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Some Theorems On Generalized Laplace Transform-II. (**)

1. VARMA [8, p. 209] gave a generalization of the well-known LAPLACE transform

$$\Phi(p) = p \int_0^\infty e^{-pt} h(t)\, dt, \tag{1.1}$$

by means of the integral equation

$$\Phi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} (pt)^{m-\frac{1}{2}} W_{k,m}(pt)\, h(t)\, dt, \tag{1.2}$$

where $W_{k,m}(z)$ denotes the WHITTAKER function.

When $k + m = \frac{1}{2}$, (1.2) reduces to (1.1) by virtue of the identity

$$W_{\frac{1}{2}-m,m}(x) \equiv x^{\frac{1}{2}-m} e^{-\frac{1}{2}x}.$$

We shall denote (1.1) and (1.2) by the symbolic expressions $\Phi(p) \doteqdot h(t)$ and $\Phi(p) \underset{k,m}{\overset{v}{=}} h(t)$ respectively.

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In a previous paper [7] some theorems connected with the transform (1.2) were given. The object of this note is to study some other properties of this generalized Laplace transform. The results obtained are quite general and include as particular cases certain theorems recenly given by RATHIE [6, p. 132] and NARAIN [5, p. 285]. In what follows n and s are positive integers.

2. Theorem 1.

If

$$\Phi(p) \underset{k,\, m}{\overset{v}{=\!=}} t^{\varrho}\, h(t),$$

$$h(p) \underset{\lambda,\, \mu}{\overset{v}{=\!=}} f(t),$$

and

$$p^{1-\frac{s\sigma}{n}+\frac{s}{n}} f(p^{s/n}) \underset{\gamma,\, \delta}{\overset{v}{=\!=}} g(t),$$

then

$$\Phi(p) = (2\pi)^{\frac{1}{2}(3-2n-s)}\, s^{\gamma+\delta}\, p^{\sigma-\varrho-1}\, n^{\lambda+\mu+k+m+\sigma} \tag{2.1}$$

$$\times \int_0^\infty G^{2s+2n,\, 2n}_{3n+s,\, 2s+3n}\left(\frac{p^n\, t^s}{s^s}\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_{3n+s}\\ \beta_1, \ldots, \beta_{2s+3n}\end{matrix}\right) g(t)\, dt,$$

where (¹) $\alpha_{i+1} = \dfrac{1-2\mu-\sigma+i}{n}$, $\alpha_{n+i+1} = \dfrac{1-\sigma+i}{n}$,

$$\alpha_{2n+i+1} = \frac{5+2\varrho+2m-2\sigma-2k+2i}{2n}, \quad \alpha_{3n+\nu+1} = \frac{1-2\gamma+2\delta+2\nu}{2s},$$

$$\beta_{\nu+1} = \frac{\nu}{s}, \quad \beta_{s+\nu+1} = \frac{\nu+2\delta}{s}, \quad \beta_{2s+i+1} = \frac{2+\varrho-\sigma+i}{n},$$

$$\beta_{2s+n+i+1} = \frac{2+\varrho+2m-\sigma+i}{n}, \quad \beta_{2s+2n+i+1} = \frac{1+2\lambda-2\mu-2\sigma+2i}{2n}$$

(¹) For the behaviour of $G^{\alpha\,\beta}_{\gamma\,\delta}\left(x\,\middle|\,\begin{matrix}a_1, \ldots, a_\gamma\\ b_1, \ldots, b_\delta\end{matrix}\right)$, see [3, p. 212].

for $i = 0, 1, \ldots, (n-1)$ and $\nu = 0, 1, \ldots, (s-1)$, provided that $R(\varrho + 2 + m \pm m + \mu \pm \mu) > 0$, $R(2s\mu + n\delta \pm n\delta + s\sigma) > 0$, $R(s\sigma + n\delta \pm n\delta) > 0$, $R(p) > 0$, the generalized Laplace transforms of $|g(t)|$ and $|f(t)|$ exist and the integral is convergent.

