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A Problem in Rectilinear Congruences Using Tensor Calculus. (**)

1. – Let the co-ordinates of a point P on the surface of reference S of a rectilinear congruence be given by $x^i = x^i(u^1, u^2)$ $(i = 1, 2, 3)$ and the direction cosines of the ray of the congruence through x^i by $\lambda^i = \lambda^i(u^1, u^2)$ $(i = 1, 2, 3)$. Since in general the rays of the congruence are not normal to S, we have

$$\lambda^i = p^\alpha x^i,\quad \alpha + qX^i \tag{1.1}$$

and

$$\lambda^i \lambda^i = 1, \tag{1.2}$$

where: p^α are the contravariant components of a vector in the surface at P; $x^i_{,\alpha}$ $(\alpha = 1, 2)$ are the direction numbers and denote covariant differentiation of x^i with regard to u^α based on the first fundamental tensor

$$g_{\alpha\beta} = x^i_{,\alpha} x^i_{,\beta} \tag{1.3}$$

of the surface S; q is a positive scalar function such that if θ is the angle between the normal to the surface at a point P and the line of the congruence through P, then

$$q = \lambda^i X^i = \cos\theta \tag{1.4}$$

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and

$$p^{\alpha}\, p^{\beta}\, g_{\alpha\beta} = \sin^2 \theta\,. \tag{1.5}$$

In what follows Latin indices take the values 1, 2, 3, and Greek indices the values 1, 2.

2. – Consider a curve C: $x^i = x^i(s)$ on a surface S. Let α^i, β^i, γ^i be the direction cosines of the tangent, principal normal and binormal at a point $P(x^i)$ of the curve through which a ray of the congruence passes. Let σ be the angle between λ^i and α^i, i.e. between λ^i and $\mathrm{d}x^i/\mathrm{d}s$, φ the angle between λ^i and β^i and ψ the angle between λ^i and γ^i. Then it is clear that we can write

$$\lambda^i = \cos\sigma\cdot\alpha^i + \cos\varphi\cdot\beta^i + \cos\psi\cdot\gamma^i \tag{2.1}$$

or

$$\gamma^i = \sec\psi\cdot\left[\lambda^i - \cos\sigma\cdot\frac{\mathrm{d}x^i}{\mathrm{d}s} - \cos\varphi\cdot\beta^i\right]. \tag{2.2}$$

By FRENET's formulae we have (K being the curvature)

$$K = \beta^i\frac{\mathrm{d}\alpha^i}{\mathrm{d}s} = \gamma^i \times \alpha^i\frac{\mathrm{d}\alpha^i}{\mathrm{d}s}\,,$$

$$K = \sec\psi\cdot\left[\lambda^i - \cos\sigma\cdot\frac{\mathrm{d}x^i}{\mathrm{d}s} - \cos\varphi\cdot\beta^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}^2x^i}{\mathrm{d}s^2}\right],$$

or

$$K = \sec\psi\cdot\left[\lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s} \quad \frac{\mathrm{d}^2x^i}{\mathrm{d}s^2}\right], \tag{2.2a}$$

on dropping out vanishing determinants since $K\beta^i = \dfrac{\mathrm{d}\alpha^i}{\mathrm{d}s} = \dfrac{\mathrm{d}^2x^i}{\mathrm{d}s^2}$. Again

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} = \varrho^{\delta}\, x^i_{,\delta} + K_n\, X^i\,, \tag{2.3}$$

where K_n is the normal curvature of the suface in the directions of the curve C

and ϱ^α is the curvature vector of the curve C. Therefore

$$K = \sec\psi \begin{vmatrix} \lambda^i & \dfrac{dx^i}{ds} & \varrho^\delta x^i_{,\delta} + K_n X^i \end{vmatrix} =$$

$$= \sec\psi \left[\begin{vmatrix} \lambda^i & \dfrac{dx^i}{ds} & \varrho^\delta x^i_{,\delta} \end{vmatrix} + K_n \begin{vmatrix} \lambda^i & \dfrac{dx^i}{ds} & X^i \end{vmatrix} \right] =$$

$$= \sec\psi \, [(p^\eta x^i_{,\eta} + q X^i \quad x^i_{,\mu} \quad x^i_{,\delta}) \varrho^\delta u'^\mu + K_n (p^\eta x^i_{,\eta} + q X^i \quad x^i_{,\mu} \quad X^i) u'^\mu] =$$

$$= \sec\psi \, [q e_{\mu\delta} \varrho^\delta u'^\mu + K_n p^\eta e_{\eta\mu} u'^\mu] ,$$

or

$$K = \sec\psi \, e_{\mu\delta} [q \varrho^\delta - K_n p^\delta] u'^\mu, \tag{2.4}$$

where

$$e_{\mu\delta} = (X^i \quad x^i_{,\mu} \quad x^i_{,\delta}) .$$

Particular Cases:

(i) If the ray of the congruence lies in the rectifying plane, we get

$$K = \operatorname{cosec}\sigma \cdot e_{\mu\delta} (q \varrho^\delta - K_n p^\delta) u'^\mu,$$

which is the same as obtained by MISHRA [1].

