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On a ring satisfying a certain condition. (**)

1. - Introduction.

The object of this paper is to study some of the properties of a ring with a principal idempotent e and satisfying the condition that for every $a \in A$, there exists a positive integer $n = n(a) > 1$, such that $(a - ae)^n = a - ae$. We start with a decomposition of the ring, analogous to the PIERCE Decomposition for Algebras. It is also desired to study the existence of other idempotents of the ring. We obtain that, in case the component R_e in the decomposition is not vacous, there exist idempotents other than e. We also give a necessary and sufficient condition that an element of the ring may be an idempotent.

2. - Preliminary.

If e is any idempotent of A we may express A as the supplementary sum, analogous to the PIERCE *decomposition for algebras,*

$$A = eAe + eL_e + R_e \cdot e + C_e .$$

Where

(i) L_e *is a left-sided ideal consisting of the set* (x) *such that* $xe = 0$, $x \in A$.

(ii) R_e *is a right-sided ideal consisting of the set* (y) *such that* $ey = 0$, $y \in A$.

(iii) C_e *is a subving consisting of the set* (z) *such that* $ez = ze = 0$, $z \in A$.

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Now we have,

$$a = eae + e(a - ae) + (a - ea)\,e + (a - ea - ae + eae)\,.$$

The quantity eae is in eAe, $e\,(a - ae)$ in eL_e, $a - ea - ae + eae$ in C_e. To prove the uniquenes of the decomposition let

$$0 = a_1 + a_2 + a_3 + a_4 \quad \text{where} \quad a_1 \in eAe\,;\ a_2 \in eL_e\,;\ a_3 \in R_e \cdot e \quad \text{and} \quad a_4 \in C_e\,.$$

We have

$$0 = e0e = ea_1\,e + ea_2e + ea_3e + ea_4e = ea_1\,e = a_1\,,$$

and

$$0 = e0 = ea_1 + ea_2 + ea_3 + ea_4 = ea_2 = a_2;$$

similarly a_3 and therefore also a_4 are zeros. This completes the proof of the result stated. We will refer to this result quite often.

3. – Theorem I:

Let A be ring with a principal idempotent e. If for every element x of the ring which is of the form $a - ac$, $a \in A$ there exists a positive integer $n = n(x) > 1$ such that $x^n = x$, then e is a right identity of the ring.

From the unique expression of the elements of A obtained above we see that any element of C_e is of the form

$$a - ae - ea + eae, \qquad \text{for some} \qquad a \in A.$$

Now, $a - ea - ae + eae = a - ea - (a - ea)e$, which is of the form, $b - be$.

Then by the hypothesis, there exits an $n > 1$ such that $(a - ea - ae + eae)^n = ae - ea - ae + eae$.

It follows that $(a - ea - ae + eae)^{n-1}$ is an idempotent or zero. In fact

$$[(a - ea - ae + eae)^{n-1}]^2 = (a - ea - ae + eae)^n \cdot (a - ea - ae + eae)^{n-2} =$$

$$= (a - ea - ae + eae) \cdot (a - ea - ae + eae)^{n-2} =$$

$$= (a - ea - ae + eae)^{n-1}\,.$$

Since $(a - ea - ae + eae)^{n-1} \in C_e$, we have,

$$e\,(a - ea - ae + eae)^{n-1} = 0 = (a - ea - ae + eae)^{n-1}\,e\,.$$

As e is a principal idempotent, it implies that $(a - ea - ae + eae)^{n-1}$ must be zero, or $(a - ea - ae + eae)^n$ and hence $a - ea - ae + eae = 0$ ... (i). Since, $a - ea - ae + eae$, is an arbitrary element of C_e, we find that $C_e = (0)$.

Now, by (i)

$$\begin{aligned}(a - ae)^2 &= a\,(a + eae) - aea - a^2 e\\ &= a\,(ea + ae) - aea - a^2 e\\ &= 0\,.\end{aligned}$$

But by hypothesis

$$(a - ae)^n = a - ae$$

for some $n > 1$. It follows that $a - ae = 0$, and hence that e is a right identity.

COROLLARY (i). It is a trivial consequence of the above theorem that $R_e = \{\, a - ea \,\}$, $a \in A$. In the unique decomposition of A, it was found that every element of $R_e \cdot e$ is of the form $(a - ea)e$. But $R_e \cdot e = R_e$, then

$$R_e = \{\, a - ea \,\}, \qquad a \in A\,.$$

(ii) For every $x \in A$ and $y \in R$, $xy = 0$. Since $xy = (xe)\,y = x\,(ey) = 0$. In particular for any $x, y \in R$, $xy = 0$ and also, therefore, $x^2 = 0$ for any x in R.

(iii) L_e is vacous. Now $L_e = \{\, x \,\}$, $xe = 0$ But $xe = 0$implies $x = 0$. Thus the result follows.

Similarly it is easy to prove

(iv) $A = \{\, ex + y \,\}$, $\quad x \in A \quad$ and $\quad y \in R_e$.

4. – We next proceed to determine the existence of idempotents other than e for the ring A. We prove

THEOREM II. *$e + y$ is an idempotent of the ring for every $y \in R_e$.*

$$(e + y)^2 = (e + y)\,(e + y) = e^2 + ey + ye + y^2 = e + y$$

$$[ey = 0 \text{ by def of } R_e \text{ and } y^2 = 0 \text{ by Cor (ii)}]\,.$$

5. – Recalling that any element of A has its unique decomposition $ex+y$, $x \in A$ and $y \in R_e$, we may prove more generally,

THEOREM III.

The necessary and sufficient condition that an element $a = ex + y$ *is to be an idempotent of the ring* A *are that* $x \notin R_e$ *and*

(i) ex *is an idempotent.*

(ii) $yx = y$ *for all* $y \in R_e$.

The condition is necessary:

The restriction $x \notin R_e$ is obvieus otherwise $ex = 0$ and then y which is in R_e is such that $y^2 = 0$ cannot be an idempotent.

Now

$$(ex+y)^2 = exex + exy + yex + y^2 = ex + yx$$

[xy and y^2 both are zero by Cor (ii)].

If $(ex+y)^2$ is an idempotent

$$ex + yx = ex + y \tag{1}$$

multiplying both sides on the left by e we get

$$e^2 x^2 + eyx = e^2 x + ey$$

or $e^2x^2 = ex$, proving (i) $[e^2 x^2 = ex^2 = exx = (ex)(ex)]$.

From (1) and $ex^2 = ex$ we get $yx = y$ proving (ii).

Conversely,

consider any element $ex+y$ of the ring A, where $x \in A$ but is not in R_e and y is in R_e.

$$(ex+y)^2 = e^2 x^2 + yx$$

$$= ex + y \text{ (by virtue of given conditions)}.$$

Hence $ex+y$ is an idempotent.

6. – We consider below a special case of Theorem I which is of importance in as much as it gives the identity in the ring when it is semi-simple. It is, in part, our main theorem.

THEOREM IV.

In case A is also semi-simple then e is the identity of the ring.

Proof: we have already seen that for any $x \in R$, $x^2 = 0$. Since the ring is semi-simple, it follows that $x = 0$. Therefore, R_e is vacous and Cor. (iv) gives $A = \{ex\}$, $x \in A$.

Hence $A = eA$, which shows that e is also the left identity of the ring. This concludes, therefore, that e is the identity of the ring A.

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