Proof.

In a recent paper [7] I have shown that if

$$h(p) \underset{\lambda,\ \mu}{\overset{v}{=}} f(t)$$

and

$$p^{1-\frac{s\sigma}{n}+\frac{s}{n}} f(p^{s/n}) \underset{\gamma,\ \delta}{\overset{v}{=}} g(t),$$

then

$$h(p) = (2\pi)^{\frac{1}{2}(2\ \ n-s)} s^{\gamma+\delta} n^{\lambda+\mu+\sigma-1} p^{1-\sigma} \tag{2.2}$$

$$\times \int_0^\infty G^{2s,2n}_{2n+s,2s+n}\left(\frac{t^s n^n}{s^s p^n}\,\middle|\,\begin{matrix}\alpha_1, \ldots, \alpha_{2n+s}\\ \beta_1, \ldots, \beta_{2s+n}\end{matrix}\right) g(t)\,\mathrm{dt},$$

where $\alpha_{i+1} = \dfrac{1-\sigma+i}{n}$, $\alpha_{n+i+1} = \dfrac{1-\sigma-2\mu+i}{n}$, $\alpha_{2n+\nu+1} = \dfrac{1+2\delta-2\gamma+2\nu}{2s}$;

$$\beta_{\nu+1} = \frac{\nu}{s}, \quad \beta_{s+\nu+1} = \frac{2\delta+\nu}{s}, \quad \beta_{2s+i+1} = \frac{1+2\lambda-2\mu-2\sigma+2i}{2n}$$

for $\nu = 0, 1, \ldots, (s-1)$ and $i = 0, 1, \ldots, (n-1)$.

By definition

$$\Phi(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}} W_{k,m}(px)\, x^\varrho\, h(x)\,\mathrm{d}x. \tag{2.3}$$

On substituting the value of $h(x)$ from (2.2) in (2.3) we have

$$\Phi(p) = (2\pi)^{\frac{1}{2}(2-n-s)}\, s^{\gamma+\delta}\, n^{\lambda+\mu+\sigma-1}\, p^{m+\frac{1}{2}}$$

$$\times \int_0^\infty x^{m+\varrho+\frac{1}{2}-\sigma}\, e^{-\frac{1}{2}px}\, W_{k,m}(px) \left\{ \int_0^\infty g(t) \right.$$

$$\left. \times\, G^{2s,2n}_{2n+s,2s+n}\left(\frac{t^s n^n}{s^s x^n} \,\middle|\, \begin{matrix} \alpha_1, \ldots, \alpha_{2n+s} \\ \beta_1, \ldots, \beta_{2s+n} \end{matrix}\right) \mathrm{d}t \right\} \mathrm{d}x$$

$$= (2\pi)^{\frac{1}{2}(2-n-s)}\, s^{\gamma+\delta}\, n^{\lambda+\mu+\sigma-1}\, p^{m+\frac{1}{2}} \int_0^\infty g(t)\, \mathrm{d}t$$

$$\times \int_0^\infty x^{m+\varrho+\frac{1}{2}-\sigma}\, e^{-\frac{1}{2}px}\, W_{k,m}(px)\, G^{2s,2n}_{2n+s,\,2s+n}\left(\frac{t^s n^n}{s^s x^n} \,\middle|\, \begin{matrix} \alpha_1, \ldots, \alpha_{2n+s} \\ \beta_1, \ldots, \beta_{2s+n} \end{matrix}\right) \mathrm{d}x$$

$$= (2\pi)^{\frac{1}{2}(2-n-s)}\, s^{\gamma+\delta}\, n^{\lambda+\mu+\sigma-1}\, p^{m+\frac{1}{2}} \int_0^\infty g(t)\, \mathrm{d}t$$