(ii) If the ray of the congruence becomes normal to the surface (i.e. the vector $p^\delta = 0$), then

$$K = \sec\psi \, e_{\mu\delta} \varrho^\delta u'^\mu .$$

(iii) If the ray of the congruence becomes tangent to the surface (i.e. $q = 0$), then

$$K = K_n \sec\psi \, e_{\delta\mu} p^\delta u'^\mu .$$

If K_u is the union curvature of the curve C [3], then

$$K_u = e_{\delta\mu} (K_n l^\delta - \varrho^\delta) u'^\mu, \tag{2.5}$$

where $l^\delta = p^\delta/q$. Then using (2.4) in (2.5), we hav

$$K = \cos\theta \cdot \sec\psi \cdot K_u .$$

Another expression for curvature can be obtained as follows: $K\beta^i = \mathrm{d}\alpha^i/\mathrm{d}s$. Using (2.1) and (2.3) this equation becomes

$$(2.6) \qquad p^\delta x^i_{,\delta} + K_n X^i = K\,\beta^i = K\,\gamma^i \times \alpha^i =$$

$$= K \sec\psi \left[\lambda^i - \cos\sigma \cdot \frac{\mathrm{d}x^i}{\mathrm{d}s} - \cos\varphi \cdot \beta^i\right] \times \frac{\mathrm{d}x^i}{\mathrm{d}s} =$$

$$= K \sec\psi \left[\lambda^i \times \frac{\mathrm{d}x^i}{\mathrm{d}s} + \cos\varphi \cdot \alpha^i \times \beta^i\right].$$

Multipliyng (2.6) with X^i and summing with respect to i:

$$K_n = K \sec\psi \left[\left[X^i \quad \lambda^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}s}\right] + \frac{\cos\varphi}{K}\left[X^i \quad \frac{\mathrm{d}x^i}{\mathrm{d}x} \quad \varrho^\delta x^i_{,\delta} + K_n X^i\right]\right] =$$

$$= \sec\psi \,[K(X^i \quad p^\delta x^i_{,\delta} + q\,X^i \quad x^i_{,\beta} u'^\beta) + \cos\varphi\,(X^i \quad x^i_{,\beta} u'^\beta \quad \varrho^\delta x^i_{,\delta} +$$

$$+ K_n X^i)] = \sec\psi\,[K\,e_{\delta\beta}\,p^\delta\,u'^\beta + \cos\varphi \cdot e_{\beta\delta}\,\varrho^\delta\,u'^\beta].$$

Hence

$$(2.7) \qquad K_n = \sec\psi\, e_{\delta\beta}(K\,p^\delta - \cos\varphi \cdot \varrho^\delta)\,u'^\beta.$$

If the line of the congruence lies in the rectifying plane, we have

$$(2.8) \qquad K_n = K \operatorname{cosec}\sigma \cdot e_{\delta\beta}\,p^\delta\,u'^\beta,$$

which is the same as obtained by MISHRA [**1**].

Multiplying the equation (2.6) by λ^i and summing up with respect to i, we obtain

$$p^\alpha \varrho_\alpha + q\, K_n = K \sec\psi \cdot \cos\varphi \cdot (\lambda^i \quad \alpha^i \quad \beta^i) =$$

$$= \sec\psi \cdot \cos\varphi \cdot (p^\alpha x^i_{,\alpha} + q\, x^i \qquad x^i_{,\beta}\, u'^\beta \qquad \varrho^\delta x^i_{,\delta} + K_n X^i) =$$

$$= \sec\psi \cdot \cos\varphi \cdot e_{\alpha\beta} [K_n\, p^\alpha - q\, \varrho^\alpha]\, u'^\beta.$$

Hence the equation

$$p^\alpha \varrho_\alpha + q K_n = \sec\psi \cdot \cos\varphi \cdot e_{\alpha\beta} [K_n\, p^\alpha - q\, \varrho^\alpha]\, u'^\beta \tag{2.9}$$

can be taken as the equation of the curve C on the surface S.

Particular cases of (2.9).

(i) If the line of the congruence lies in the osculating plane, equation (2.9) reduces to

$$e_{\alpha\beta} [K_n\, p^\alpha - q\, \varrho^\alpha]\, u'^\beta = 0,$$

which is the same as the equation of union curves as obtained by SPRINGER [3].

(ii) If the line of the congruence lies in the rectifying plane, then equation (2.9) reduces to

$$p^\alpha \varrho_\alpha + q\, K_n = 0,$$

which is the same as obtained by MISHRA [1].

