 1949, § 5.3.

([5]) TITCHMARSH, *ibid.*, § 2.52.

To justify this argument we need to know two things. Firstly that $N(r)$ is large when r is large, so that the use of Stirling's formula is legitimate; and secondly, that if $\log m(r) = O(r^k)$ for a fixed k, $f(z)$ is of finite order so that we may infer the magnitude of $M(r)$ from that of $m(r)$. Both of these propositions can be proved easily with the material at hand.

Theorem 4. $N(r)$ is an increasing function of r. Unless $f(z)$ is a polynomial, $N(r) \to \infty$ as $r \to \infty$.

Proof. From (1) we have for all n

$$| a_n | (\lambda r)^n \leqslant m(\lambda r) \leqslant \lambda^{N(\lambda r)} m(r),$$

and so

$$| a_n | r^n \leqslant \lambda^{N(\lambda r) - n} m(r).$$

By putting $n = N(r)$, we obtain

$$\lambda^{N(\lambda r) - N(r)} \geqslant 1;$$

and so if $\lambda > 1$, $N(\lambda r) \geqslant N(r)$. Thus $N(r)$ is an increasing function of r.

Suppose now that $N(r)$ is bounded; $N(r) < A$, say. Since the Taylor coefficients (a_n) are clearly bounded, $m(r) = O(r^A)$; and then from (3), which is true whether the order of $f(z)$ is finite or not, $M(r) = O(r^A)$. This shows that $f(z)$ is a polynomial and so completes the proof of the theorem.

Theorem 5. If $\log m(r) = O(r^k)$ as $r \to \infty$, where k is some constant, $f(z)$ is of finite order.

Proof. From (2), which is true of all integral functions and does not pre-suppose that the order of $f(z)$ is finite, $N(r) = O(r^k)$. If we substitute this into (3), where $k = N(2r) = O(r^k)$, we find $\log M(r) = O(r^k)$ thus proving that the order of $f(z)$ is finite.