$$\times \int_0^\infty x^{m+\varrho+\frac{1}{2}-\sigma}\, e^{-\frac{1}{2}px}\, W_{k,m}(px)\, G^{2n,2s}_{2s+n,\,2n+s}\left(\frac{x^n s^s}{n^n t^s} \,\middle|\, \begin{matrix} 1-\beta_1, \ldots, 1-\beta_{2s+n} \\ 1-\alpha_1, \ldots, 1-\alpha_{2n+s} \end{matrix}\right) \mathrm{d}x,$$

on changing the order of integration and and using a well-known property of G-function [**3**, p. 212]

$$G^{\alpha\,\beta}_{\gamma\,\delta}\left(x^{-1} \,\middle|\, \begin{matrix} a_i \\ b_j \end{matrix}\right) = G^{\alpha\,\beta}_{\gamma\,\delta}\left(x \,\middle|\, \begin{matrix} 1-b_j \\ 1-a_i \end{matrix}\right).$$

Now evaluate the x-integral by means of the formula [**7**]

(2.4)
$$\int_0^\infty (px)^{m-\frac{1}{2}}\, x^{-n\varrho}\, e^{-\frac{1}{2}px}\, W_{k,m}(px)\, G^{\alpha\,\beta}_{\gamma\,\delta}\left(zx^n \,\middle|\, \begin{matrix} a_1, \ldots, a_\gamma \\ b_1, \ldots, b_\delta \end{matrix}\right) \mathrm{d}x$$

$$= p^{n\varrho-1}\, (2\pi)^{\frac{1}{2}-\frac{1}{2}n}\, n^{k+m-n\varrho}$$

$$G^{\alpha,\,\beta+2n}_{\gamma+2n,\,\delta+n}\left(\frac{zn^n}{p^n} \,\middle|\, \begin{matrix} \varrho, \ldots, \varrho+\frac{n-1}{n}, \frac{n\varrho-2m}{n}, \ldots, \frac{n\varrho-2m+n-1}{n}, a_1, \ldots, a_\gamma \\ b_1, \ldots, b_\delta, \frac{2n\varrho+2k-2m-1}{2n}, \ldots, \frac{2n\varrho+2k-2m+2n-3}{2n} \end{matrix}\right),$$

where $R\ (nb_h + m \pm m + 1 - n\varrho) > 0$, for $h = 1, \ldots, \alpha$;

$$|\arg z| < \left(\alpha + \beta - \frac{1}{2}\gamma - \frac{1}{2}\delta\right)\pi,\ \alpha + \beta > \frac{1}{2}\gamma + \frac{1}{2}\delta,\ R\,(p) > 0\,,$$

then we arrive at the result.

The change of the order of integration can be justified by the application of de la Vallèe Poussin's theorem [**1**, p. 504] when the generalised Laplace transforms of the functions $|g\,(t)|$ and $|f\,(t)|$ both exist and the integral is convergent.

2.1 Corollary

On taking $\gamma + \delta = \frac{1}{2}$ in (2.1) it takes the following form. If

$$\Phi\,(p) \underset{k,\ m}{\overset{v}{=\!=}} t^{\varrho}\, h\,(t),$$

$$h\,(p) \underset{\lambda,\ \mu}{\overset{v}{=\!=}} f\,(t)$$

and

$$p^{1-\frac{s\sigma}{n}+\frac{s}{n}} f\,(p^{s/n}) \doteqdot g\,(t)\,,$$

then

(2.5) $$\Phi\,(p) = (2\pi)^{\frac{1}{2}(3-2n-s)}\, s^{\frac{1}{2}}\, p^{\sigma-\varrho-1}\, n^{\lambda+\mu+k+m+\sigma}$$

$$\times \int_0^\infty G^{s+2n,\ 2n}_{3n,\ s+3n}\left(\frac{p^n\ t^s}{s^s}\ \middle|\ \begin{matrix}\alpha_1, \ldots, \alpha_{3n}\\ \beta_1, \ldots, \beta_{s+3n}\end{matrix}\right) g\,(t)\,\mathrm{d}t\,,$$

where $\alpha_{i+1} = \dfrac{1-2\mu-\sigma+i}{n}$, $\alpha_{n+i+1} = \dfrac{1-\sigma+i}{n}$,

