Riv. Mat. Univ. Parma (2) 2 (1961), 203-206

S. K. JAIN (*)

On the existence of identity. (**)

The present note is intended to consider the existence of identity in a primitive ring which satisfies a certain condition. It is analogous to the condition given by HERSTEIN in his paper for the commutativity of a ring. By a primitive ring we mean that A contains a right-sided maximal ideal F whose quotient $(F : A) = (0)$.

THEOREM

Let A be a primitive ring having no non-zero nil potent elemnts and if

$$(xe - x)^{n(x)} = xe - x .$$

for every $x \in A$ and e, a fixed element, $\in Z$, centre of A.

Then A has identity.

Before we prove the theorem we prove

Lemma: *If a ring a has no non-zero nilpotent elements then any idempotent in R must belong to Z.*

Let $E^2 = E$, $E \in R$, is therefore an idempotent
Take

$$x \in R$$

(*) Indirizzo: Dept. of Math. Deshbandhu Colege, Kalkaji, New-Delhi (India).
(**) Ricevuto il 18 marzo 1961.

Now

$$(Ex - Ex\,E)^2 = (Ex - Ex\,E)\,(Ex - Ex\,E) =$$
$$= Ex\,Ex - Ex\,E^2x\,E - Ex\,E^2x + Ex\,Ex\,E =$$
$$= Ex\,Ex - Ex\,Ex\,E - Ex\,Ex + Ex\,Ex\,E =$$
$$= 0\,.$$

Similarly, $(xE - Ex\,E)^2 = 0$.
Since it has no non-zero nilpotent elements, then

$$Ex = Ex\,E = xE \quad \text{i.e} \quad E \in Z\,.$$

Since A is a primitive ring it possesses a maximal right ideal ϱ which contains no non-zero two sided ideal of A. Thus $\varrho \cap Z = (0)$. (Z being its centre) since if $x \in \varrho \cap Z$ then $xA = Ax$ is a two sided ideal of of A located in ϱ, so must be (0).
By the primitivity of A we must conclude that $x = 0$.
Now let $x \in \varrho$.
By the hypothesis

$$(xe - x)^{n(x)} = xe - x\,.$$

Then

$$[(xe - x)^{n-1}]^2 = (xe - x)^{n+n-2} = (xe - x)^n\,(xe - x)^{n-2} =$$
$$= (xe - x)\,(xe - x)^{n-2} = (xe - x)^{n-1}.$$

$\therefore$ $(xe - x)^{n-1}$ is an idempotent.

$(xe - x) \in \varrho$ and also, therefore $(xe - x)^{n-1} \in \varrho$.

$\therefore$ $(xe - x)^{n-1} \in \varrho \cap Z$.

It implies $(xe - x)^{n-1} = 0$.

Now

$$0 = (xe - x)^{n-1}\,(xe - x) = (xe - x)^n = (xe - x)^n$$
$$= (xe - x)$$

$\therefore$ For every $x \in \varrho$,

$$xe = x$$

Suppose $a \in \varrho$; $r \in A$ $\therefore$ $ar \in \varrho$.

We then have

$$(ar)\, e = ar$$

or

$$a\,(re - r) = 0\,, \qquad \varrho\,(re - r) = 0$$

which in a primitive ring implies that either $\varrho = (0)$ or $re - r = 0$.

If $re - r = 0$ then it follows that e is the identity of A.

In case $\varrho = (0)$, then A must be a division ring, meaning thereby, that A has identity.

The author wishes to thank Dr. P. B. BHATTACHARYA and Dr. P. K. MENON for the kind inspiration and encouragement.