Riv. Mat. Univ. Parma (2) **2** (1961), 281-286

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A note on Bessel function of the first kind. (**)

1. The BESSEL function of the first kind $J_n(x)$ of order n is defined for $n > -1$ and $-\infty < x < +\infty$ by the power series:

$$J_n(x) = \sum_{\nu=0}^{\infty} (-)^\nu \frac{\left(\frac{x}{2}\right)^{n+2\nu}}{\Gamma(n+\nu+1)\,\nu!} \tag{1.1}$$

It satisfies the recurrence formula

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \quad (n > 0). \tag{1.2}$$

Recently SZÁSZ [1] has established the following inequality for the BESSEL function $J_n(x)$:

$$J_n^2(x) - J_{n-1}(x)\, J_{n+1}(x) > \frac{1}{n+1} J_n^2(x); \tag{1.3}$$

$$n > 0, \quad -\infty < x < +\infty.$$

THIRUVENKATACHAR and NANJUNDIAH [2] pointed out that the inequality (1.3) becomes an equality only for $x = 0$, $(n > 0)$. Now if we define the following function of the real variable x:

$$\Delta_{n,h,k;x}(f) \equiv \begin{vmatrix} f_{n+h}(x) & f_n(x) \\ f_{n+h+k}(x) & f_{n+k}(x) \end{vmatrix}$$

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(**) Ricevuto il 13 gennaio 1961.

for a function $f_n(x)$, where h and k are integers such that $k \geqslant h \geqslant 1$, (1.3) can be put in the elegant form:

$$(1.4) \qquad (n+1)\, \Delta_{n-1,1,1;x}(J) > J_n^2(x); \qquad n > 0, \; -\infty < x < +\infty .$$

First we wish to point out that

$$(1.5) \qquad \left.\begin{aligned} \Delta_{n,1,2;x}(J) &< 0, \quad -\infty < x < 0 \\ &= 0, \quad x = 0 \\ &> 0, \quad 0 < x < +\infty \end{aligned}\right\} ; \qquad (n > 0)$$

For

$$x^2 \Delta_{n,1,2;x}(J) = \begin{vmatrix} xJ_{n+1}(x) & xJ_n(x) \\ 2(n+2)J_{n+2}(x) - xJ_{n+1}(x) & 2(n+1)J_{n+1}(x) - xJ_n(x) \end{vmatrix}$$

$$= (2x) \cdot \begin{vmatrix} J_{n+1}(x) & J_n(x) \\ (n+2)J_{n+2}(x) & (n+1)J_{n+1}(x) \end{vmatrix}$$

$\therefore$ we get

$$x\, \Delta_{n,1,2;x}(J)$$

$$= 2\left[(n+1)J_{n+1}^2(x) - (n+2)J_n(x)J_{n+2}(x)\right]$$

$$(1.6) \qquad = 2\left[(n+2)\Delta_{n,1,1;x}(J) - J_{n+1}^2(x)\right].$$

Thus it follows from (1.4) and (1.6) that

$$(1.7) \qquad x\, \Delta_{n,1,2;x}(J) \geqslant 0; \qquad n > 0, \qquad -\infty < x < +\infty .$$

and a glance at (1.7) now convinces us that $\Delta_{n,1,2;x}(J) \gtrless 0$, according as $0 < x < +\infty$, or, $-\infty < x < 0$.

To prove that $\Delta_{n,1,2;x}(J) = 0$, for $x = 0$, we write $J_n(x) = x^n j_n(x)$, where $j_n(x)$ is an entire function. Now we have from (1.6)

$$\Delta_{n,1,2;x}(J) = 2x^{2n+1}\{(n+1)\, j_{n+1}^2(x) - (n+2)\, j_n(x)\, j_{n+2}(x)\}$$

$$\therefore\ \Delta_{n,1,2;x}(J) = 0, \quad \text{for} \quad x = 0\ (n > 0)\,.$$

2. In this section we shall first derive a recurrence formula for $\Delta_{n,1,2;x}(J)$, whereby certain properties of $\Delta_{n,i,2;x}(J)$ are rendered intuitive.

To this end, we have

$$\begin{aligned} & x\,\Delta_{n,1,2;x}(J) \\ & = x\,J_{n+1}(x)\,J_{n+2}(x) - J_n(x)\{2(n+2)\,J_{n+2}(x) - x\,J_{n+1}(x)\} \\ & = x\,J_n(x)\,J_{n+1}(x) - 2\,(n+2)\,J_n(x)\,J_{n+2}(x) + J_{n+2}(x)\{2n\,J_n(x) - J_{n-1}(x)\} \\ & = x\,\Delta_{n-1,1,2;x}(J) - 4\,J_n(x)\,J_{n+2}(x)\,, \end{aligned}$$

whence, on using the identity (1.6) just proved, we get

$$x\,[(n+2)\,\Delta_{n-1,1,2;x}(J) - n\,\Delta_{n,1,2;x}(J)] = 4\,(n+1)\,J_{n+1}^2(x)\,. \tag{2.1}$$

We can re-write the relation (2.1) in the form

$$x\left[\frac{\Delta_{n-1,1,2;x}(J)}{n\,(n+1)} - \frac{\Delta_{n,1,2;x}(J)}{(n+1)\,(n+2)}\right] = \frac{4}{n\,(n+2)}\,J_{n+1}^2(x) \tag{2.2}$$