$\alpha_{2n+i+1} = \dfrac{5+2\varrho+2m-2\sigma-2k+2i}{2n}$, $\beta_{\nu+1} = \dfrac{\nu}{s}$, $\beta_{\nu+i+1} = \dfrac{2+\varrho-\sigma+i}{n}$,

$\beta_{s+n+i+1} = \dfrac{2+\varrho+2m-\sigma+i}{n}$, $\beta_{s+2n+i+1} = \dfrac{1+2\lambda-2\mu-2\sigma+2i}{2n}$

for $i = 0, 1, \dots, (n-1)$ and $\nu = 0, 1, \dots, (s-1)$; provided that the integral is convergent and Laplace transforms of $|g(t)|$ and generalized Laplace transform of $|f(t)|$ exist, $R(\sigma + \mu \pm \mu) > 0$, $R(\varrho + 2 + m \pm m + \mu \pm \mu) > 0$ and $R(p) > 0$.

(2.5) was obtained by NARAIN [5, p. 285] in an entirely different form.

3. Theorem 2.
If

$$h(t) \doteqdot \Phi(p^{n/s})$$

and

$$t^{-l} \exp(-a t^{s/n}) \, \Phi(t) \underset{\lambda,\ \mu}{\overset{v}{=\!=}} \psi(p, a),$$

then

$$\psi(p, a) = (2\pi)^{\frac{1}{2}(2-n-s)} \, n^{\frac{1}{2}} \, s^{\lambda+\mu+s/n-l} \, p^{l-s/n} \tag{3.1}$$

$$\times \int_0^\infty G^{n,2s}_{2s,n+s}\left[\frac{s^s (t+a)^n}{p^s n^n} \,\middle|\, \begin{matrix} \alpha_1, \dots, \alpha_{2s} \\ \beta_1, \dots, \beta_{n+s} \end{matrix}\right] h(t)\, dt,$$

provided that the generalized Laplace transform of $|t^{-l} \exp(-a t^{s/n}) \, \Phi(t)|$ and Laplace transform of $|h(t)|$ exist and the integral is convergent, $R(n\mu \pm n\mu + n + s - nl) > 0$, $R(p) > 0$, $R(a) > 0$,

$$\alpha_{\nu+1} = \frac{nl - s + n\nu}{ns}, \quad \alpha_{s+\nu+1} = \frac{nl - 2n\mu - s + n\nu}{ns}, \quad \beta_{i+1} = \frac{i}{n},$$

$$\beta_{n+\nu+1} = \frac{2n\lambda + 2nl - 2n\mu - n - 2s + 2n\nu}{2ns}; \text{ for } \nu = 0, 1, \dots, (s-1)$$

and $i = 0, 1, \dots, (n-1)$.

Proof.
We have

$$h(t) \doteqdot \Phi(p^{\frac{n}{s}}) \tag{3.2}$$

and [7]

$$(3.3)\qquad t^{\frac{2n\mu+n-2nl}{2s}} \exp\left(-at-\frac{1}{2}zt^{n/s}\right) W_{\lambda,\mu}(zt^{n/s})$$

$$\doteqdot (2\pi)^{\frac{1}{2}(2-n-s)}\, n^{-\frac{1}{2}}\, p\, s^{1+\lambda+\mu+s/n-l}\, z^{\,l-\mu-\frac{1}{2}-s/n}$$

$$\times\, G^{n,2s}_{2s,s+n}\left[\frac{z^s (p+a)^n}{s^s\, n^n}\,\middle|\,\begin{matrix}\alpha_1, \dots, \alpha_{2s}\\ \beta_1, \dots, \beta_{s+n}\end{matrix}\right],$$

where $R(n\mu \pm n\mu + n + s - nl) > 0$, $R(p) > 0$, $R(a) > 0$ and $R(z) > 0$.