3. – We have as before

$$\gamma^i = \sec\psi \cdot \left[\lambda^i - \cos\sigma \cdot \frac{dx^i}{ds} - \cos\varphi \cdot \beta^i\right].$$

Therefore

$$\frac{d\gamma^i}{ds} = \sec\psi \cdot \tan\psi \cdot \frac{d\psi}{ds} \cdot \left[\lambda^i - \cos\sigma \cdot \frac{dx^i}{ds} - \cos\varphi \cdot \beta^i\right] + \sec\psi \cdot \left[\frac{d\lambda^i}{ds}\right.$$

$$\left. - \cos\sigma \cdot \frac{d^2x^i}{ds^2} + \sin\sigma \cdot \frac{d\sigma}{ds}\frac{dx^i}{ds} - \cos\varphi \cdot \frac{d\beta^i}{ds} + \sin\varphi \cdot \frac{d\varphi}{ds} \cdot \beta^i\right].$$

By FRENET's formulae we have

$$\tau = \beta^i \cdot \frac{d\gamma^i}{ds} =$$

$$= \beta^i \cdot \left\{ \sec\psi \cdot \tan\psi \cdot \frac{d\psi}{ds} \left[\lambda^i - \cos\sigma \cdot \frac{dx^i}{ds} - \cos\varphi \cdot \beta^i \right] + \right.$$

$$\left. + \sec\psi \cdot \left[\frac{d\lambda^i}{ds} - \cos\sigma \cdot \frac{d^2x^i}{ds^2} + \sin\sigma \cdot \frac{d\sigma}{ds} \cdot \frac{dx^i}{ds} - \cos\varphi \cdot \frac{d\beta^i}{ds} + \sin\varphi \cdot \frac{d\varphi}{ds} \cdot \beta^i \right] \right\} =$$

$$= \gamma^i \times \alpha^i \cdot \left\{ \sec\psi \cdot \tan\psi \cdot \frac{d\psi}{ds} \cdot (\lambda^i - \cos\varphi \cdot \beta^i) + \right.$$

$$\left. + \sec\psi \cdot \left[\frac{d\lambda^i}{ds} - \cos\sigma \cdot \frac{d^2x^i}{ds^2} + \sin\varphi \cdot \frac{d\varphi}{ds} \cdot \beta^i \right] \right\} .$$

Since $\beta^i \dfrac{dx^i}{ds} = 0$, $\beta^i \dfrac{d\beta^i}{ds} = 0$. Therefore

$$\tau = \left[\sec\psi \cdot \left[\lambda^i - \cos\sigma \cdot \frac{dx^i}{ds} - \cos\varphi \cdot \beta^i \right] \quad \frac{dx^i}{ds} \right.$$

$$+ \sec\psi \cdot \tan\psi \cdot \frac{d\psi}{ds} \cdot (\lambda^i - \cos\varphi \cdot \beta^i) +$$

$$\left. + \sec\psi \cdot \left[\frac{d\lambda^i}{ds} - \cos\sigma \cdot \frac{d^2x^i}{ds^2} + \sin\varphi \cdot \frac{d\varphi}{ds} \beta^i \right] \right] .$$

On dropping out vanishing determinants, we have

$$\tau = \sec^2\psi \cdot \left[\left[\lambda^i \quad \frac{dx^i}{ds} \quad \frac{d\lambda^i}{ds} \right] - \cos\sigma \cdot \left[\lambda^i \quad \frac{dx^i}{ds} \quad \frac{d^2x^i}{ds^2} \right] + \right.$$

$$\left. + \sin\varphi \cdot \frac{d\varphi}{ds} \left[\lambda^i \quad \frac{dx^i}{ds} \quad \beta^i \right] - \cos\varphi \cdot \left[\beta^i \quad \frac{dx^i}{ds} \quad \frac{d\lambda^i}{ds} \right] \right] .$$

Now, using (2.2a),

$$\left[\lambda^i \quad \frac{dx^i}{ds} \quad \beta^i\right] = \frac{1}{K}\left[\lambda^i \quad \frac{dx^i}{ds} \quad \frac{d^2x^i}{ds^2}\right] = \cos\psi\,,$$

$$\left[\lambda^i \quad \frac{dx^i}{ds} \quad \frac{d\lambda^i}{ds}\right] = e_{\delta\nu}\,(p^\delta\, v\, \varphi - q\, \mu^\delta_\varphi)\, u'^\nu\, u'^\varphi\,,$$

$$\left[\beta^i \quad \frac{dx^i}{ds} \quad \frac{d\lambda^i}{ds}\right] = \frac{1}{K}\, e_{\delta\nu}\,(\varrho^\delta\, v_\varphi - K_n\, \mu^\delta_\varphi)\, u'^\nu\, u'^\varphi\,,$$

(cfr. [2]). Therefore we have

$$\tau = \sec^2\psi\cdot\left[e_{\delta\nu}\cdot(p^\delta\, v_\varphi - q\,\mu^\delta_\varphi)\; u'^\nu\cdot u'^\varphi - \cos\psi\cdot\left[\cos\sigma - \sin\varphi\cdot\frac{d\varphi}{ds}\right] - \right.$$

$$\left. - \frac{\cos\varphi}{K}\cdot e_{\delta\nu}(\varrho^\delta v_\varphi - K_n\,\mu^\delta_\varphi)\cdot u'^\nu . u'^\varphi\right].$$

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