Thus it follows from (1.5) and (2.2) that the sequence

$$\sigma_{n+\nu}(x) \equiv \frac{\Delta_{n+\nu,1,2;x}(J)}{(n+\nu+1)\,(n+\nu+2)}\ ;$$

$$(n > 0,\ \nu = 0,\ 1,\ 2,\ \ldots,\ x \text{ is fixed})$$

which remaims positive in the interval $0 < x < +\infty$, is monotone descending in the same interval as ν increases and ultimately $\sigma_{n+\nu}(x) \to 0$ as $\nu \to \infty$, because when $\nu \to \infty$, $J_{n+\nu}(x)$ and consequently $\Delta_{n+\nu,1,2;x}(J) \to 0$. Similarly the sequence $\sigma_{n+\nu}(x)$ which remains negative in the interval $-\infty < x < 0$, is monotone ascending in the same interval as ν increases and ultimately $\sigma_{n+\nu}(x) \to 0$ as $\nu \to \infty$. Obviously $\sigma_{n+\nu}(x) \to 0$ as $\nu \to \infty$, for $x = 0$ $\therefore$ we have

$$\sigma_{n+\nu}(x) \to 0, \text{ as } \nu \to \infty;\ n > 0,\ -\infty < x < +\infty . \tag{2.3}$$

Turning to the relation (2.2), we immediately derive

$$\frac{\Delta_{n-1,1,2;x}(J)}{n(n+1)} - \frac{\Delta_{n+\nu,1,2;x}(J)}{(n+\nu+1)(n+\nu+2)} = \frac{4}{x}\sum_{r=0}^{\nu} \frac{J^2_{n+r+1}(x)}{(n+r)(n+r+2)} \tag{2.4}$$

Since $J_{n+\nu}(x) \to 0$ as $\nu \to \infty$, (2.4) yields the following series for $\Delta_{n-1,1,2;x}(J)$:

$$\Delta_{n-1,1,2;x}(J) = \frac{4n(n+1)}{x}\sum_{r=0}^{\infty} \frac{J^2_{n+r+1}(x)}{(n+r)(n+r+2)} \tag{2.5}$$

Again returning to the relation (2.1), we have

$$(n+2)\,\Delta_{n-1,1,2;x}(J) - n\Delta_{n,1,2;x}(J) \gtrless 0,$$

according as

$$0 < x < +\infty \qquad \text{or,} \qquad -\infty < x < 0 ,$$

whence we derive at once

$$3\cdot\sum_{i=0}^{n-1} \Delta_{i,1,2;x}(J) \gtrless n\cdot\Delta_{n,1,2;x}(J) \tag{2.6}$$

according as $0 < x < +\infty$ or, $-\infty < x < 0$.

3. Finally we shall discuss certain properties of $\Delta_{n,1,1;x}(J)$, similar to those studied in the case of $\Delta_{n,1,2;x}(J)$.

We notice that

$$\Delta_{n,1,1;x}(J) = \frac{2}{x}\begin{vmatrix} J_{n+1}(x) & J_n(x) \\ (n+1)\,J_{n+1}(x) & nJ_n(x) \end{vmatrix} + \begin{vmatrix} J_n(x) & J_{n-1}(x) \\ J_{n+1}(x) & J_n(x) \end{vmatrix}$$

$$\therefore\ n\,\Delta_{n,1,1;x}(J) = n\,\Delta_{n-1,1,1;x}(J) - \frac{2n}{x}J_n(x)\,J_{n+1}(x),$$

whence, on using the recurrence relation (1.2), we get

$$n \Delta_{n,1,1;x}(J) = (n+1) \Delta_{n-1,1,1;x}(J) - J_n^2(x) - J_{n+1}^2(x),$$

which is due to Thiruvenkatachar and Nanjundiah.

Thus it follows that

$$\frac{\Delta_{n-1,1,1;x}(J)}{n} > \frac{\Delta_{n,1,1;x}(J)}{n+1} \tag{3.1}$$

Hence it is clear that the sequence

$$A_{n+\nu}(x) \equiv \frac{\Delta_{n+\nu,1,1;x}(J)}{n+\nu+1};$$

$$(n > 0, \ \nu = 0, 1, 2, \ldots, \ x \text{ is fixed})$$

is positive monotone descending in the interval $-\infty < x < +\infty$ $(x \neq 0)$ and ultimately $A_{n+\nu}(x) \to 0$ as $\nu \to \infty$. Obviously $A_{n+\nu}(x) \to 0$ as $\nu \to \infty$ for $x = 0$. Therefore we get

$$A_{n+\nu}(x) \to 0 \text{ as } \nu \to \infty; \ n > 0, \quad -\infty < x < +\infty. \tag{3.2}$$

Turning to the inequality (3.1), we immediately derive

$$2 \cdot \sum_{i=0}^{n-1} \Delta_{i,1,1,x}(J) > n \Delta_{n,1,1;x}(J), \tag{3.3}$$

$$(-\infty < x < +\infty, \quad x \neq 0).$$

Towards the close of the paper the author wishes to express his sincere thanks to the late lamented professor Dr. H. M. Sengupta for his encouragement in preparing this work.

This paper was first communicated on 16th.Nov. 1959.

But unfortunately this was lost in mails. (Added in proof).

References.

[1] O. Szász: Proc. Amer. Math. Soc., 1950, **1**, 256.

[2] V. R. Thiruvenkatachar and Nanjundiah: Proc. Ind. Acad. Sci., 1951, **33 A**, 377.