Using (3.2) and (3.3) in Parseval-Goldstein theorem of Operational Calculus, namely.

If

$$f_1(p) \doteqdot g_1(t) \qquad \text{and} \qquad f_2(p) \doteqdot g_2(t),$$

then

$$\int_0^\infty f_1(t)\, g_2(t)\, t^{-1}\, dt = \int_0^\infty g_1(t)\, f_2(t)\, t^{-1}\, dt,$$

we obtain

$$\int_0^\infty t^{\frac{2n\mu+n-2nl-2s}{2s}} \exp\left\{-at-\frac{1}{2}zt^{n/s}\right\} W_{\lambda,\mu}(zt^{n/s})\, \Phi(t^{n/s})\, dt$$

$$= (2\pi)^{\frac{1}{2}(2-n-s)}\, n^{-\frac{1}{2}}\, s^{1+\lambda+\mu+s/n-l}\, z^{l-\mu-\frac{1}{2}-s/n}$$

$$\times \int_0^\infty G^{n,2s}_{2s,2s+n}\left[\frac{z^s (t+a)^n}{s^s\, n^n}\,\middle|\,\begin{matrix}\alpha_1, \dots, \alpha_{2s}\\ \beta_1, \dots, \beta_{2s+n}\end{matrix}\right] h(t)\, dt.$$

The theorem follows immediately from above on multiplying both the sides by $z^{\mu+\frac{1}{2}}$, putting $t = u^{s/n}$ and lastly replacing z by p.

3.1. Corollary 1.

If

$$h(t) \doteqdot \Phi(p^{n/s})$$

and

$$t^{-l} \exp(-at^{s/n})\, \Phi(t) \doteqdot \psi(p, a),$$

then

(3.4) $$\psi(p, a) = (2\pi)^{\frac{1}{2}(2-n-s)}\, n^{\frac{1}{2}}\, s^{\frac{1}{2}+s/n-l}\, p^{l-s/n}$$

$$\times \int_0^\infty G^{n,s}_{s,n}\left[\frac{(t+a)^n s^s}{n^n p^s} \,\middle|\, \begin{matrix}\alpha_1, \dots, \alpha_s \\ \beta_1, \dots, \beta_n\end{matrix}\right] h(t)\, dt,$$

provided that the Laplace transforms of $|h(t)|$ and $|t^{-l} \exp(-at^{s/n})\, \Phi(t)|$ exist, the integral is convergent, $R(p) > 0$, $R(a) > 0$, $R(n + s - nl) > 0$, where $\alpha_{\nu+1} = \dfrac{nl - s + n\nu}{ns}$, $\beta_{i+1} = \dfrac{i}{n}$, for $i = 0, \dots, (n-1)$ and $\nu = 0, 1, \dots, (s-1)$.

For $\lambda + \mu = \frac{1}{2}$, (3.1) reduces to Corollary 1.

3.2. Corollary 2.

If

$$h(t) \doteqdot \Phi(p)$$

and

$$t^{\nu-2}\, e^{-at}\, \Phi(t) \underset{\lambda,\mu}{\overset{\nu}{=\!=}} \psi(p, a),$$

then

(3.5) $$\psi(p, a) = p^{1-\nu} \frac{\Gamma(\nu)\, \Gamma(\nu + 2\mu)}{\Gamma(\frac{1}{2} + \mu + \nu - \lambda)}$$

$$\int_0^\infty {}_2F_1\left(\nu, \nu + 2\mu; \frac{1}{2} + \mu + \nu - \lambda; -\frac{t+a}{p}\right) h(t)\, dt,$$

provided that the integral is convergent and Laplace transform of $|h(t)|$ and generalized Laplace transform of $|t^{\nu-2}\, e^{-at}\, \Phi(t)|$ exist, $R(p) > 0$, $R(a) > 0$, $R(\nu) > 0$, $R(\nu + 2\mu) > 0$.

Put $n = s = 1$ and $l = 2 - \nu$ in (3.1) to obtain Cor. 2.

4. Theorem 3.

If

$$h(t) \frac{v}{\overline{k,\ m}} \Phi(p)$$

and

$$t^{\sigma} e^{-\alpha t} h(t) \frac{v}{\overline{\lambda,\ \mu}} \psi(p,\ \alpha : \sigma),$$

then

(4.1)
$$\Phi(p) = \frac{\Gamma(\frac{1}{2} + c - \lambda - \mu)}{\Gamma(c)\,\Gamma(c-2\mu)}\, p^{k+m+\frac{1}{2}}$$

$$\times \int_0^\infty t^{c-2\mu-2}\, {}_3F_2 \left[\begin{matrix} \frac{1}{2} - k \pm m,\ \frac{1}{2} + c - \lambda - \mu \\ c,\ c - 2\mu \end{matrix};\ -\frac{t}{p}\right] \psi\left(t, p : c + k + m - 2\mu - \frac{1}{2}\right) \mathrm{d}t,$$

provided that the integral is convergent and generalised Laplace transform of $|h(t)|$ and $|t^{\sigma} e^{-\alpha t} h(t)|$ exist, $R(c) > 0$, $R(c - 2\mu) > 0$ and $R(p) > 0$.

Proof.

ERDÉLYI [2, p. 50] has shown that

(4.2)
$$W_{k,m}(z) = \frac{\Gamma(\frac{1}{2} + c - \lambda - \mu)}{\Gamma(c)\,\Gamma(c-2\mu)}\, e^{-\frac{1}{2}z} z^{k}$$

$$\times \int_0^\infty e^{-\frac{1}{2}x} x^{c-\mu-3/2}\, W_{\lambda,\mu}(x)\, {}_3F_2 \left(\begin{matrix} \frac{1}{2} - k \pm m,\ \frac{1}{2} + c - \lambda - \mu \\ c,\ c - 2\mu \end{matrix};\ -\frac{x}{z}\right) \mathrm{d}x,$$

$$R(c) > 0,\ R(c - 2\mu) > 0 .$$

On putting $z = pu$ and $x = tu$ this takes the form

$$W_{k,m}(pu) = \frac{\Gamma(\frac{1}{2} + c - \lambda - \mu)}{\Gamma(c)\,\Gamma(c-2\mu)}\, e^{-\frac{1}{2}pu} (pu)^{k}$$

$$\times \int_0^\infty e^{-\frac{1}{2}tu} (tu)^{c-\mu-3/2}\, W_{\lambda,\mu}(tu)\, {}_3F_2 \left(\begin{matrix} \frac{1}{2} - k \pm m,\ \frac{1}{2} + c - \lambda - \mu \\ c,\ c - 2\mu \end{matrix};\ -\frac{t}{p}\right) u\ \mathrm{d}t .$$

Substituting the above value of $W_{k,m}(pu)$ in

$$(4.3) \qquad \Phi(p) = p \int_0^\infty e^{-\frac{1}{2}pu} (pu)^{m-\frac{1}{2}} W_{k,m}(pu)\, h(u)\, du,$$

we obtain

$$\Phi(p) = p^{k+m+\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+c-\lambda-\mu)}{\Gamma(c)\,\Gamma(c-2\mu)} \int_0^\infty \exp(-pu)\, u^{k+m-\frac{1}{2}} h(u)\, du$$

$$\times \int_0^\infty e^{-\frac{1}{2}tu} (tu)^{c-\mu-3/2} W_{\lambda,\mu}(ut)\, {}_3F_2\left(\begin{matrix}\frac{1}{2}-k\pm m,\ \frac{1}{2}+c-\lambda-\mu \\ c,\ c-2\mu\end{matrix};\ -\frac{t}{p}\right) u\, dt\,.$$

$$= \frac{\Gamma(\frac{1}{2}+c-\lambda-\mu)}{\Gamma(c)\,\Gamma(c-2\mu)}\, p^{k+m+\frac{1}{2}} \int_0^\infty t^{c-\mu-3/2}$$

$$\times\, {}_3F_2\left(\begin{matrix}\frac{1}{2}-k\pm m, \frac{1}{2}+c-\lambda-\mu \\ c, c-2\mu\end{matrix};\ -\frac{t}{p}\right) dt \int_0^\infty u^{c+k+m-\mu-1} \exp\left(-pu-\frac{tu}{2}\right) W_{\lambda,\mu}(tu)\, h(u)\, du.$$

$$= \frac{\Gamma(\frac{1}{2}+c-\lambda-\mu)}{\Gamma(c)\,\Gamma(c-2\mu)}\, p^{k+m+\frac{1}{2}} \int_0^\infty t^{c-2\mu-2}$$

$$\times\, {}_3F_2\left(\begin{matrix}\frac{1}{2}-k\pm m,\ \frac{1}{2}+c-\lambda-\mu \\ c,\ c-2\mu\end{matrix};\ -\frac{t}{p}\right) \psi\left(t, p: c+k+m-2\mu-\frac{1}{2}\right) dt\,.$$

The change of the order of integration is permissible by virtue of de la Vallée Poussin's theorem [**1**, p. 504] when the generalised Laplace transforms of the functions $|h(t)|$ and $|t^\sigma e^{-\alpha t} h(t)|$ exist and the resulting integral is convergent.

4.1 Corollary.

If

$$h(t) \underset{k,\, m}{\overset{v}{=\!=}} \Phi(p)$$

and

$$t^\sigma e^{-\alpha t} h(t) \doteqdot \psi(p,\ \alpha: \sigma),$$

then

(4.4)
$$\Phi(p) = \frac{p^{k+m+\frac{1}{2}}}{\Gamma(c-2\mu)} \int_0^\infty t^{c-2\mu-1} {}_2F_1\left(\begin{matrix}\frac{1}{2}-k \pm m \\ c-2\mu\end{matrix}; -\frac{t}{p}\right)$$

$$\times \frac{\psi(p+t : c+k+m-2\mu-\frac{1}{2})}{p+t} \, dt ,$$

provided that the integral is convergent, $R(c-2\mu) > 0$, $R(p) > 0$.

For $\lambda + \mu = \frac{1}{2}$, (4.1) reduces to (4.4).

(4.4) was given by RATHIE [**6**, p. 132] in a slightly different form.

5. Theorem 4.

If

$$h(t) \underset{k,\ m}{\overset{v}{=}} \Phi(p)$$

and

$$e^{-\alpha^2 t^2} t^\sigma h(t^2) \underset{l,\ \nu}{\overset{v}{=}} \psi(p,\ \alpha : \sigma) ,$$

then

(5.1)
$$\Phi(p^2) = \frac{2p^{2k+2m+1}\,\Gamma(\frac{1}{2}+\nu-l+2c)}{\Gamma(2c)\,\Gamma(2c+2\nu)} \int_0^\infty t^{2c-2}$$

$$\times {}_4F_4\left[\begin{matrix}\frac{1}{2}-k \pm m,\ \frac{1}{4}+c+\frac{1}{2}(\nu-l),\ \frac{3}{4}+c+\frac{1}{2}(\nu-l) \\ c,\ c+\frac{1}{2},\ c+\nu,\ c+\nu+\frac{1}{2}\end{matrix}; -\frac{t^2}{4p^2}\right]$$

$$\times \psi(t,\ p : 2k+2m+2c) \, dt ,$$

provided that the generalized Laplace transform of $|h(t)|$ and $|t^\sigma e^{-\alpha^2 t^2} h(t^2)|$ exist and the integral is convergent, $R(c) > 0$, $R(c+\nu) > 0$ and $R(p) > 0$.

Proof.

We have, SHANKER [4, p. 454]

$$(5.2)\qquad W_{k,m}\left(\frac{1}{2}\,p^2\right)=\frac{2^{-k}\,\Gamma\left(\frac{1}{2}+\nu-l+2c\right)}{\Gamma(2c)\,\Gamma(2c+2\nu)}\,e^{-\frac{1}{4}p^2}\,p^{2k}$$

$$\times\int_0^\infty e^{-\frac{1}{2}u}\,u^{\nu+2c-\frac{3}{2}}\,W_{l,\nu}(u)$$

$$\times\,{}_4F_4\left(\begin{matrix}\frac{1}{2}-k\pm m,\ \frac{1}{4}+c+\frac{1}{2}(\nu-l),\ \frac{3}{4}+c+\frac{1}{2}(\nu-l)\\ c,\ c+\frac{1}{2},\ c+\nu,\ c+\nu+\frac{1}{2}\end{matrix};\,-\frac{u^2}{2p^2}\right)\mathrm{d}u,$$

where $R(c)>0$ and $R(c+\nu)>0$.

Replacing p by $\sqrt{2}px$, u by xt in (5.2) subsituting for $W_{k,m}(p^2x^2)$ in

$$\Phi(p^2)=2p^2\int_0^\infty x\,(p^2x^2)^{m-\frac{1}{2}}\,e^{-\frac{1}{2}p^2x^2}\,W_{k,m}(p^2x^2)\,h(x^2)\,\mathrm{d}x,$$

and changing the order of integration which can easily be seen to be permissible under the conditions stated with the theorem, we obtain the result.

5.1 Corollary.

When $l+\nu=\frac{1}{2}$, the theorem takes the following form.

If

$$h(t)\,\underset{k,\ m}{\overset{v}{=\!=}}\,\Phi(p)$$

and

$$t^\sigma\,e^{-\alpha^2t^2}\,h(t^2)\doteqdot\psi(p,\ \alpha:\sigma)\,,$$

then

$$(5.3)\qquad \Phi(p^2)=\frac{2p^{2k+2m+1}}{\Gamma(2c)}\int_0^\infty t^{2c-2}$$

$$\times\,{}_2F_2\left(\begin{matrix}\frac{1}{2}-k\pm m\\ c,\ c+\frac{1}{2}\end{matrix};\,-\frac{t^2}{4p^2}\right)\psi(t,\ p:2k+2m+2c)\,\mathrm{d}t,$$

provided that the generalized Laplace transform of $| h(t) |$ and Laplace transform of $| t^{\sigma} e^{-\alpha^2 t^2} h(t^2) |$ exist and the integral is convergent, $R(c) > 0$, $R(p) > 0$.

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References

[1] T. J. I'A. BROMWICH, *An introducion to the theory of infinite series*, MacMillan, London, 1955.

[2] A. ERDELYI, *Integral representations for Whittaker function*, Proc. Benares Math. Soc., (2), **1** (1939), 39-53.

[3] A. ERDELYI, *Higher Transcendental Functions*, Vol. I, Mc Graw-Hill, New York (1953).

[4] HARI SHANKER, *Certain Integral representaions for Whittaker functions*, Proc. Camb. Phil. Soc., **44**, Part 3 (1947), 453-455.

[5] ROOP NARAIN, *Some properties of generalized Laplace transform-I*, Riv. Mat. Univ. Parma, **8**, (1957), 283-306.

[6] C. B. RATHIE, *Some results involving Hypergeometric and E-functions*, Proc. Glasg. Math. Assoc., **2** (1955) 132-138.

[7] R. K. SAXENA, *Some theorems on Generalized Laplace transform-I*, Proc. Nat. Inst. Sci., India, **26**, (1960), 400-413.

[8] R. S. VARMA, *On a generalization of Laplace transform*, Proc. Nat. Acad. Sci. India, Sec. A, **20**, (1951), 209-216.